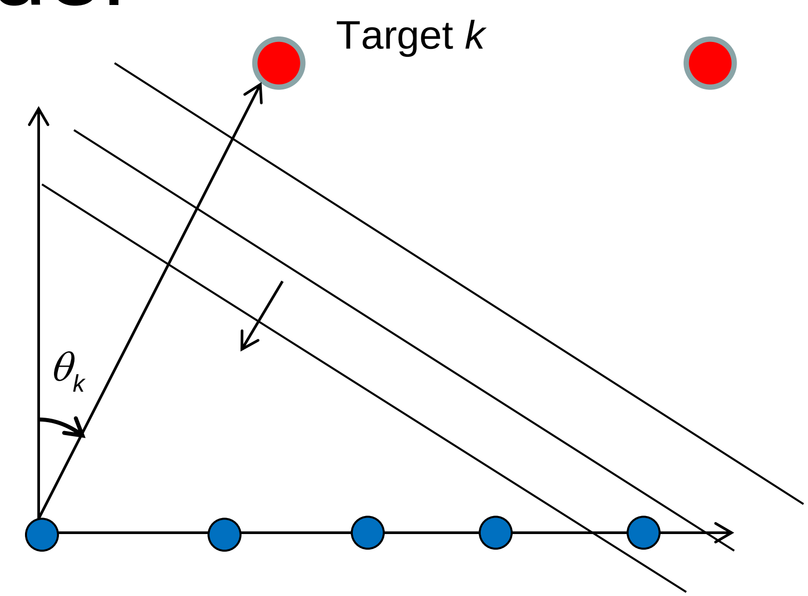
Direction of Arrival Estimation without Phase Measurements

Haley Kim
Advisor – Alexander Haimovich

Outline

- Signal Model
- Direction of Arrival with phase and motivation
- Direction of Arrival without phase
- Numerical Results

Signal Model



Antenna array of N sensors

At sensor *l:*

$$y_l = \sum_{k=1}^{K} \exp(\mathrm{j}2\pi z_l \sin\theta_k) + n$$

Unknowns are $\sin(\theta_k)$ for all targets

Signal Model

- In matrix format: $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$
- **A** is a collection of steering vectors $\mathbf{A} = [\mathbf{a}(\theta_1) \quad \mathbf{a}(\theta_2) \ldots \quad \mathbf{a}(\theta_N)]$

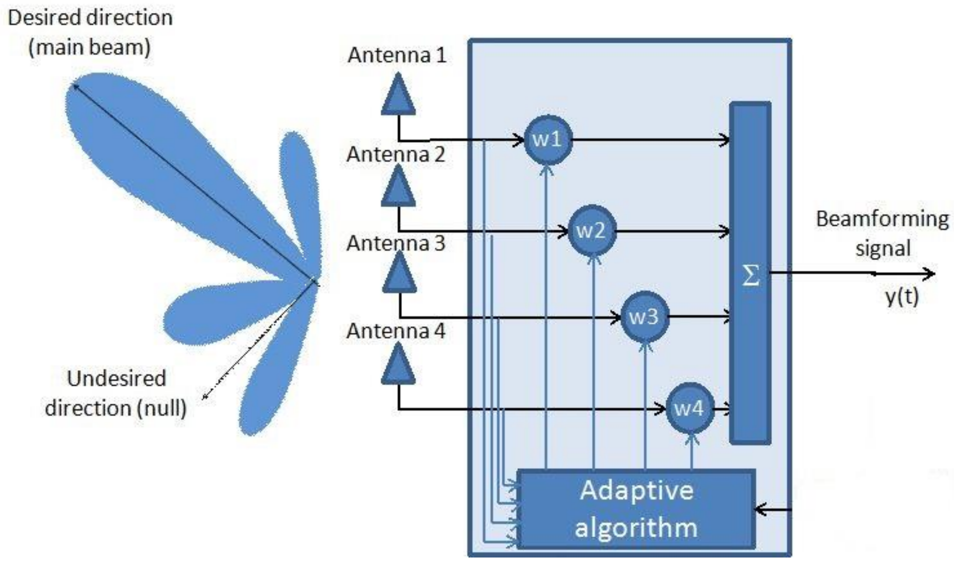
  $\mathbf{a}(\theta_k) = [1 \ \exp(j2\pi z_2 \sin(\theta_k) \ \ldots \exp(j2\pi z_N \sin(\theta_k)]^{\mathrm{T}}$
- Each column of **A** is a steering vector for specified angle
- In this representation every non-zero element in **x** represents a target

Direction of Arrival With Phase

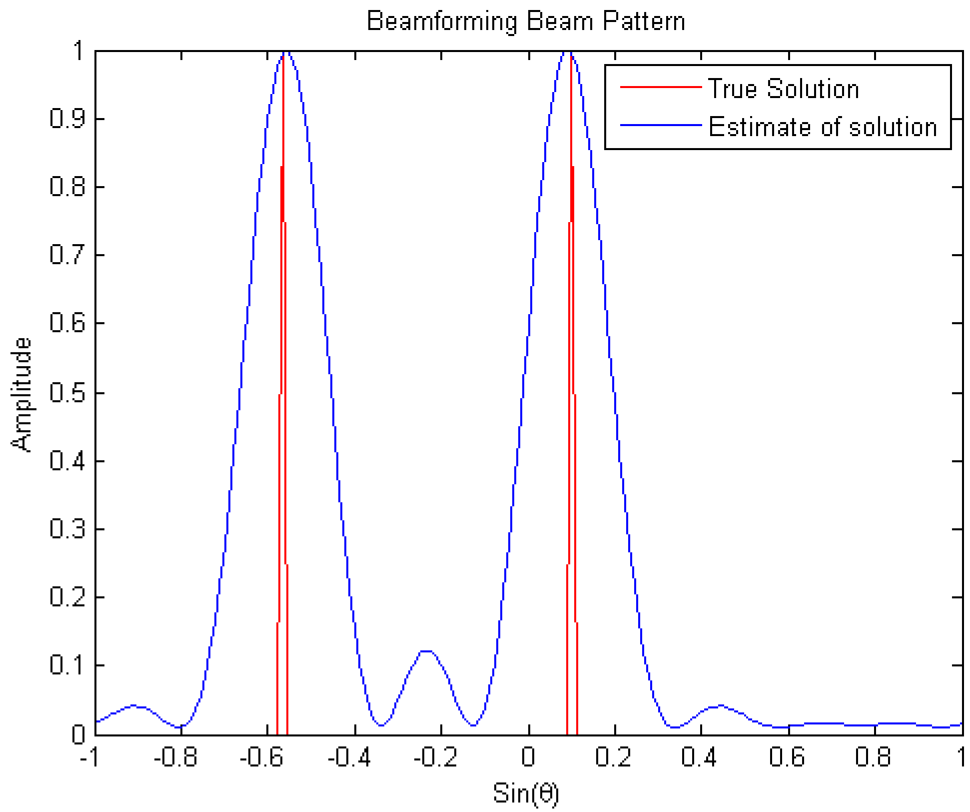
- Angle of the targets are embedded in the phase:

$$y_l = \sum_{k=1}^{K} \exp(j 2\pi z_l \sin\theta_k) + n$$

- One method to solving for the angles is beamforming



Direction of Arrival with Phase



Example of DoA estimation using beamforming

Motivation

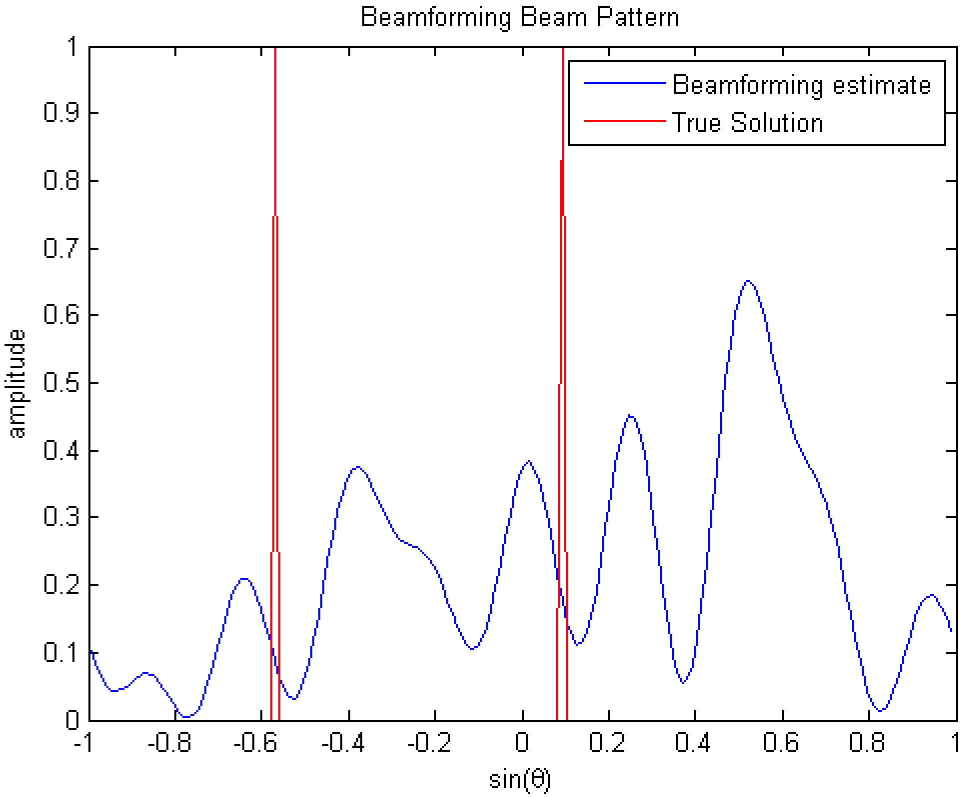
- So far, we assumed we have perfect knowledge and synchronization of the phase at each sensor
- Maintaining this synchronization is not trivial
- If each sensor has a random phase error

$$y_l = \exp(j\phi_l)(\sum_{k=1}^{K} \exp(\mathrm{j}2\pi z_l \sin\theta_k)) + n$$

- Random phase error given by uniform distribution

$$\phi \sim \mathrm{U}\ [0, 2\pi]$$

Beamforming approach



Beamforming estimate when perfect synchronization is not available.

Direction of Arrival without phase

- Use magnitude squared of **y** to avoid phase errors

$$y_l = \left| \exp(j\phi_l)(\sum_{k=1}^{K} \exp(\mathrm{j}2\pi z_l \sin\theta_k)) \right|^2 = \left| \sum_{k=1}^{K} \exp(\mathrm{j}2\pi z_l \sin\theta_k) \right|^2$$

- Removes phase errors in signal.
- However conventional methods that exploit phase are not applicable

Direction of Arrival without Phase

- Phase information determined the target position where is the information if the magnitude squared is taken?
- Simple case, 2 targets at sensor *l*:

$$y_l = \left| x_1 \exp(j2\pi z_l \sin(\theta_1) + x_2 \exp(j2\pi z_l \sin(\theta_2) \right|^2$$

$$y_l = |x_1|^2 + |x_2|^2 + 2|x_1||x_2| \cos(2\pi z_l (\sin\theta_1 - \sin\theta_2))$$

- Information in the magnitudes are the **difference** or distances of the angle positions.

Inherent Ambiguities

- Unfortunately lack of phase gives rise to ambiguities in our solution, looking at the equation for two targets:

$$y_l = |x_1|^2 + |x_2|^2 + 2|x_1||x_2|\cos(2\pi z_l(\sin\theta_1 - \sin\theta_2))$$

- Magnitude at sensor *l* depends on the cosine, which gives rise to two ambiguities of interest. Circular shift and signal mirroring

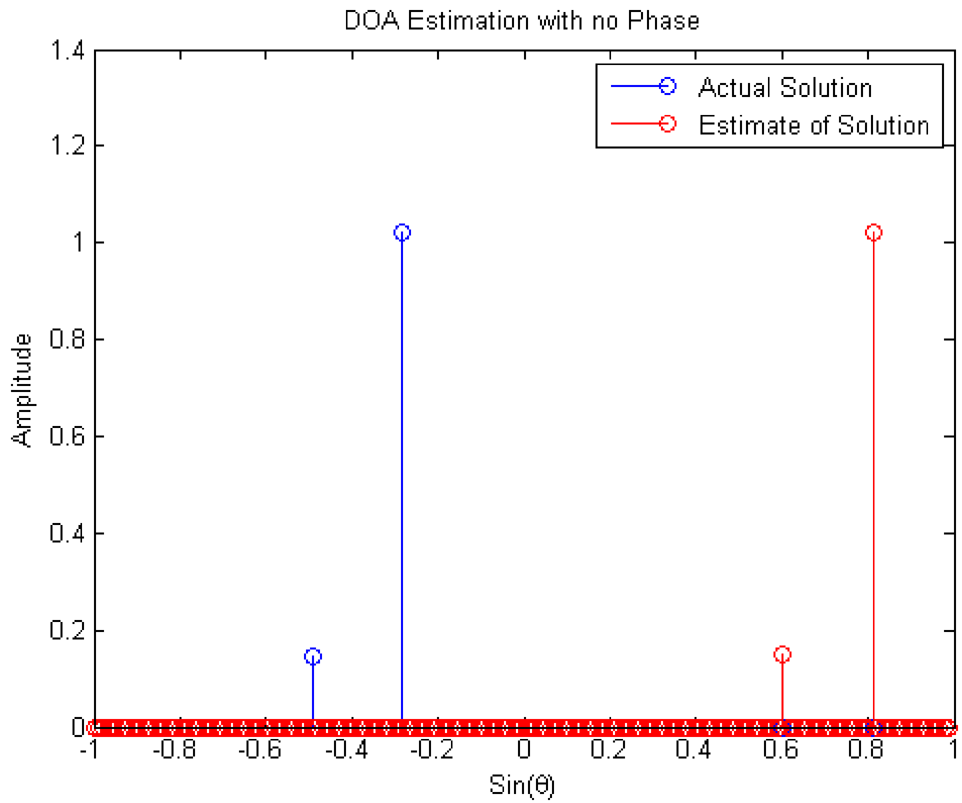
Circular Shift Ambiguity

- The circular shift can be seen by shifting both target angles by some amount $\sin(\phi)$

$$\sin\bar{\theta}_1 = \sin\theta_1 + \sin\phi$$
$$\sin\bar{\theta}_2 = \sin\theta_2 + \sin\phi$$
$$\bar{y}_l = |x_1|^2 + |x_2|^2 + 2|x_1||x_2|\cos(2\pi z_l(\sin\bar{\theta}_1 - \sin\bar{\theta}_2))$$
$$\bar{y}_l = y_l$$

- Any constant shift to both targets results in the same magnitude squared value

Circular Shift Ambiguity



Example of the circular shift ambiguity. The estimate of the solution is a shifted version of the true solution

Signal Mirroring Ambiguity

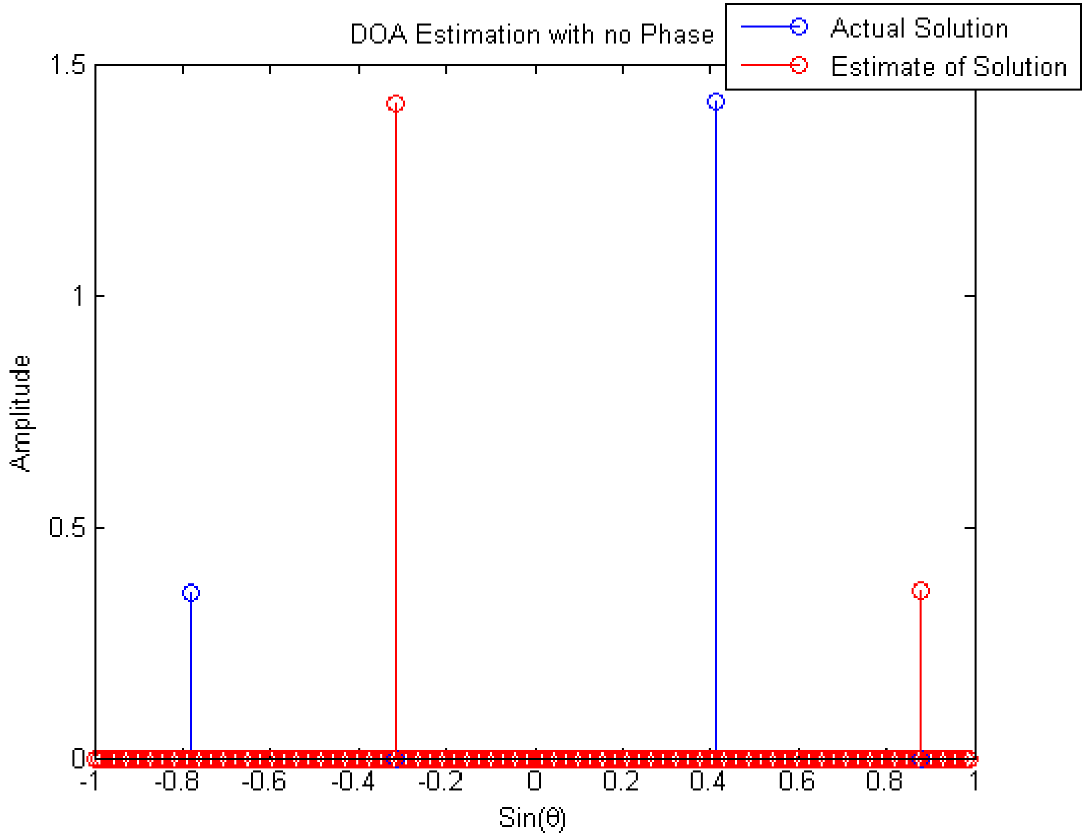
- Signal mirroring is an axis reversed version of the solution.
- Signal mirroring occurs due to the cosine term being even

$$\cos(2\pi z_l \Delta) = \cos(-2\pi z_l \Delta)$$

$$\Delta = \left|\sin\theta_1 - \sin\theta_2\right|$$

- Equivalent to saying if we know one of the targets we can find the distance between the targets, but not the direction.

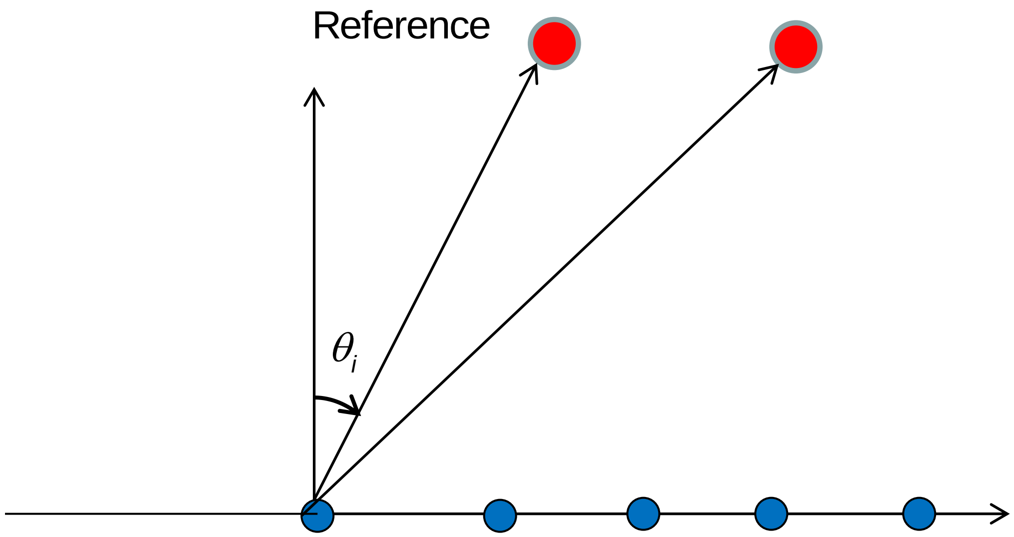
Signal Mirroring Ambiguity



Example of signal mirroring ambiguity. The estimate of the solution is an axis reversed version of the signal

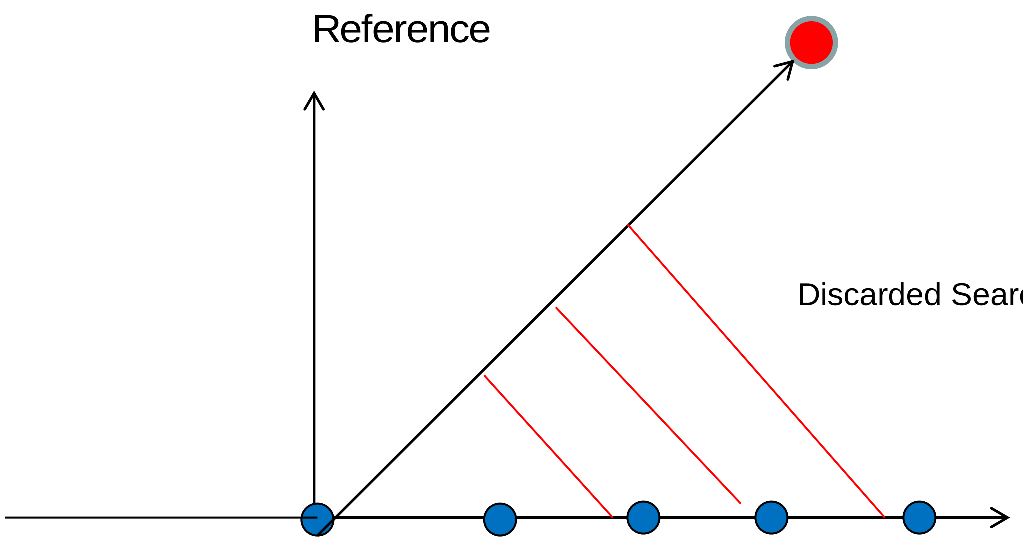
Resolving Ambiguities

- Additional information is required to solve this problem.
- We propose the use of a reference target to eliminate ambiguities



Resolving Ambiguities

- Resolves the circular shift by fixing one point.
- By specifying a target signal mirroring is restricted to a certain region



Phase Retrieval

- The solution to our problem can be mathematically expressed as

$$\min_x \left\| \mathbf{y} - \left|\mathbf{Ax}\right|^2 \right\|_2^2$$

$$\text{s.t. } \|x\|_0 \le K + 1$$

$$x_i = x_{ref}$$

- Problems of this kind are known as the phase retrieval problem
- This optimization problem is not convex

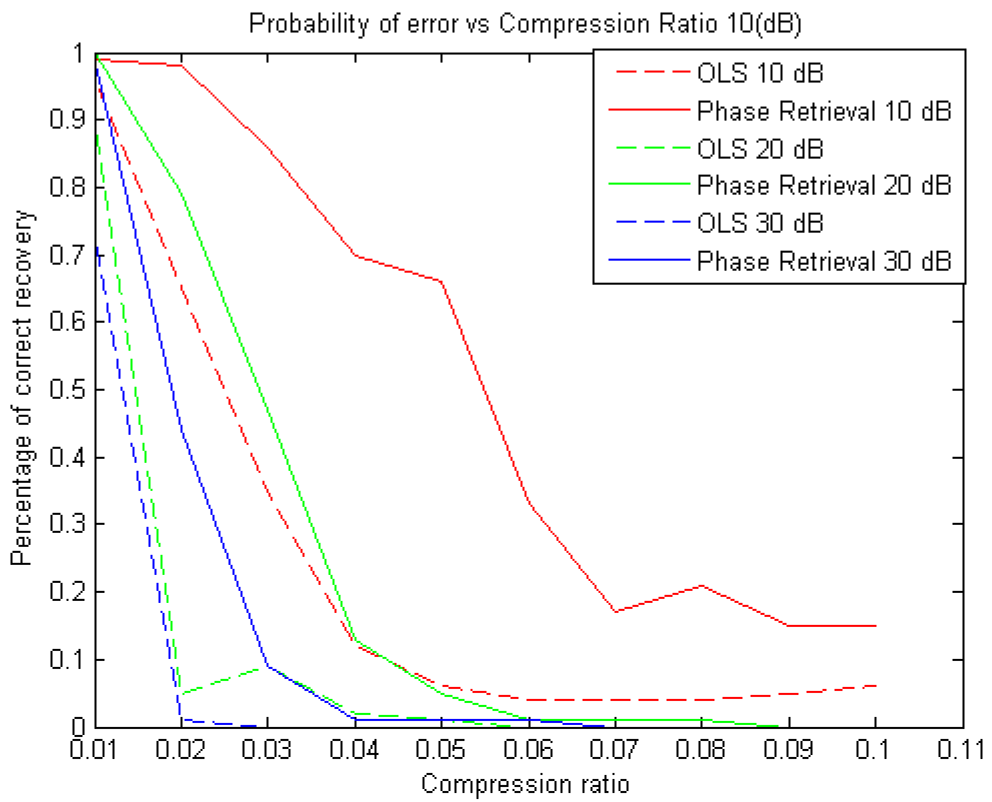
Phase Retrieval Algorithms

- Three classes of algorithms exist to solve this problem

- Alternating Projections
- Phaselifting (SDP based algorithms)
- Combinatorial searches

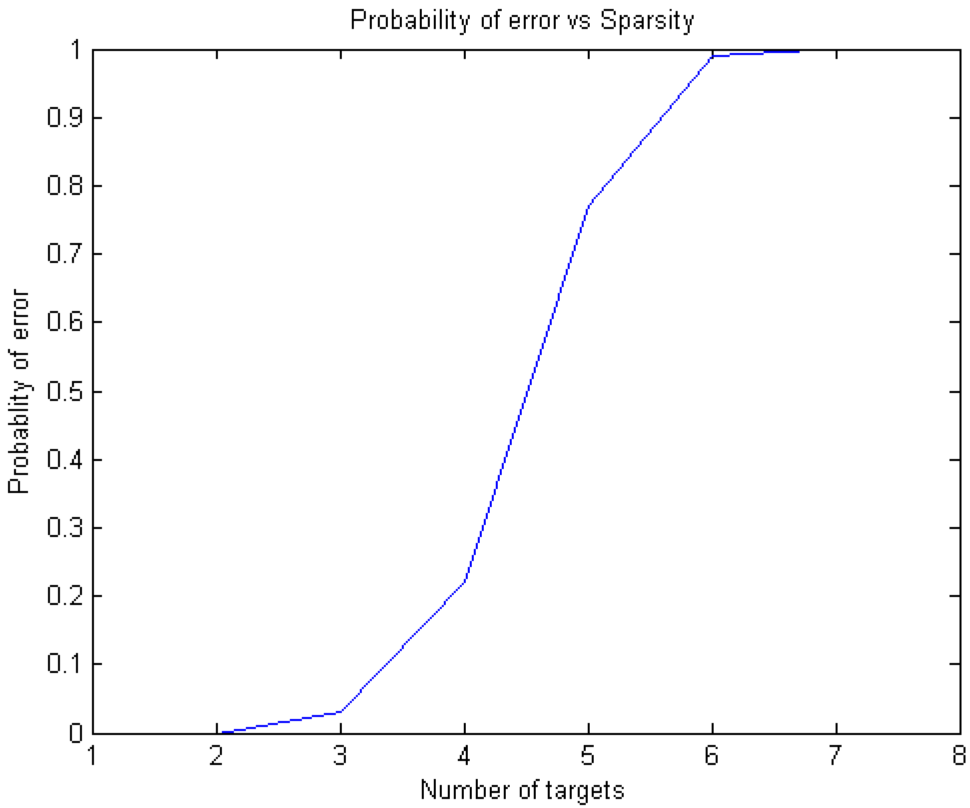
- For this presentation we have used the GESPAR [1] algorithm, which falls into the combinatorial search algorithm

Probability of Error vs Compression



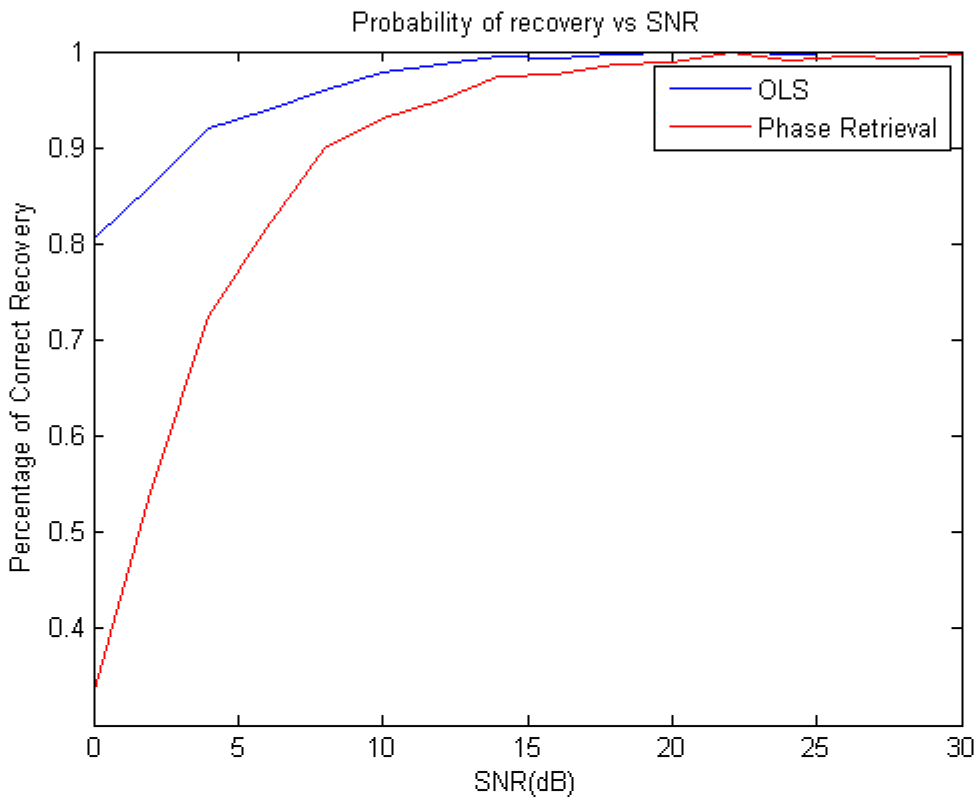
Probability of error vs compression ratio for various SNR. Compression ratio is defined as $CR = {n}/{n_{full}}$

Recovery vs Sparsity



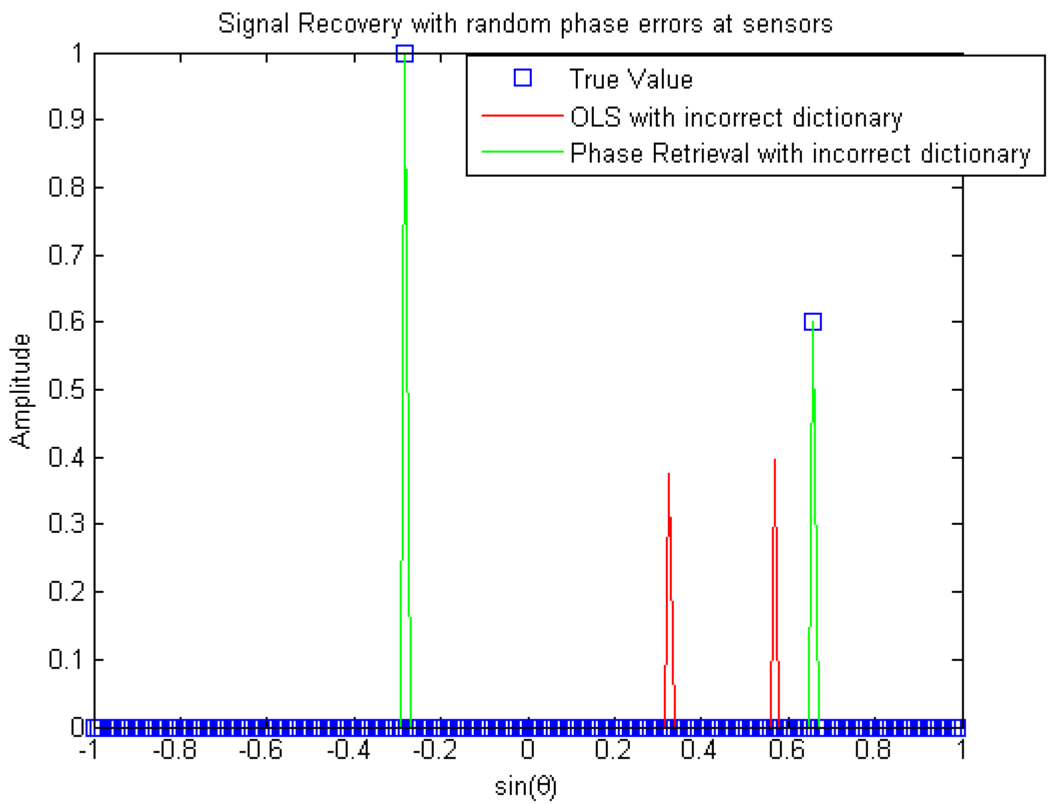
Percentage of error vs number of targets for a CR = 0.17

Probability of Recovery



Recovery probability vs SNR for OLS (Phase available) and phase retrieval CR = 0.16

Phase Errors



Signal recovery using an incorrect steering vector dictionary when phase is and is not available

References

- [1] Shechtman, Yoav, Amir Beck, and Yonina C. Eldar. "GESPAR: Efficient Phase Retrieval of Sparse Signals." *arXiv preprint arXiv:1301.1018* (2013).

Thank you!

Any Questions?