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A GEOMETRIC APPROACH TO SINGULARLY PERTURBED NONLOCAL REACTION-DIFFUSION EQUATIONS*

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Abstract. In the context of a microwave heating problem, a geometric method to construct a spatially localized, 1-pulse steady-state solution of a singularly perturbed, nonlocal reaction-diffusion equation is introduced. The 1-pulse is shown to lie in the transverse intersection of relevant invariant manifolds. The transverse intersection encodes a consistency condition that all solutions of nonlocal equations must satisfy. An oscillation theorem for eigenfunctions of nonlocal operators is established. The theorem is used to prove that the linear operator associated with the 1-pulse solution possesses an exponentially small principal eigenvalue. The existence and instability of n-pulse solutions is also proved. A further application of the theory to the Gierer–Meinhardt equations is provided.

Key words. nonlocal reaction-diffusion equation, geometric singular perturbation theory, transversality, metastability, eigenvalues

AMS subject classifications. 35K57, 35K60, 34C10

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1. Introduction.

This paper is concerned with establishing a geometric method to analyze singularly perturbed, nonlocal reaction-diffusion equations. Such equations arise in microwave heating applications [18], activator-inhibitor chemical systems [20], thermistor [19], and ballast resistor problems [3], among other places. Nonlocal equations also are of interest because a higher-dimensional system can often be recast into a lower-dimensional nonlocal system [5, 7, 9, 14]. Existence of stationary solutions for scalar nonlocal equations has been considered by [3, 7, 9, 18, 19]. Stability of solutions for scalar nonlocal equations has been studied by [1, 3, 5, 10, 11, 12, 14].

Various methods for showing existence of solutions to nonlocal, boundary value problems have been employed. In [3, 9], it is shown that a homogeneous steady-state solution becomes unstable and a certain bifurcation occurs, thereby yielding a new steady-state or time-periodic solution. Lacey uses Picard iteration to show that nonhomogeneous steady-state solutions exist for equations modeling thermistors [19]. These methods establish the existence of solutions, but are not constructive in the sense that they give little information about the structure of solutions and where they lie in an appropriate phase space. Alternatively, asymptotic methods can be used to formally construct solutions [18]. However, asymptotic analysis does not actually yield a proof that a solution exists, and without a proof, one cannot be certain that what is constructed by formal asymptotics actually corresponds to the asymptotics of the true solution. Indeed, there are examples in the literature in which what appear to be asymptotic approximations of solutions are derived, but in fact there are no true solutions nearby at all. In the spirit of the approach that we shall take, Doelman and Rottschäfer [7] have studied a nonlocal reduction of the Ginzburg–Landau equations from a geometric point of view. The main purpose of their work is to compare a nonlocal model to a singularly perturbed one to determine whether the former is a

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good approximant of the latter. As a result, their nonlocal equations do not involve singular perturbations.

In this paper, in the context of a microwave heating problem, it is shown how to use geometric singular perturbation theory to construct spatially localized 1-pulse solutions for a nonlocal boundary value problem. A 1-pulse solution contains one spatially localized local maxima. This solution is shown to lie in the transverse intersection of relevant invariant manifolds. Two manifolds intersect transversely, if at a point of intersection, the tangent spaces of these manifolds span the ambient space. Solutions to nonlocal equations must satisfy a so-called consistency condition, which we discuss in detail below. An important byproduct of this work is the development of a geometric method to determine which trajectories in phase space satisfy the consistency condition. We show how to replace the scalar nonlocal boundary value problem with a higher-dimensional local boundary value problem in which the consistency constraint has been embedded.

Transversality is obtained as a direct result of the higher dimensionality. In fact, our analysis shows that transversality implies satisfaction of the consistency condition. Furthermore, since transversality of manifolds implies local uniqueness of intersections, we also obtain local uniqueness of the 1-pulse solution. This means that in a neighborhood of phase space of a 1-pulse solution, there are no other steady-state solutions. We also prove the existence of spatially localized n-pulse solutions.

Freitas [10, 11, 12] has obtained extensive results on the stability of scalar nonlocal reaction-diffusion equations. He shows how to locate the spectrum of a nonlocal linear operator by seeing how the spectrum of a related local operator changes under perturbations. Here, we locate the spectrum of a nonlocal version of a standard Sturm–Liouville operator. We show that the 1-pulse is metastable in that the nonlocal operator possesses an exponentially small principal eigenvalue. For $n \geq 2$, the n-pulses are unstable. The primary tool we employ is an oscillation theorem found in Bose and Kriegsmann [1]. Using Freitas' results, we show that this theorem holds under more general circumstances than those found in [1].

The equation of interest arises in microwave heating applications. A spatially localized hot spot forms in a thin ceramic fiber when it is microwave heated in a highly resonant, single mode cavity. The spot forms along the axis of the sample and begins to propagate [21]. In most cases the spot eventually becomes stationary, thereby leaving a localized region of the fiber at a dramatically higher temperature than the rest. Kriegsmann [18] derives the following nonlocal reaction-diffusion equation to model this phenomena:

$$U_t = \epsilon^2 U_{xx} + \frac{pf(U)}{1 + c^2(\int_0^1 f(U)\ dx)^2} - h(U), \tag{1.1}$$

$$U_x(0,t) = U_x(1,t) = 0.$$

Here $0 \leq x \leq 1$, U corresponds to the dimensionless temperature along the fiber and is assumed to be nonnegative; p is the dimensionless power which is proportional to the square of the amplitude of the mode which excites the cavity; and c lumps several physical parameters together. The function $h(U)$ models heat loss at the surface of the fiber due to convection and radiation and satisfies $h(0) = 0$, $h'(U) > 0$. The function $f(U)$ represents the effective electrical conductivity of a low-loss ceramic, such as alumina, and satisfies $f(0) = 1$, $f'(U) > 0$ and $f(U)$ grows faster than $h(U)$

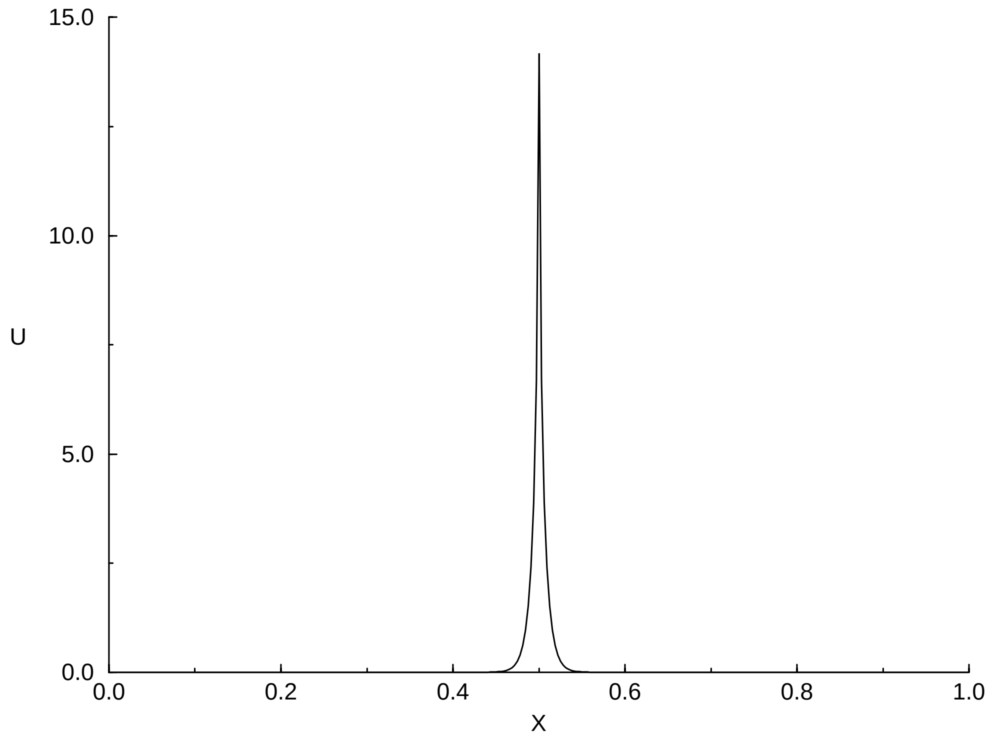

FIG. 1.1. *Kriegsmann's* 1*-pulse solution with* $\epsilon = 0.01$*,* $p = 1.0$*,* $c = 0.01$*,* $\beta = 0.01$*,* $c_1 = 1.0$.

for sufficiently large values of U. Both f and h are assumed to be sufficiently smooth. The nonlocal term models the detuning effect the heated fiber has upon the cavity. The diffusion constant ϵ is the aspect ratio of the fiber and in practice is much less than one, thereby making (1.1) singularly perturbed.

In [18], Kriegsmann chooses $f(U) = e^{c_1 U}$, $c_1 > 0$ and $h(U) = 2(U+\beta[(U+1)^4-1])$, $\beta \ll 1$. There, he formally constructs a localized, 1-pulse, steady-state solution ($U_t = 0$) using matched asymptotic expansions (see Figure 1.1). Kriegsmann shows that for ϵ sufficiently small, the 1-pulse possesses the following characteristics:

(C1) The solution is symmetric about $x = 1/2$.
(C2) It is nearly constant and close to zero valued on most of $[0, 1]$.
(C3) On a small interior layer centered around $x = 1/2$, the solution attains a maximum value that tends to ∞ as $\epsilon \to 0$. The value of the nonlocal integral term also tends to ∞ in this limit.

Recently, Bose and Kriegsmann [1] proved that this solution is metastable. The solution is, in fact, unstable, but perturbations of the solution decay to the solution or a translate of it, which then persists for exponentially long amounts of time.

In this paper, we study (1.1) with $f(U) = 1 + U^2$ and $h(U) = 2U$. Our choice of these functions is motivated by the fact that they are the simplest functions for which solutions of (1.1) retain the qualitative features of those found in [18]. Moreover, this choice simplifies the analysis so as to focus on the geometric approach. See Remark 3.6 for a discussion on the effect of including $O(\beta)$ terms in the function $h(U)$. To construct a steady-state 1-pulse solution of (1.1), we study the following boundary value problem:

$$\epsilon^2 U_{xx} + \frac{p(1+U^2)}{1+c^2(\int_0^1 1+U^2\ dx)^2} - 2U = 0, \tag{1.2}$$

$$U_x(0) = U_x(1) = 0.$$

Let $I = \int_0^1 U^2\ dx$. Replacing $\int_0^1 1+U^2\ dx$ by $1+I$, we note that the value I is determined by the solution itself. Thus a solution of (1.2) must satisfy the *consistency condition*

$$I_\star = \int_0^1 U^2(x, I_\star)\ dx. \tag{1.3}$$

The prescribed value $I_\star$ must be the value that is determined by computing the integral of U^2 along a trajectory of (1.2). This motivates the introduction of an auxiliary variable $V(x) = \int_0^x U^2\ dx$.

We recast (1.2) as the following system of first-order equations, where $' = d/dx$:

$$\begin{aligned} \epsilon U' &= W, & V' &= U^2, \\ \epsilon W' &= 2U - \tfrac{p(1+U^2)}{1+c^2(1+I)^2}, & I' &= 0. \end{aligned} \tag{1.4}$$

Note that $V(1) = \int_0^1 U^2\ dx$. This formulation, in a very natural way, recasts (1.2)–(1.3) into the boundary value problem (1.4), subject to the boundary conditions

$$\begin{aligned} (U, W, V, I) &= (U(0), 0, 0, I_\star) \text{ at } x = 0, \\ (U, W, V, I) &= (U(1), 0, I_\star, I_\star) \text{ at } x = 1. \end{aligned} \tag{1.5}$$

The new consistency condition is $V(1) = I_\star$. The analysis will show that $I_\star$ is unique, as a priori, this is not obvious. The values of $U(0)$ and $U(1)$ will also need to be determined. Since $\int_0^1 1+U^2\ dx = 1+I$ is simply a number, by phase plane considerations, any pulse solution of (1.4)–(1.5) will necessarily be symmetric about $x = 1/2$. Thus $U(0) = U(1)$. Also, the trivial equation $I' = 0$ in (1.4) is necessary, as the unique value $I_\star$ will be determined by transversality with respect to I.

We employ geometric singular perturbation theory to construct a 1-pulse solution of (1.4)–(1.5) [8, 15]. This theory involves finding solutions to sets of reduced equations obtained by formally setting $\epsilon = 0$ in appropriately scaled versions of (1.4). These solutions are then pieced together to form a singular solution. If the singular solution lies in certain manifolds which satisfy relevant transversality conditions, then an actual solution for $\epsilon \ll 1$ is obtained. Transverse intersections persist under perturbation which allows the $\epsilon = 0$ results to be extended to ϵ small. The first theorem that we prove is the following.

THEOREM 1.1. (a) *For ϵ sufficiently small, there exists a locally unique, symmetric* 1*-pulse solution U_1 of* (1.4)–(1.5)*. The maximum value U_{max} of this solution and the unique value $I_\star$ of the nonlocal term are given by*

$$U_{max} = 3c^2 I_\star^2/p + O(\epsilon), \qquad I_\star = \left(\frac{p^2}{12\sqrt{2}c^4\epsilon}\right)^{1/3} + O(\epsilon).$$

(b) *Let ϵ_1 be sufficiently small such that a symmetric* 1*-pulse solution exists as in* (a)*. Then for $\epsilon = \epsilon_1/n$, there exists a symmetric n-pulse solution U_n of* (1.4)–(1.5).

Stability of these solutions with respect to arbitrary perturbations is an important property for any physically realizable solution. For nonlocal equations of the type under consideration, Chafee [3] has shown that linear stability implies asymptotic stability. If a solution is asymptotically stable, then arbitrary perturbations of the solution decay in an appropriate function space. The main result to be proved here is that the 1-pulse solution is metastable, as is the case for the 1-pulse constructed in [18] and analyzed in [1].

THEOREM 1.2. (a) *The* 1*-pulse solution* U_1 *is a metastable solution of* (1.1).
(b) *For* $n \geq 2$*, n-pulse solutions* U_n *are unstable solutions of* (1.1) *with principal eigenvalue bounded away from the origin as* $\epsilon \to 0$*.*

We now give an outline of the paper. Due to criterion (C3) of Kriegsmann's solution, it turns out that (1.4) is not the correctly scaled version of the equations with which to work. In section 2, we rescale (1.4) to obtain the correct set of equations, together with relevant sets of reduced equations. In section 3, we construct a singular 1-pulse solution and show that it perturbs to yield an actual 1-pulse solution for ϵ sufficiently small. We also obtain n-pulse solutions using a simple rescaling argument. In section 4, we prove Theorem 1.2 concerning the stability of solutions. Section 5 contains numerical simulations of the full time-dependent equations. The numerically obtained values for U_{max} and $I_\star$ are shown to agree closely with the theoretically predicted ones. In section 6, we give a second application of our theory to a special limit of the Gierer–Meinhardt equations [13] which describe biological pattern formation. In this limit, the system of two reaction-diffusion equations can be reduced to a scalar nonlocal reaction-diffusion equation of the type considered above. A brief discussion concludes the paper.

2. The singular solution.

2.1. Scalings. The asymptotic analysis of Kriegsmann [18] shows that the maximum value of the 1-pulse solution of (1.1) tends to infinity as ϵ tends to 0. Moreover, the value of the nonlocal term I also tends to infinity in this limit. Numerical simulations of (1.1), with either set of nonlinearities discussed above, show that the main contribution to the nonlocal term occurs on the small interior layer around $x = 1/2$. A set of outer and inner equations associated with (1.4) can naively be derived in an attempt to capture this behavior. An outer set of equations is found simply by setting $\epsilon = 0$ in (1.4). An inner set of equations is obtained by rescaling the spatial variable x in a neighborhood of $1/2$ by using $\xi = (x - 1/2)/\epsilon$ and then setting $\epsilon = 0$ in the ensuing equations. The problem with this scaling is that $dV/d\xi = 0$. Thus there would be no contribution to the nonlocal integral term over the inner solution.

Following ideas similar to those found in [6], we rescale both the spatial and dependent variables in (1.4). Let $u = \epsilon^a U$, $w = \epsilon^a W$, $v = \epsilon^b V$, $Z = \epsilon^b I$, and $\xi = (x - 1/2)/\epsilon$. We use a capital Z for the new scaling of I to emphasize that transversality occurs with respect to this parameter. It is clear why U and W need to be scaled by the same power of ϵ. That V and I also need to be scaled by equal powers of ϵ follows from the fact that the consistency condition requires $V(1) = I$. Introducing these scalings in (1.4) with $\dot{} = d/d\xi$ yields

$$
\begin{aligned}
\dot{u} &= w, & \dot{v} &= \epsilon^{b+1-2a} u^2, \\
\dot{w} &= 2u - p\, \frac{\epsilon^{a+2b} + \epsilon^{2b-a} u^2}{\epsilon^{2b} + c^2(\epsilon^b + Z)^2}\,, & \dot{Z} &= 0, \\
\dot{Z} &= 0.
\end{aligned}
\tag{2.1}
$$

We require that all nonconstant terms in (2.1) be $O(1)$. We do this in order to make sure that both the linear (u) and the nonlinear (u^2) terms in the w component of the vector field are $O(1)$ so that the fast subsystem possesses a homoclinic orbit. Doing so implies $2b = a$ and $2a = b + 1$. Thus $a = 2/3$ and $b = 1/3$. Next, scale back to the x-variable, introduce $y = x$, and append the equation $dx/dy = 1$ to allow the invariant manifolds, defined below, to explicitly contain a spatial component. The set of equations then becomes

$$\begin{aligned} &\epsilon du/dy = w, && dZ/dy = 0, \\ &\epsilon dw/dy = 2u - p\tfrac{\epsilon^{4/3}+u^2}{\epsilon^{2/3}+c^2(\epsilon^{1/3}+Z)^2}, && dx/dy = 1, \\ &\epsilon dv/dy = u^2. && \end{aligned} \tag{2.2}$$

In these new scalings, the consistency condition (1.3) becomes

$$\epsilon Z_\star = \int_0^1 u^2(y, Z_\star)\, dy. \tag{2.3}$$

The boundary conditions (1.5) transform to

$$\begin{aligned} (u, w, v, Z, x) &= (u(0), 0, 0, Z_\star, 0), \\ (u, w, v, Z, x) &= (u(1), 0, Z_\star, Z_\star, 1). \end{aligned} \tag{2.4}$$

In terms of these boundary conditions, the consistency condition is also recognized as $v(1) = Z_\star$. As before the symmetry of the 1-pulse implies $u(0) = u(1)$, and these values along with $Z_\star$ will need to be determined by the analysis.

2.2. Solutions to reduced equations and singular solutions. The inner and outer sets of equations associated with (2.2) are now easy to obtain. To derive the outer equations, set $\epsilon = 0$ in (2.2):

$$\begin{aligned} &0 = w, && dZ/dy = 0, \\ &0 = 2u - p\tfrac{u^2}{c^2Z^2}, && dx/dy = 1, \\ &0 = u^2. && \end{aligned} \tag{2.5}$$

To obtain the inner inequations which incorporate the symmetry of the 1-pulse, condition (C1), we rescale (2.2) in a neighborhood of $y = 1/2$ using $\xi = (y - 1/2)/\epsilon$, and set $\epsilon = 0$:

$$\begin{aligned} &\dot u = w, && \dot Z = 0, \\ &\dot w = 2u - p\ \tfrac{u^2}{c^2Z^2}\ , && \dot x = 0, \\ &\dot v = u^2. && \end{aligned} \tag{2.6}$$

Solutions to (2.5) are easily found as both u and w are forced to be zero by the vector field. The values of v and Z, however, are unspecified, but constant for the outer equations. Solutions to the outer equations capture the behavior described in condition (C2). Solutions to (2.6) are also easily obtained. Note that the u and w equations are decoupled from the v equation and that Z acts as a parameter in these equations. For each value of Z, the $u - w$ equations are Hamiltonian with critical points $(0, 0)$ and $(2c^2Z^2/p, 0)$. The origin is a saddle point and the other critical point is a center. The $u - w$ phase plane is shown in Figure 2.1. The value of v along an

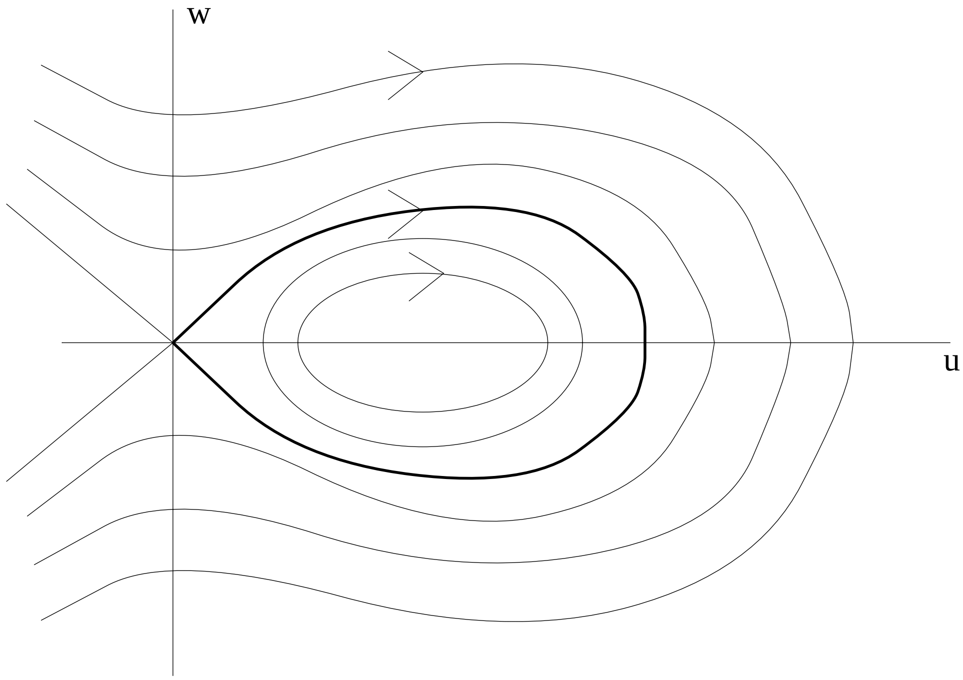

FIG. 2.1. *The phase plane for the* $u - w$ *equations of* (2.6). *The homoclinic solution is the darker curve.*

inner solution is determined by integrating u^2 along a trajectory in the $u - w$ phase plane. The homoclinic solution of the inner equations (2.6) captures the behavior described in (C3).

The singular solution can now be constructed. By definition, $v(0) = 0$. Thus at $y = 0$, only Z is unspecified. Since Z can vary, there exists a one-parameter family of singular solutions. We describe one particular member of this family. Fix $Z > 0$. The first piece of the singular trajectory is a solution of (2.5) that connects $(0, 0, 0, Z, 0)$ to $(0, 0, 0, Z, 1/2)$ in (u, w, v, Z, x) space. The second piece is the solution of (2.6) that connects $(0, 0, 0, Z, 1/2)$ at $\xi = -\infty$ to $(0, 0, v, Z, 1/2)$ at $\xi = \infty$. In the u and w components, this singular piece corresponds to the homoclinic orbit pictured in Figure 2.1. Since the value of v at $\xi = -\infty$ is different than at $\xi = +\infty$, the inner piece is actually a heteroclinic orbit in the full five-dimensional phase space. The third and final piece of the singular trajectory is a solution of (2.5) from $(0, 0, v, Z, 1/2)$ to $(0, 0, v, Z, 1)$.

With these scalings, the consistency condition (2.3) reduces to a particularly simple form. In the outer equations (2.5), $u = 0$. Thus, the outer solutions contribute nothing to the integral. The $\epsilon = 0$ value of $Z_\star$, denoted by Z_0, is determined solely by the inner equations and is given by

$$Z_0 = \int_{-\infty}^{\infty} u^2(\xi, Z_0)\, d\xi. \tag{2.7}$$

3. Invariant manifolds and transversality. We now pick out a unique singular solution which satisfies (2.7) from the one-parameter family of singular 1-pulse solutions. We then extend the analysis to ϵ small. To do both, we recast the above analysis into the language of invariant manifolds. Our analysis relies on the seminal work of Fenichel [8] on the persistence of invariant manifolds. See Jones [15] and the references therein for a thorough exposition of the theory and some of its applications.

Following Tin, Kopell, and Jones [22], we define manifolds of points which, respectively, satisfy the boundary conditions (2.4) at $y = 0$ and 1. We then flow the $y = 0$ boundary manifold forward to determine whether it intersects the boundary manifold at $y = 1$. Denote the flows of (2.5) and (2.6) as the outer and inner flows, respectively. These flows are used to track the evolution of the $y = 0$ boundary manifold over different pieces of the singular solution.

The $y = 0$ boundary manifold is defined by

$$B_0 = \{(u, w, v, Z, x) : u = w = v = x = 0\}.$$

Thus B_0 is a one-dimensional curve consisting solely of different Z values along which u, w, v, and x are restricted to 0. At $y = 1$, we define two different boundary manifolds. First, let

$$B_R = \{(u, w, v, Z, x) : u = w = 0, x = 1\}.$$

The manifold B_R is two-dimensional as both v and Z are free and by definition positive. It contains no information about the consistency condition. Enforcing $v(1) = Z$ restricts B_R to the following one-dimensional submanifold:

$$B_1 = \{(u, w, v, Z, x) : u = w = 0, v = Z, x = 1\}.$$

At $y = 1/2$, the jump off and touch down curves are defined. These are curves along which the outer and inner flows must match. The jump off curve is

$$J_0 = \{(u, w, v, Z, x) : u = w = v = 0, x = 1/2\}$$

and the touch down curve is

$$T_0 = \left\{(u, w, v, Z, x) : u = w = 0, v = \int_{-\infty}^{\infty} u^2(\xi, Z)\, d\xi, x = 1/2\right\}.$$

The touch down curve is determined by flowing J_0 forward under the inner flow. In particular, it contains no information about the consistency condition. In the next subsection, we will define an analogous curve T_c which will encode the consistency condition.

Denote by $B_0 \cdot y$ the two-dimensional manifold formed by flowing B_0 forward. Note that $J_0 = B_0 \cdot y|_{y=1/2^-}$, so the outer flow transversely intersects the jump off curve on the slow manifold. This is essential for the perturbation result later for ϵ small. Now use the inner flow to follow $J_0 = B_0 \cdot 1/2^-$. For each point on J_0, there exists a homoclinic solution of the inner $u - w$ equations (2.6). As mentioned earlier, considering the full set of inner equations in a five-dimensional phase space, this is actually a heteroclinic solution in that the values of v and Z on the jump off curve J_0 and touch down curve T_0 are dramatically different; see Figure 3.1. Thus the inner flow defines a two-dimensional (sheet) manifold of heteroclinic orbits, which connects J_0 to T_0. Therefore T_0 is also one-dimensional. Lastly, flow T_0 forward to $y = 1$ using the outer flow to obtain $T_0 \cdot 1$. The curve $T_0 \cdot 1$ is the image of B_0 flowed under the appropriate outer and inner equations. Thus we define $B_0 \cdot 1 = T_0 \cdot 1$. By

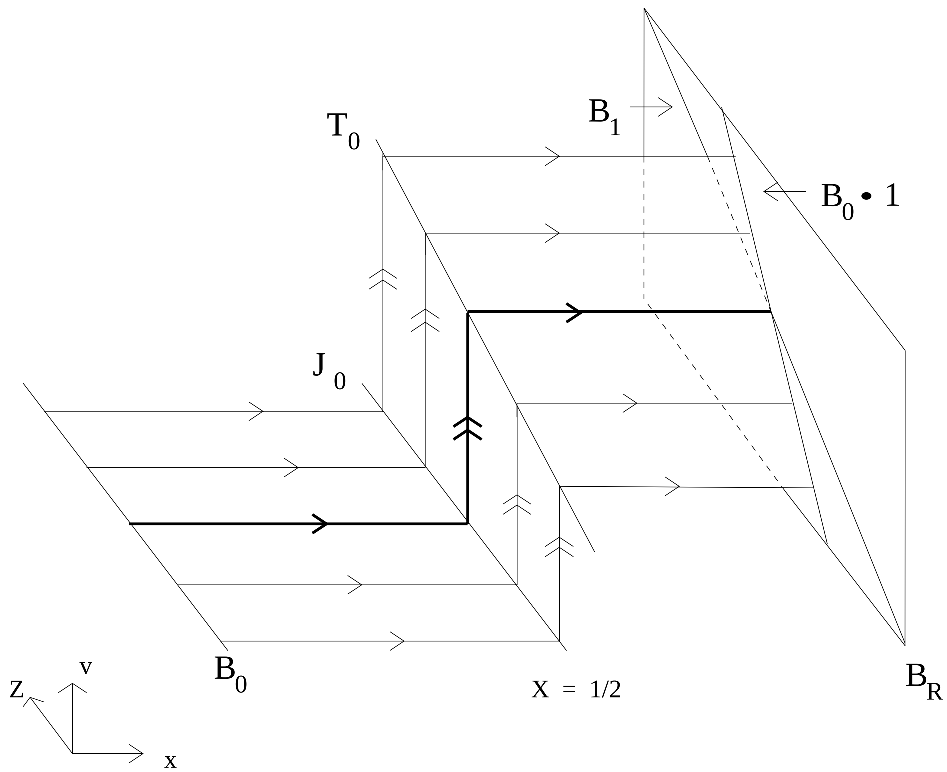

FIG. 3.1. *A graphical representation of the singular manifolds, curves, and flows, projected into (v, Z, x) space. The darker curve depicts the unique singular solution that satisfies the consistency condition.*

construction, $B_0 \cdot 1 \cap B_R \neq \emptyset$. If there is a singular 1-pulse solution, then it must satisfy the following geometric version of (2.7):

$$B_0 \cdot 1 \cap B_1 \neq \emptyset. \tag{3.1}$$

Further, if $B_0 \cdot 1$ transversely intersects B_1 in $\mathbf{R^2}$, then there exists a unique singular 1-pulse solution which satisfies (2.7). The following lemma establishes the transversality.

LEMMA 3.1. *The curve T_0 transversely intersects the line $v = Z$ at a unique point in (Z, v) space.*

Proof. It is easy to verify that

$$u(\xi) = \frac{3c^2 Z^2}{p} \operatorname{sech}^2 \frac{\xi}{\sqrt{2}}, \quad w(\xi) = \dot{u} \tag{3.2}$$

solves the $u - w$ equations of (2.6). Let $v(Z)$ be defined by the first equation of (3.3), below. It is easily checked that

$$v(Z) = \int_{-\infty}^{+\infty} u^2(\xi, Z)\, d\xi = \tfrac{12\sqrt{2}c^4 Z^4}{p^2}. \tag{3.3}$$

The graph of (3.3) is exactly T_0 projected onto (Z, v) space; see Figure 3.2. Note that it intersects the line $v = Z$ at exactly one point. This intersection value is calculated

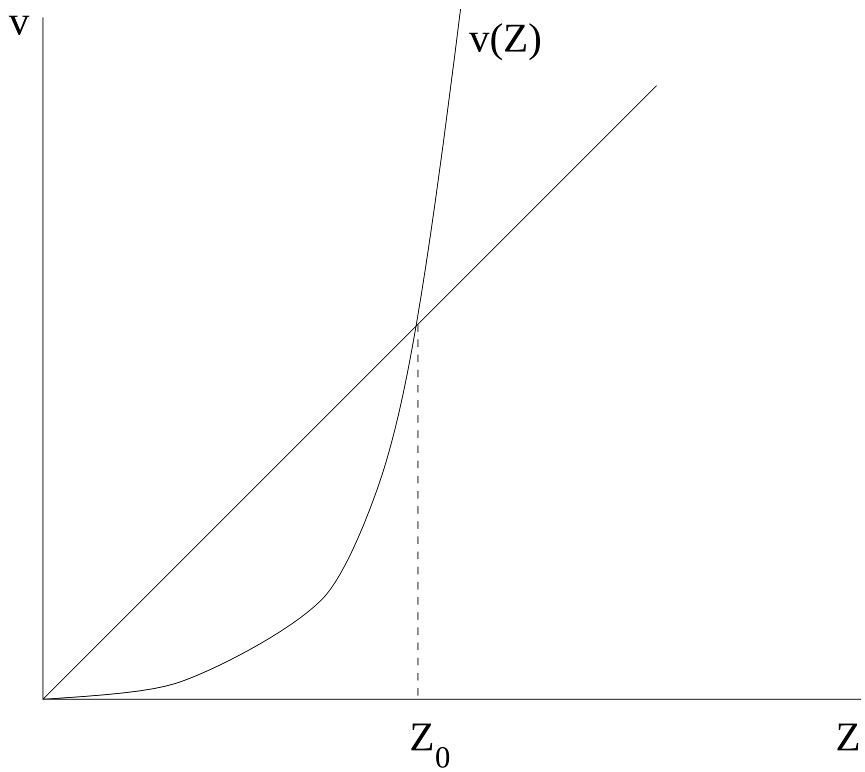

FIG. 3.2. *The transverse intersection of $v(Z)$ and $v = Z$ in (Z, v) space.*

by solving $Z_0 = v(Z_0)$, which yields

$$Z_0 = \left(\frac{p^2}{12\sqrt{2}c^4}\right)^{1/3}. \tag{3.4}$$

Therefore (3.4) determines the unique value Z_0 that satisfies (2.7). Transversality follows since the slope of T_0 at the point Z_0 is not equal to one, which is the slope of $v = Z$. ☐

Remark 3.2. While we use the closed form of $u(\xi)$ to obtain transversality, it will be clear from the estimates for the Gierer–Meinhardt equations in section 6 that the transversality can be obtained in the absence of a closed form solution.

Lemma 3.1 is sufficient to prove that $B_0 \cdot 1$ transversely intersects B_1, the restriction of B_R to the line $v = Z$. Since v changes only along the inner solution, in the (Z, v) plane, the curves T_0 and $B_0 \cdot 1$ are identical. Thus transversality follows from the lemma and is stated in the following corollary.

COROLLARY 3.3. *The curve $B_0 \cdot 1$ transversely intersects B_1 at $v = Z_0$ in (Z, v) space.*

3.1. The argument for $0 < \epsilon \ll 1$. A region of an invariant manifold M_0 is normally hyperbolic if all of the eigenvalues corresponding to eigenvectors normal to the manifold are bounded away from the imaginary axis. Fenichel [8] showed that a normally hyperbolic invariant manifold persists under perturbations. Moreover, the perturbed manifold M_ϵ is $O(\epsilon)$ close to M_0 and also retains this hyperbolic structure.

Finally he showed that the flow on M_ϵ is $O(\epsilon)$. For any Z, recall that $(u, w) = (0, 0)$ is a saddle point for the inner flow and that u and w are restricted to these values on B_0 and B_1. It follows then that the boundary manifolds B_0 and B_1 are normally hyperbolic. Therefore, they perturb to nearby manifolds B_0^ϵ and B_1^ϵ. We need to show that the forward evolution of B_0^ϵ transversely intersects B_1^ϵ at $x = 1$. Instead of making the calculation at $x = 1$, it is more convenient to check this intersection at some intermediate value $x = a$ by flowing B_1^ϵ backwards in space. We will assume that $x = a$ is in some sufficiently small deleted neighborhood of $x = 1/2$.

We need one more critical result concerning the singular flows. Figure 3.1 gives a picture of how the singular flows evolve, but it is deceptive in that it does not fully reveal the transversality that exists in the equations. Indeed, in Figure 3.1, it appears that there is no transversality due to the inner equations. While it is certainly true that the manifold leaving J_0 does not transversely intersect the manifold approaching T_0, we are not actually interested in T_0.

Instead, flow B_1 backwards to $x = 1/2$ under the outer flow. Define a new consistency curve by

$$T_c = \{(u, w, v, Z, x) : u = w = 0, v = Z, x = 1/2\}.$$

Consider once again the inner equations (2.5). Let $W^u(J_0)$ denote the two-dimensional center-unstable manifold of J_0 composed of the union of the one-dimensional unstable manifolds of the critical point $(0, 0)$ over different values of Z. Similarly, let $W^s(T_c)$ denote the two-dimensional center-stable manifold of T_c. Both of these manifolds exist since $(0, 0)$ is a hyperbolic critical point. Note that $W^u(J_0) \to T_0$ as $\xi \to \infty$, and that T_c projected into (Z, v) space is the line $v = Z$. The following is a corollary of Lemma 3.1 and Corollary 3.3.

COROLLARY 3.4. *The manifold $W^u(J_0)$ transversely intersects $W^s(T_c)$ in (u,w,Z) space at $Z = Z_0$.*

The corollary shows that the inner flow induces a transverse intersection of *the* manifolds needed to actually construct the 1-pulse solution. This transversality encodes the consistency condition. Finally, for later use, let $M_0 = \cup_{y\in[0,1/2]} B_0 \cdot y$ and $\mathcal{M}_0 = \cup_{y\in[1/2,1]} T_c \cdot y$, under the outer flow (2.5).

Tin, Kopell, and Jones [22] give conditions for general boundary value problems for which the existence of a singular solution implies the existence of an actual solution for ϵ small. Consistent with the major simplifications offered by geometric singular perturbation theory, these conditions are on the $\epsilon = 0$ singular manifolds. Thus the verification of these conditions occurs in lower-dimensional reduced settings. Tin, Kopell, and Jones' work is based on the exchange lemma of Jones and Kopell [16], which itself relies on Fenichel's invariant manifold theory. The hypotheses (H1)–(H3) below, which provide sufficient conditions to prove the existence of an actual solution for ϵ sufficiently small, are all based on transversality at $\epsilon = 0$ [22]. Stated in notation adapted for this paper, they are the following:

(H1) The outer flow on M_0 transversely intersects J_0.

(H2) $W^u(J_0)$ transversely intersects $W^s(T_c)$ in (u, w, Z) space.

(H3) The outer flow on $\mathcal{M}_0$ transversely intersects T_c.

For our situation, (H1) and (H3) are trivial to verify as can be seen in Figure 3.1. Note that these transversality calculations need to be verified in only a two-dimensional ambient space. Hypothesis (H2) follows directly from Corollary 3.4. Although the ambient space for this intersection is three-dimensional, the needed calculation occurs in a two-dimensional space.

Tin, Kopell, and Jones' results imply the existence of an actual solution for ϵ sufficiently small for the following reason. Let $\mathcal{B}_0^\epsilon$ and $\mathcal{B}_1^\epsilon$ denote the manifolds obtained by flowing B_0^ϵ forward and B_1^ϵ backward under (2.2), respectively. The perturbed manifold $\mathcal{B}_0^\epsilon$ is $O(\epsilon)$ close to M_0, up to a neighborhood of $x = 1/2$. Similarly $\mathcal{B}_1^\epsilon$ is $O(\epsilon)$ close to $\mathcal{M}_0$. Based on the exchange lemma [16], Tin, Kopell, and Jones show that when $\mathcal{B}_0^\epsilon$ and $\mathcal{B}_1^\epsilon$ veer away from these outer manifolds, they are $C^1 - O(\epsilon)$ close to $W^u(J_0)$ and $W^s(T_c)$, respectively. Thus not only are the perturbed manifolds $O(\epsilon)$ close to relevant singular manifolds, so are their tangent spaces. Since transversality is determined by the behavior of tangent spaces, the C^1 closeness is important. Therefore, since $W^u(J_0)$ and $W^s(T_c)$ intersect transversely at Z_0 independent of ϵ, the $C^1 - O(\epsilon)$ closeness of the perturbed manifolds $\mathcal{B}_0^\epsilon$ and $\mathcal{B}_1^\epsilon$ to these manifolds implies that they also intersect transversely for some $Z_\star$ $O(\epsilon)$ close to Z_0.

Therefore, we have shown that $B_0^\epsilon \cdot 1$ transversely intersects B_1^ϵ for ϵ sufficiently small. In $\mathbf{R}^2$, the unique point of intersection of these two curves determines $Z_\star$. This value of $Z_\star$ determines $I_\star$ which is then used in (1.2) to obtain $U_1(x)$. The value of $U_1(0) = U_1(1)$ can then be found. To determine U_{max}, we use the Hamiltonian associated with the inner equations (2.6):

$$H(u, w) = w^2/2 - u^2 + \frac{pu^3}{3c^2 Z_0^2}. \tag{3.5}$$

Since the value of H is conserved along trajectories, and, in particular, along the homoclinic of Figure 2.1, we can solve $H(0, 0) = H(u_{max}, 0)$. This yields $u_{max} = 3c^2 Z_0^2/p$. Finally, rescaling (u, v, w, Z) back to the original (U, W, V, I) variables and using Fenichel [8], the values U_{max} and $I_\star$ stated in Theorem 1.1 are obtained.

Remark 3.5. The geometric argument presented above fails to uniquely pick out the point at which the 1-pulse is centered. Indeed, it is only by using symmetry of the 1-pulse solution that we define the inner equations at $x = 1/2$ and not at some other point in the domain. See Ward [23] for a thorough discussion on this indeterminacy. As he shows, the metastability of solutions is a direct consequence of it.

Remark 3.6. In (1.2), we have not included the $O(\beta)$ terms which actually appear in Kriegsmann's model [18]. The geometric results presented here can be extended to the case where these terms are included in the vector field provided that β is $O(\epsilon^{2+\alpha})$, $\alpha > 0$. The reason why this restriction is needed is precisely because of characteristic (C3) of Kriegsmann's solution. On the interior layer, $U \to \infty$. Thus the included quartic U^4 would be the dominant term of the rescaled equations (2.1) unless β is $O(\epsilon^{2+\alpha})$. As a result, the existence and transverse intersection of $B_0^\epsilon \cdot 1$ with B_1^ϵ persists provided β is small enough.

3.2. Existence of *n*-pulse solutions. Equation (1.2) also admits n-pulse solutions for small values of ϵ. These solutions contain n equal, local maxima and are also symmetric about $x = 1/2$. A similar geometric argument as above could be given to construct these solutions. Instead, we present a simple rescaling argument which exploits the symmetry requirements of the solutions.[1]

Denote the symmetric 1-pulse solution by $\Phi_1(x)$. This solution satisfies

$$\epsilon^2 \Phi_{1_{xx}} + \frac{pf(\Phi_1)}{1 + c^2(\int_0^1 f(\Phi_1)\, dx)^2} - h(\Phi_1) = 0,$$

[1] I thank Gregory Kriegsmann for suggesting the rescaling argument to me.

$$\Phi_1'(0) = \Phi_1'(1) = 0.$$

Next define

$$\Phi_2(x) = \begin{cases} \Phi_1(x), & 0 < x < 1, \\ \Phi_1(x-1), & 1 < x < 2. \end{cases}$$

Then $\Phi_2(2x)$ satisfies

$$\frac{\epsilon^2}{4}\Phi_{2_{xx}} + \frac{pf(\Phi_2)}{1 + c^2(\int_0^1 f(\Phi_2)\,dx)^2} - h(\Phi_2) = 0,$$

$$\Phi_2'(0) = \Phi_2'(1) = 0.$$

Thus

$$\Phi_2(2x) = \begin{cases} \Phi_1(2x), & 0 < x < 1/2, \\ \Phi_1(2x-1), & 1/2 < x < 1, \end{cases}$$

and $\Phi_2(2x)$ has two local maximum at $x = 1/4$ and $x = 3/4$. Thus $\Phi_2(2x)$ is a symmetric 2-pulse solution of (1.2) which has been obtained by reflecting and rescaling the 1-pulse solution $\Phi_1(x)$. The diffusion constant is half that of the corresponding 1-pulse solution. Now continue the process to obtain a symmetric 4-pulse solution and so on. It is seen that a symmetric $\Phi_{2^m}(2^m x)$ solution exists which satisfies (1.2).

There is nothing special about base 2. In fact, defining

$$\Phi_3(x) = \begin{cases} \Phi_1(x), & 0 < x < 1, \\ \Phi_1(x-1), & 1 < x < 2, \\ \Phi_1(x-2), & 2 < x < 3, \end{cases}$$

we see that $\Phi_3(3x)$ satisfies (1.2) and is a symmetric 3-pulse solution. Continuing in this fashion, it is easy to obtain a symmetric n-pulse solution for any value of n.

In closing, we do not obtain local uniqueness of these solutions by this rescaling method. However, in the next section, we show that any n-pulse solutions for $n \geq 2$ are unstable, so this lack of information concerning local uniqueness is not so important.

4. Stability. In this section, we prove Theorem 1.2. In particular, we show that the 1-pulse solution constructed above is metastable, with its principal eigenvalue being exponentially small in ϵ. The n-pulse solutions, however, are unstable, with principal eigenvalues bounded away from the origin as $\epsilon \to 0$.

Our analysis relies on two key ingredients. The first is an oscillation theorem for the nonlocal eigenfunctions and their corresponding eigenvalues. This theorem is similar to the one found in Bose and Kriegsmann [1], but is established here under more general conditions. We need a different argument to prove it due to the choice of $f(u)$. The second ingredient will be an analysis of Ward [23], which shows the existence of an exponentially small eigenvalue for equations of the type that we consider.

The oscillation theorem that we prove below is quite general. It holds for a large class of nonlinearities, provided that the underlying pulse solution is symmetric about the midpoint of the domain of interest. In particular, the theorem also holds outside of the singular perturbation parameter regime. However, using the oscillation theorem to prove the metastability of the 1-pulse uses the singular structure in two important ways. First, it is needed to establish the existence of an exponentially small eigenvalue. Second, it is used to rule out the existence of an eigenfunction of strictly one sign. For local equations, it is well known that pulse solutions have O(1) with respect to ϵ unstable principal eigenvalues. The corresponding eigenfunction, after normalization, is strictly positive. Using our oscillation theorem and information about the structure of solutions as $\epsilon \to 0$, we show that no such eigenfunction can exist for the nonlocal problem. Thus, the nonlocal term can be viewed as stabilizing an unstable solution of the local problem.

In the standard manner, assume $U(x,t) = U_1(x) + \phi(x)e^{-\lambda t}$ and linearize (1.1) about U_1 to obtain the following nonlocal eigenvalue problem:

$$\epsilon^2\phi'' + (A(x) + \lambda)\phi = B(x)\int_0^1 C(x)\phi\,dx, \tag{4.1}$$

$$\phi'(0) = \phi'(1) = 0, \tag{4.2}$$

where

$$A(x) = -2 + \frac{2pU_1}{1 + c^2(1 + I_\star)^2},\ B(x) = \frac{2pc^2(1 + I_\star)(1 + U_1^2)}{(1 + c^2(1 + I_\star)^2)^2},\ C(x) = 2U_1. \tag{4.3}$$

Let L_1 be the linear operator associated with (4.1), which is given by

$$L_1\phi = -\epsilon^2\phi'' - A(x)\phi + B(x)\int_0^1 C(x)\phi\ dx. \tag{4.4}$$

Denote the spectrum of L_1 by $\sigma(L_1)$. If $Re\ \sigma(L_1) > 0$, then U_1 will be an asymptotically stable solution of (1.1) [3]. In [1], $f(u) = e^{c_1 u}$ which implies that L_1 is self-adjoint. Now, since $f(u) = 1 + u^2$, L_1 is not a self-adjoint operator. Thus there is no a priori guarantee that the eigenvalues of L_1 are real. We show below that due to the symmetry of U_1, the eigenvalues of L_1 must in fact be real.

The spectrum of L_1 can be related to the eigenvalues of the following Sturm–Liouville equation:

$$\epsilon^2\psi'' + (A(x) + \nu)\psi = 0, \tag{4.5}$$

$$\psi'(0) = \psi'(1) = 0. \tag{4.6}$$

Denote by L_0 the corresponding linear operator. For L_0, there exists a sequence of eigenvalues $\{\nu_n\}$ such that $\nu_0 < \nu_1 < \nu_2 \dots$ and corresponding eigenfunctions $\{\psi_n\}$ such that each eigenfunction has exactly n interior zeros [4]. Due to the fact that U_1 is symmetric about $x = 1/2$, it turns out that the eigenfunctions $\{\psi_n\}$ break up into two subsets: $\{\psi_{2n}\}$ which are even about $x = 1/2$ and $\{\psi_{2n+1}\}$ which are odd about $x = 1/2$. This occurs because $A(x)$, $B(x)$, and $C(x)$ all must be even about $x = 1/2$. As a result, note that

$$\int_0^1 C(x)\psi_{2n+1}\ dx = 0. \tag{4.7}$$

Therefore the odd local eigenpairs $(\nu_{2n+1}, \psi_{2n+1})$ also turn out to be nonlocal eigenpairs. This observation forms the basis for the following oscillation theorem for the nonlocal eigenvalues and eigenfunctions.

Oscillation Theorem. *Let λ be a nonlocal eigenvalue of L_1 with corresponding eigenfunction ϕ. For $n \geq 1$,*

(a) $\lambda = \nu_{2n-1}$ *if and only if* $\phi = \psi_{2n-1}$ *has* $2n-1$ *interior zeros.*

(b) $\nu_{2n-1} < \lambda < \nu_{2n+1}$ *if and only if* ϕ *has* $2n$ *interior zeros.*

(c) *Every interval* (ν_{2n-1}, ν_{2n+1}) *contains exactly one nonlocal eigenvalue except possibly one such interval which may contain at most two nonlocal eigenvalues.*

Remark 4.1. The oscillation theorem, along with Lemma 4.2 below, gives a complete description of $\sigma(L_1)$. Moreover, as will be apparent from the proof, the exact forms of the nonlinearities $f(u)$ and $h(u)$ are never used. As a result, the oscillation theorem holds quite generally.

Proof. Part (a) of the theorem is obvious. Part (b) is proved by Bose and Kriegsmann in [1]. To prove part (c), we need a different argument than in [1]. There, because $f(u) = e^{c_1 u}$, we were able to explicitly show that in the interval (ν_{2n-2}, ν_{2n}), there exists exactly one nonlocal eigenvalue. In the present situation, we have no information about the interval (ν_{2n-2}, ν_{2n}). We show, however, that the symmetry of U_1 forces part (c) to hold.

We need the following result.

Lemma 4.2. *The eigenvalues of L_1 are strictly real.*

Proof. Following Freitas [10], we introduce the parameter δ to define

$$L_\delta \phi = L_0 \phi + \delta B(x) \int_0^1 C(x) \phi \, dx. \tag{4.8}$$

Note that $\delta = 1$ yields L_1 as defined in (4.4) and $\delta = 0$ yields L_0. Freitas shows that the eigenvalues $\lambda(\delta)$ of L_δ vary continuously with δ. If $\lambda \in \sigma(L_\delta)$ for all δ, then Freitas calls the eigenvalue a fixed eigenvalue, otherwise the eigenvalue is a moving eigenvalue. Generically, as δ is varied from zero, some of the eigenvalues $\lambda(\delta)$ begin to move along the real axis. Freitas shows that eigenvalues cannot suddenly appear or disappear as δ is varied. Moreover, he shows that the only way complex conjugate eigenvalues can be created is if for some value of δ, two moving eigenvalues collide while traveling in opposite directions. Then for some nonzero δ interval, perhaps semi-infinite, they remain complex. Finally, Freitas shows that moving eigenvalues which are traveling in the same direction cannot cross over one another. We see here that the odd subscripted eigenvalues $\lambda_{2n+1}(\delta) = \nu_{2n+1}$ are fixed eigenvalues and that the even ones $\lambda_{2n}(\delta)$, where $\lambda_{2n}(0) = \nu_{2n}$, are moving eigenvalues.

In [1], the Prüfer transformation, $\tan \epsilon\phi'/\phi$, is used to show that for $\delta = 1$, there must exist a (moving) nonlocal eigenvalue in every subinterval (ν_{2n-1}, ν_{2n+1}). That same argument actually shows that this must be true for all δ. This observation is sufficient to rule out the existence of complex conjugate eigenvalues. We argue by contradiction. Suppose that there exists $\hat{\delta}$ at which $\lambda_0(\hat{\delta})$ and $\lambda_2(\hat{\delta})$ collide and become complex. Then since there must exist a real nonlocal eigenvalue in (ν_1, ν_3), it follows that $\lambda_4(\hat{\delta}) \in (\nu_1, \nu_3)$. In particular, there exists $\delta_4 < \hat{\delta}$ such that $\lambda_4(\delta_4) = \nu_3$. Since for any value of δ, (ν_3, ν_5) must also contain a real nonlocal eigenvalue, it follows that $\lambda_6(\delta_4) \in (\nu_3, \nu_5)$. Thus there exists a $\delta_6 < \delta_4 < \hat{\delta}$ such that $\lambda_6(\delta_6) = \nu_5$. Continuing by induction, there exists a monotone decreasing sequence $\{\delta_{2n}\}$ such that $\lambda(\delta_{2n}) = \nu_{2n-1}$ for $n \geq 2$.

The sequence $\{\delta_{2n}\}$ denotes the "crossing times" at which a moving eigenvalue crosses over the fixed eigenvalue immediately to its left. Since the sequence $\{\delta_{2n}\}$ is monotone decreasing, let us assume that $\lim_{n\to\infty} \delta_{2n} = \delta^\star$, where $\delta^\star \geq 0$. If $\delta^\star < 0$, then we are done. At $\delta = \delta^\star$, no moving eigenvalues can have crossed fixed ones.

But for $\delta = \delta^\star + \alpha$, where $\alpha > 0$ but arbitrarily small, an *infinite* number of moving eigenvalues must have crossed over fixed ones. The following lemma, which is a direct consequence of Proposition 3.8 in Freitas [10], shows that this situation cannot occur.

LEMMA 4.3. *For any fixed $\delta > 0$, only a finite number of moving eigenvalues can have crossed fixed ones.*

Proof. We sketch the proof here. The details of the proof can be found in Freitas [10]. Let $\eta \in \rho(L_0)$, where ρ is the resolvent set of L_0. Let $R(\eta, L_0)$ denote the resolvent of L_0. Using a general result of Kato [17], Freitas shows that if $\delta \int_0^1 B(x)\phi_{2n}\ dx \int_0^1 C(x)\phi_{2n}\ dx \cdot ||R(\eta, L_0)|| < 1$, then $\eta \in \rho(L_\delta)$. Moreover, again using Kato, he shows that $||R(\eta, L_0)|| = (\mathrm{dist}(\eta, \sigma(L_0)))^{-1}$. Therefore if

$$\delta \int_0^1 B(x)\phi_{2n}\ dx \int_0^1 C(x)\phi_{2n}\ dx < \mathrm{dist}(\eta, \sigma(L_0)), \tag{4.9}$$

then $\eta \in \rho(L_\delta)$, i.e., $\eta \notin \sigma(L_\delta)$. Freitas shows that since $\nu_{2n+1} - \nu_{2n} \to \infty$ as $n \to \infty$, the $\mathrm{dist}(\eta, \sigma(L_0))$ can be made arbitrarily large for sufficiently large n. Since $B(x)$ and $C(x)$ are functions associated with the linearization about U_1 and are independent of n and since ϕ_{2n} is increasingly oscillatory as $n \to \infty$, $\int_0^1 B(x)\phi_{2n}\ dx \int_0^1 C(x)\phi_{2n}\ dx$ is bounded as $n \to \infty$. Thus, for any fixed value of δ, if n is sufficiently large, (4.9) holds. This means that if δ is fixed, if n is sufficiently large, and if $\mathrm{Re}\ \eta \in (\nu_{2n-1}, \nu_{2n})$, then $\lambda_{2n}(\delta) > \mathrm{Re}\ \eta$. Therefore only a finite number of moving eigenvalues can have crossed fixed ones. □

Lemma 4.3 stands in direct contradiction to the construction of our sequence $\{\delta_{2n}\}$. Recall, that the existence of such a sequence is a necessary condition for complex conjugate eigenvalues to exist. Thus, we conclude that no complex eigenvalues can exist for $\delta > 0$ and, in particular, for $\delta = 1$. □

Using Lemma 4.2, we can now prove part (c) of the oscillation theorem. From the proof of Lemma 4.2, a moving eigenvalue $\lambda_{2n}(\delta)$ can only cross over the fixed eigenvalue ν_{2n+1} that lies immediately to its right. Moreover, for all δ every interval (ν_{2n-1}, ν_{2n+1}) must contain at least one nonlocal eigenvalue. Thus if $\lambda_0(\delta) < \nu_1$ for all δ, then for $n \geq 1$, every interval (ν_{2n-1}, ν_{2n+1}) must contain exactly one nonlocal eigenvalue. Next suppose $\lambda_0(1)$ and $\lambda_2(1)$ are both in (ν_1, ν_3), then for $n \geq 2$, (ν_{2n-1}, ν_{2n+1}) contains exactly one nonlocal eigenvalue. If (ν_3, ν_5) contains two nonlocal eigenvalues, then for $n \geq 3$, (ν_{2n-1}, ν_{2n+1}) contains exactly one nonlocal eigenvalue. It also implies that $\lambda_2(1)$ and $\lambda_4(1)$ are elements of (ν_3, ν_5). The eigenvalue $\lambda_0(1)$ must remain in (ν_1, ν_3). Continuing in this manner, we see that at most one interval (ν_{2n-1}, ν_{2n+1}) contains two nonlocal eigenvalues. □

See [1] for further remarks concerning the oscillation theorem.

4.1. Analysis for the 1-pulse U_1. The specific analysis to prove the metastability of the 1-pulse is very similar to that in Bose and Kriegsmann [1]. We sketch only the results.

LEMMA 4.4. *The nonlocal eigenvalue $\lambda_1 = \nu_1 < 0$ but is exponentially small in ϵ. Moreover, it is the principal eigenvalue of L_1.*

Proof. Consider (4.1) with Dirichlet boundary conditions. The derivative of the 1-pulse, $\zeta = U_1'$, is an eigenfunction of this equation with a corresponding eigenvalue of zero. Thus ζ satisfies

$$\epsilon^2\zeta'' + A(x)\zeta = 0, \tag{4.10}$$

$$\zeta(0) = \zeta(1) = 0. \tag{4.11}$$

Next, consider (4.1) for the eigenpair (ϕ_1, λ_1). A standard trick is to multiply (4.1) by ζ, (4.10) by ϕ_1, subtract the two ensuing equations, and then integrate by parts. Doing so yields that λ_1 is $O(\epsilon^2 U_1''(1))$ and negative; see [1] for specifics. Ward [23] shows that $\epsilon^2 U_1''(1)$ is $O(e^{-a/\epsilon})$, with $a > 0$, for the class of pulse like solutions under consideration here. See [23] for a detailed derivation of this result. Thus λ_1 is exponentially small in ϵ.

We next show that L_1 has no eigenfunction of strictly one sign. Thus, part (b) of the oscillation theorem will imply that λ_1 is the principal eigenvalue. We argue by contradiction. Assume that there exists $\phi_0 > 0$ for all $x \in [0, 1]$ which satisfies (4.1–4.2). Let λ_0 be its associated eigenvalue. Let $J = \int_0^1 2U_1\phi_0 \, dx$. Note that $J > 0$. Integrating (4.1) on $[0, 1]$ yields

$$(\lambda_0 - 2) \int_0^1 \phi_0 \, dx + \frac{pJ}{1 + c^2(1 + I_\star)^2} = \frac{2pc^2 J(1 + I_\star)^2}{(1 + c^2(1 + I_\star)^2)^2}. \tag{4.12}$$

Rearranging and factoring common terms yields

$$(\lambda_0 - 2) \int_0^1 \phi_0 \, dx = \frac{pJ}{(1 + c^2(1 + I_\star)^2)^2} \left[c^2(1 + I_\star)^2 - 1\right]. \tag{4.13}$$

From Theorem 1.1, we know that $I_\star \to \infty$ as $\epsilon \to 0$. Thus for ϵ sufficiently small, the right-hand side of (4.13) is strictly positive. Since $\int_0^1 \phi_0 \, dx > 0$ by assumption, this implies $\lambda_0 > 2$. This however contradicts part (b) of the oscillation theorem as λ_0 cannot by greater than ν_1. Thus there is no eigenfunction of strictly one sign. □

4.2. Analysis for the n-pulse solutions U_n. The eigenvalue equation for the n-pulse is identical to (4.1) except that U_1 is replaced by U_n. Recall that the n-pulse solutions were generated using the symmetry of the 1-pulse. As a result, information about the principal eigenvalue for the linearization around these pulses can be obtained from the 1-pulse. Assume first that $n = 2$. The analysis for $n \geq 3$ is identical to this case. As shown in [1], it turns out that λ_1 is $O(\epsilon^2 U_1''(1/2))$. Notice the difference between this relationship and that for the 1-pulse presented above. Since $U_1(1/2) = U_{max}$, by (1.2),

$$\epsilon^2 U_1''(1/2) = 2U_{max} - \frac{p(1 + U_{max}^2)}{1 + c^2(1 + I_\star)^2}. \tag{4.14}$$

Using the estimates from Theorem 1.1, we obtain

$$\epsilon^2 U_1''(1/2) = \frac{3c^2 I_\star^2}{p} \left(2 - \frac{3c^2 I_\star^2}{1 + c^2(1 + I_\star)^2}\right). \tag{4.15}$$

As ϵ tends to zero, the term in parentheses tends to negative one, while the factor multiplying this tends to infinity. Thus the right-hand side tends to negative infinity. Therefore $\lambda_1 \to -\infty$ as $\epsilon \to 0$. This shows that the principal eigenvalue of the 2-pulse (and analogously the n-pulses) is unstable and bounded away from the origin for ϵ sufficiently small.

5. Numerical simulations. In this section we present a few numerical simulations. We solved the time dependent equation (1.1) using an implicit Crank–Nicholson scheme as in [1, 18]. For all simulations we ran the code for around 20,000 time steps, which corresponds to about 12 seconds. To create a 1-pulse solution, we evolved an

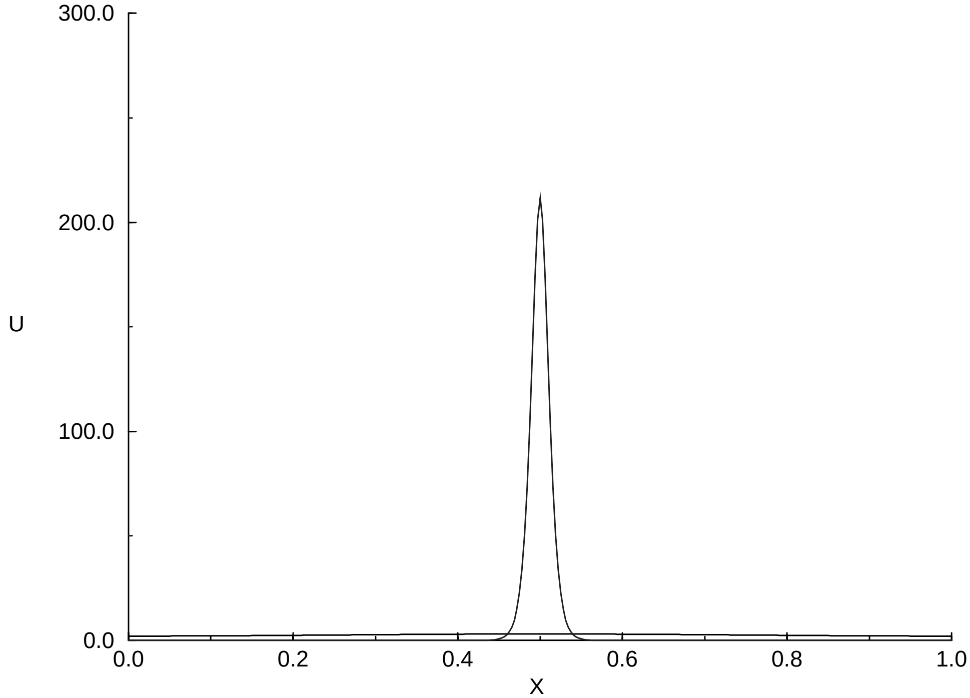

FIG. 5.1. *The* 1*-pulse solution with* $\epsilon = 0.01$, $p = 1.0$, $c = 0.01$.

initial condition that had one small local maxima at $x = 1/2$; see Figure 5.1. To test whether our analytic predictions of U_{max} and $I_\star$ are reasonable, we ran the code with several different choices of p and c and a few different values of ϵ and numerically calculated these values. In Table 5.1, we show results for $p = 1.0$. Different values for ϵ and c are presented. Kriegsmann [18] uses $\epsilon = 0.01$ and $c = 0.01$ in his simulations, which are both physically relevant parameter values. As can be seen, the numerical values agree closely with the theoretical predictions, thus implying a consistency between our numerical and analytic results.

TABLE 5.1
A comparison of analytically predicted and numerically calculated values of U_{max} *and* $I_\star$.

ϵ	c	Numerical value of U_{max}	Analytic value of U_{max}	Numerical value of $I_\star$	Analytic value of $I_\star$
0.01	0.01	211.77	210.85	831.41	838.37
0.03	0.01	100.32	101.37	568.52	581.29
0.01	0.1	43.96	45.42	36.89	38.9

Given such close agreement, we further pursued the ramifications of the metastability of the 1-pulse by evolving different initial conditions. Figure 5.2 shows the evolution of a symmetric 2-bump perturbation of a homogeneous solution for $\epsilon = 0.01$. We show only the initial condition and the ensuing 1-pulse. Notice that it is not centered at the midpoint of the domain, so it is not a steady-state solution. When we let the code run for much longer times, the 1-pulse remained fixed. This is a manifestation of the metastability of the solution. This non–steady-state 1-pulse is moving exponentially slowly towards one of the boundaries. We do not know why the pulse

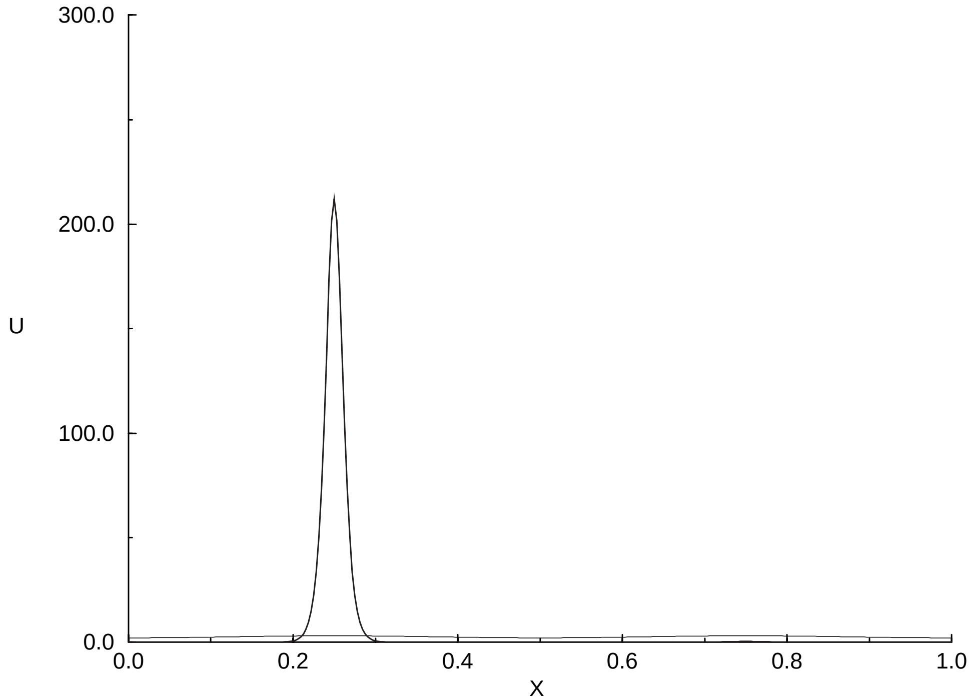

FIG. 5.2. *An exponentially slowly moving* 1*-pulse with* $\epsilon = 0.01$*,* $p = 1.0$*,* $c = 0.01$*.*

forms at $x = 1/4$ and not at $x = 3/4$. Using other initial conditions, we can generate a pulse at any location in the domain (simulations not shown). The exponentially small eigenvalue introduces a near translational invariance, so this is not surprising. Incidentally, it is possible to obtain a 1-pulse centered at $x = 1/2$ by providing an initial condition with a sufficiently large local maximum at $x = 1/2$. These results are consistent with [1].

Finally, using the rescaling argument in section 3.2, we can obtain so-called 1/2-pulse steady-state solutions. These solutions are close to zero valued on most of $(0, 1)$, but contain a layer near one of the boundaries where the value of U grows inversely to some power of ϵ. We call them 1/2-pulses since they look like the 1-pulse solution over the interval $[0, 1/2]$ (or alternatively, $[1/2, 1]$). We also observed such solutions numerically, which leads us to believe that they are stable solutions. We do not, however, pursue their stability here.

6. Application to the Gierer–Meinhardt equations. The Gierer–Meinhardt equations [13] arise in biological pattern formation. They fall in the general class of activator-inhibitor systems. They can be written in nondimensional form in one spatial dimension as

$$U_t = \epsilon^2 U_{xx} - U + \frac{U^p}{H^q}, \tag{6.1}$$

$$\tau H_t = D_H H_{xx} - \mu H + \frac{U^m}{H^s}, \tag{6.2}$$

$$U_x(0,t) = U_x(1,t) = 0, \;\; H_x(0,t) = H_x(1,t) = 0. \tag{6.3}$$

The variable U represents the activator concentration and H represents the inhibitor concentration. Here $p > 1$, $q, m > 0$, and $s \geq 0$. These equations were studied numerically in [13] and localized pulses were observed. Recently, Iron and Ward [14] used numerical and asymptotic techniques to demonstrate the metastability of the observed 1-pulse solution of a nonlocal reduction of (6.1) for $x \in \Omega$, where Ω is a closed subset of $\mathbf{R^N}$. They derived a nonlocal reaction-diffusion equation which was valid in the limit as $\tau \to 0$ and $D_H \to \infty$. They did not rigorously construct solutions nor prove their stability. Here, for $x \in [0, 1]$ we derive a slightly different scaled version of the nonlocal equation that appears in [14] and then construct a 1-pulse solution. As before n-pulses can be obtained by the rescaling argument. Moreover, the metastability analysis will be exactly as in section 4.

As $D_H \to \infty$, H becomes spatially homogeneous. We then integrate (6.2) from zero to one and set $\tau = 0$ to obtain the following algebraic equation:

$$\mu H^{s+1} = \int_0^1 U^m \, dx. \tag{6.4}$$

Setting $\mu = 1$, for convenience, we obtain the following scalar nonlocal reaction-diffusion equation:

$$U_t = \epsilon^2 U_{xx} - U + \frac{U^p}{(\int_0^1 U^m \, dx)^{q/(s+1)}}, \tag{6.5}$$

$$U_x(0, t) = U_x(1, t) = 0.$$

Iron and Ward first scale the system (6.1–6.2) to reflect that the height of the pulse goes to infinity as ϵ tends to zero. They then integrate (6.2). The difference is that they have a factor of $(1/\epsilon)^{q/(s+1)}$ multiplying the nonlocal term.

As before, we rewrite (6.5) as a system of first-order equations using $I = \int_0^1 U^m \, dx$ and the auxiliary variable $V(x) = \int_0^x U^m \, dx$. Set $n = q/(s + 1)$.

$$\begin{aligned} \epsilon U' &= W, & V' &= U^m, \\ \epsilon W' &= U - \tfrac{U^p}{I^n}, & I' &= 0. \end{aligned} \tag{6.6}$$

Introducing the scalings $u = \epsilon^a U$, $w = \epsilon^a W$, $v = \epsilon^b V$, and $Z = \epsilon^b I$ as before and balancing terms yields the following values for a and b:

$$a = \frac{q}{qm - p(s + 1)}, \qquad b = \frac{p(s + 1)}{qm - p(s + 1)}.$$

Thus we require $qm > p(s + 1)$, which is consistent with [13, 14]. This yields

$$\begin{aligned} \epsilon u' &= w, & \epsilon v' &= u^m, \\ \epsilon w' &= u - \tfrac{u^p}{Z^n}, & Z' &= 0. \end{aligned} \tag{6.7}$$

If we rescale in a neighborhood of $x = 1/2$ using $\xi = (x - 1/2)/\epsilon$, then the ensuing system has one saddle point and one center point as before. Similarly, there exists a homoclinic solution connecting the saddle to itself. Using the change of variable $y = x$ and appending $dx/dy = 1$, the boundary manifolds, jump off, and touch down curves are defined exactly as before. Consistency is again checked on the inner equations.

As before, the outer flow transversely intersects J_0 and T_c, so (H1) and (H3) are satisfied. To verify (H2), we must check transversality of $B_0 \cdot 1$ with B_1. We use the inner equations associated with (6.7) which are

$$\dot{u} = w, \qquad \dot{v} = u^m, \qquad \dot{w} = u - \frac{u^p}{Z^n}, \qquad \dot{Z} = 0. \tag{6.8}$$

The first two equations have Hamiltonian given by

$$H(u, w) = w^2/2 - u^2/2 + \frac{u^{p+1}}{(p+1)Z^n}. \tag{6.9}$$

The maximum value of u is given by $u_{max} = ((p+1)Z^n/2)^{1/(p-1)}$. Recall that the curve $v(Z) = \int_{-\infty}^{\infty} u^m(\xi, Z)\, d\xi$ is the projection of the touch down curve T_0 on the (Z, v) plane. We must show that this curve transversely intersects the line $v = Z$. To do this we obtain estimates for $\int_{-\infty}^{\infty} u^m\, d\xi$ for $Z \ll 1$ and $Z \gg 1$ and use the intermediate value theorem.

Recall that the homoclinic solution of the first two equations of (6.8) corresponds to the pulse. The value of the Hamiltonian on this solution is 0. Using (6.9), the integral of interest can be rewritten as

$$\int_{-\infty}^{\infty} u^m\, d\xi = 2\int_0^{u_{max}} \frac{u^{m-1}}{(1 - \frac{2u^{p-1}}{(p+1)Z^n})^{1/2}}\, du. \tag{6.10}$$

Moreover,

$$\int_0^{u_{max}} \frac{u^{m-1}}{\left(1 - \frac{2u^{p-1}}{(p+1)Z^n}\right)^{1/2}}\, du > \int_0^{u_{max}} u^{m-1}\, du = \frac{1}{m}\left(\frac{(p+1)Z^n}{2}\right)^{m/(p-1)}. \tag{6.11}$$

It is easily seen that if $Z \gg 1$, then the right-hand side of the above inequality is greater than Z, since $nm/(p-1) = qm/(p-1)(s+1) > qm/p(s+1) > 1$, by assumption.

Depending on the choices of the parameters m, p, q, and s there are many ways to obtain an upper bound for $\int_{-\infty}^{\infty} u^m\, d\xi$. We assume that $n > 1$ and $p \leq m+1$ in the following. Note that $Z < 1$ implies that u_{max} and thus u are less than one. Then

$$\int_0^{u_{max}} \frac{u^{m-1}}{\left(1 - \frac{2u^{p-1}}{(p+1)Z^n}\right)^{1/2}}\, du < \int_0^{u_{max}} \frac{u^{p-2}}{\left(1 - \frac{2u^{p-1}}{(p+1)Z^n}\right)^{1/2}}\, du = \frac{(p+1)Z^n}{(p-1)}. \tag{6.12}$$

Clearly, if $Z \ll 1$, then the right-hand side of (6.12) is less than Z. For different choices of the parameters m, p, q, and s, other estimates such as (6.12) can be obtained, but we do not pursue them here.

Combining the appropriate estimates above, it is seen that if $Z \ll 1$, then $\int_{-\infty}^{\infty} u^m\, d\xi < Z$, and if $Z \gg 1$, then $\int_{-\infty}^{\infty} u^m\, d\xi > Z$. Thus by the intermediate value theorem, there exists at least one value of Z_0 for which $\int_{-\infty}^{\infty} u^m(\xi, Z_0)\, d\xi = Z_0$. This value is unique as can be inferred from the above estimates. Specifically, for the particular choices $m = p-1$ and $m = 2p-2$, it is easy to check by direct integration that $v(Z)$ is $O(Z^n)$ and $O(Z^{2n})$, respectively. For any choice of $m > 2p-2$, the function $v(Z)$ can be obtained by using a suitable number of integration by parts and eventually reducing to the calculation of an integral of the form (6.10) where $p-1 \leq m \leq 2p-2$. In each integration by parts, the boundary terms disappear and an additional factor of Z^n is introduced. By rewriting the relevant integral as the sum of two integrals for the cases $u < 1$ and $u > 1$, if necessary, and using the results obtained from the special cases $m = p-1$ or $m = 2p-2$, we can obtain a lower bound for $v(Z)$ which is at least $O(Z^n)$. This result holds for all $m \geq p-1$, which is the parameter regime of interest. Since $n > 1$, this clearly shows that $V(Z)$ is not linear in Z. This establishes transversality of T_0 with the line $v = Z$.

The stability analysis for this 1-pulse solution is analogous to what we presented in section 4. The nonexistence of a positive eigenfunction can be established under the further restriction that $p \leq m$. We leave the details to the interested reader. Also, time-dependent simulations of (6.5) using different choices of the parameters, produced metastable 1-pulses as in section 5.

Finally, Iron and Ward [14] discuss the metastability of the 1-pulse in higher spatial dimensions. Moreover, they derive an equation of motion for the metastable pulse and discuss how it interacts with the boundary in which it is enclosed. We refer the interested reader to their work.

7. Discussion. In this paper, we have developed a systematic geometric method to construct spatially localized pulse like solutions for singularly perturbed nonlocal boundary value problems. While we have not stated a general theorem concerning the construction of solutions, it is clear that the procedure outlined above is applicable to a large variety of scalar nonlocal equations. Moreover, the analysis presented above is not restricted to singularly perturbed equations or to the construction of pulse-like solutions. These methods can also be used to construct front-type (heteroclinic) solutions.

We showed how to recast the scalar nonlocal problem as a higher-dimensional local problem. The geometric framework provided above can also extend to higher-dimensional nonlocal systems. In higher dimensions, the jump off and touch down curves may become surfaces, but the abstract description using manifolds accounts easily for this possibility. One aspect of the present low-dimensional analysis that will remain in the higher-dimensional setting will be the transversality of two one-dimensional curves in a two-dimensional ambient space needed to establish the consistency condition (1.3). No matter what the dimensionality of the full system is, this two-dimensional problem will always persist. Thus one of the challenges of any nonlocal analysis is to recast the system into a form where this consistency condition becomes apparent.

A general oscillation theorem was also presented. A sufficient condition for which this theorem holds is the symmetry of the underlying pulse solution. We note, however, that this condition is a very natural one to impose since any steady-state pulse solution of (1.1) must necessarily by symmetric. Modifications of the oscillation theorem should also hold in circumstances where the operator L_1 has an infinite number of fixed eigenvalues. It is hard to give a general example where this may occur, but such a situation may arise when the underlying solution is not strictly of one sign.

We have also proved the metastability of the 1-pulse solution and the instability of n-pulse solutions. As discussed in [1], this type of metastability is qualitatively different than the metastability found in, say, Carr and Pego [2]. There, the authors construct metastable solutions that contain an arbitrarily high number of interior layers. These layers consist of heteroclinic, and not homoclinic, solutions to the appropriate set of inner equations. They show that each interior layer contributes an exponentially small eigenvalue. In our work, we have shown that additional localized pulses contribute $O(1)$ unstable eigenvalues. The nonlocal term can be viewed as being strong enough to remove at most one local unstable eigenvalue.

Finally, the applications considered in this paper are of interest in their own right. Because of the time scales of physical interest, metastability is tantamount to stability. Thus metastability of the 1-pulse of the microwave heating model due to Kriegsmann [18] suggests that localized heating of ceramic materials can be achieved in a stable and reliable manner. The nonlinearities chosen for the present study yield maximum heating rates that are too high and beyond the melting point of the fiber. However, for the nonlinearities used in [1, 18], physically acceptable maximal heating rates are obtained. Other nonlocal models arising in different microwave heating applications are the focus of further research.

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