Linear stability analysis of nematic liquid crystals flowing down an incline plane

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1 Introduction

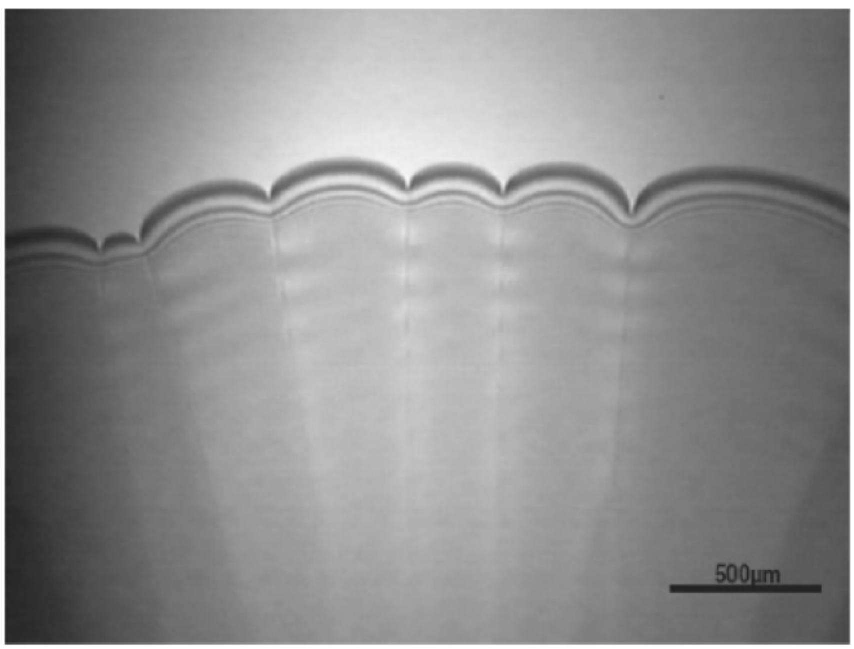

Figure 1: Fingering instabilities develop at the contact line of a nematic liquid crystal (NLC) drop spreading under the influence of gravity alone on a flat silicone wafer at a relative humidity greater than 95%. Experimental photo taken by Poulard and Cazabat [6].

Calamitic liquid crystals (LCs) are rod-like anisotropic organic molecules found vastly in nature and electronic displays. When in the nematic phase, LCs prefer self-organize roughly parallel to their long axes giving rise to directional order [7]. The absence of positional order contributes to the fluidity of these molecules, similar to that of an isotropic medium. In a series of experiments conducted by Poulard and Cazabat [6], wave-like fingering instabilities were observed at the contact line of LC drops spreading on a silicone substrate (see Fig. 1). Topological defects such as disclinations can also be observed in LCs, however the focus of this letter is only instabilities present at the contact line.

The case of a Newtonian fluid flowing down an incline plane is well understood, however one wonders about the behavior of the system if a thin-film of nematic liquid crystals (NLCs) are placed on the incline instead. The focus of this letter, is to observe this very phenomena and analyze the instabilities developing in the flow. The behavior of the flat film is first analyzed in a two-dimensional (2D) model where the critical wavenumber and the growth rate are derived. Next, we look for traveling-wave solutions in a limiting case and derive the wave speed. Lastly, linear stability analysis (LSA) is used to analyze the dynamics of LCs flowing down an incline. The constant-flux driven case is considered in the limit of a small wavenumber to predict the development of wave-like instabilities as shown in Fig. 1. The theory is then compared to numerical results obtained by Olarinire, Seric, and Singh and experimental results observed by Dupiano, Naughton, Pineda, and Shah as a senior project for applied mathematics.

2 Lubrication theory

In free film lubrication theory, a thin film of a viscous fluid is in contact with solid surface where the no-slip condition dictates that the tangential direction of the fluid velocity vanishes, hence creating a shear flow. At the "free" boundary, the fluid is exposed to the ambient gas and experiences surface tension. This liquid-gas boundary is subject to deformations and the propagating fluid gives rise to traveling-waves and other interesting dynamics such as ruptures exposing the substrate to the ambient gas and fingering instabilities [5].

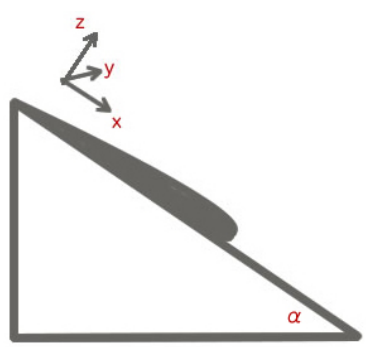

Figure 2: Thin-film of fluid flowing down the incline plane.

The lubrication approximation can be applied when the film of characteristic thickness H is small compared to the length L of the substrate allowing us to introduce a small parameter $\epsilon = H/L$. Consider the example of fluid flowing down an incline plane as illustrated in Fig. 2 with velocity $\mathbf{u} = (u, v, w)$. Let U be the characteristic flow speed in the x-direction. Since the velocity gradients in the x, y-directions are of order U/L, and of order $U/H = U/\epsilon L$ in the z-direction, the gradients in the x, y-directions are negligible.

Balancing terms in the continuity equation for an incompressible fluid, we see that w if of order ϵU and hence negligible. Lastly, it can be shown that for the lubrication approximation, inertial terms may be neglected if $\epsilon^2 Re \ll 1$ where Re is the Reynolds number [1]. Lubrication theory allows us to greatly simplify the governing equations making theoretical and computational analysis much more feasible.

3 Evolution equation

A model was developed by Ben Amar and Cummings (2001) [2] describing the evolution of the film height h for NLCs on a rigid substrate driven by a constant flux at the far-field $x = -\infty$.

NLCs have a property where they like to align parallel to the substrate and orthogonal to the free surface. The letter by Ben Amar and Cummings assumes a strong anchoring case where the LCs linearly increase in angle throughout the height of the film. However, a caveat occurs at the contact line where strong anchoring dictates that the LC must take upon both a parallel and orthogonal orientation at a single point. This contradiction was addressed in a weak anchoring model developed by Cummings et al. (2011) [3] and is summarized below:

$$h_t + \nabla \cdot [h^3(C\nabla\nabla^2 h - B\nabla h) + N(m^2 - hmm')\nabla h] = 0 \tag{1}$$

$$m = \frac{h^{3/2}}{\beta^{3/2} + h^{3/2}}\,. \tag{2}$$

The lubrication approximation was adopted for the flow equations and prewetting of the substrate was assumed with a precursor layer of thickness $b \ll h$ extending to $x = \infty$ [2]. Here, h is the fluid thickness, C is the inverse capillary number, B is the Bond number, and N is the inverse Ericksen number. The equation is non-dimensionalized using the parameter groupings specified in Cummings et al. [3].

For the incline case, the downhill component of gravity is added to (1) resulting in the following formulation:

$$h_t + \nabla \cdot [h^3(C\nabla\nabla^2 h - B\nabla h) + N(m^2 - hmm')\nabla h] + U(h^3)_x = 0\,. \tag{3}$$

Here, U is scaled using the same parameter grouping as B. In short, the fourth-order term describes the forces due to surface tension, the last two terms multiplied by B and U are due to the normal and tangential components of gravity respectively, and the term multiplying N arises from the properties of NLCs [4].

4 Analytical results

The evolution equation is still quite complicated and hence difficult to analyze. Below, we present several scenarios where perturbations and approximations are applied to predict the overall behavior of the system and conditions necessary for developing instabilities.

4.1 Linear stability analysis of the flat film

Let us first consider the simple case of a flat film of thickness h_o and assume that h is independent of y. Therefore (3) reduces to the equation:

$$h_t + \partial_x[h^3(Ch_{xxx} - Bh_x) + N(m^2 - hmm')h_x] + U(h^3)_x = 0\,. \tag{4}$$

Perturbing the profile by a small amplitude of order ϵ where $0 < \epsilon \ll 1$, we have

$$h(x,t) = h_o + \epsilon h_1(x,t) + O(\epsilon^2),\ , \tag{5}$$

and substituting (5) into (4), we find that the $O(\epsilon)$ equation is

$$h_{1t} + Ch_o^3 h_{1xxxx} - \big(B - NM(h_o)\big)h_o^3 h_{1xx} + 3Uh_o^2 h_{1x} = 0 \tag{6}$$

$$M(h_o) = \frac{ho^{3/2} - \beta^{3/2}/2}{(ho^{3/2} + \beta^{3/2})^3}\,. \tag{7}$$

The dispersion relation can be obtained by assuming solutions of the form $h_1 \propto e^{\sigma t + ikx}$ and substituting into (6) [3]:

$$\sigma = -Ch_o^3 k^4 + \big(B - NM(h_o)\big)h_o^3 k^2 - i3Uh_o^2 k\,. \tag{8}$$

The imaginary part of the growth rate σ predicts oscillations and does not affect stability. Surface tension is responsible for stabilizing the system, hence a decaying curve for $Re\,\sigma$ is a result of perturbations with short wavelengths. When $NM(h_o) > B$, $Re\,\sigma > 0$ for wavenumbers $0 < k < k_c = \sqrt{(NM(h_o) - B)/C}$ as shown in Fig. 3, hence solutions are unstable and instabilities develop. The critical or fastest growing wavenumber, which occurs at the maximum of (21) and the corresponding growth-rate are

$$k_m = \sqrt{\frac{NM(h_o) - B}{2C}} \tag{9}$$

$$\sigma_m = \frac{(NM(h_o) - B)^2}{4C} h_o^3\,. \tag{10}$$

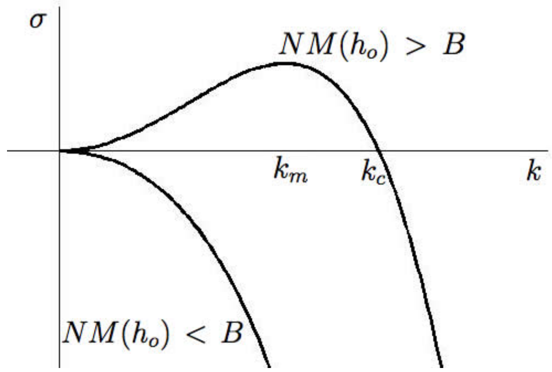

Figure 3: The growth-rate as a function of the wave-number ploted for the two cases $NM(h_o) < B$ and $NM(h_o) > B$. Note that instabilities occur for only the later case when $0 < k < k_c$.

In experiments, it is most likely that the observed growth-rate will be close to σ_k due to the growing exponential nature of the problem. In numerical calculations performed by Olarinire, Seric, and Singh, the growth-rate for the constant volume case was 0.2386 in non-dimensional units where $h_o = B = C = \beta = 1$. Hence, N here is found to be 3.91. In the experiments carried out by Dupiano, Naughton, Pineda, and Shah, the growth-rate was calculated to be $0.547/s$, where $h_o = 0.1$ after being scaled by the length of the glass-slide. β were taken to be one and B, C were calculated to be one. Using the experimental parameters, N was found to be 54. N is quite difficult to determine and was predicted to lie in the large range of many orders of magnitude apart [3].

4.2 Traveling-wave solutions for the two-dimensional equation

To gain a basic understanding of this complicated phenomena, let us analyze the flow in the limit of a large β such that $m(h)$ reduces to

$$m(h) = \frac{h^{3/2}}{\beta^{3/2}}\,. \tag{11}$$

Another simplification is to assume that the profile is fairly uniform in the transverse direction. Hence, the 2D evolution equation given by (4) can be used with the boundary conditions (BCs):

$$h \to b \text{ as } x \to -\infty\,, \quad h \to 1 \text{ as } x \to \infty\,, \quad h_x \to 0 \text{ as } x \to -\infty\,, \quad h_x \to 0 \text{ as } x \to \infty\,. \tag{12}$$

The fluid is of thickness $h_o = 1$ at the front and as thick as the precursory layer b far behind the front. The Neumann BCs are used since far behind and far in front of the contact line the fluid thickness is approximately constant [4].

Expecting traveling-wave solutions, we analyze (4) in a moving reference frame by making the change of variable $\xi = x - Vt$ where V is the velocity of the wave. Hence substituting $h_o(\xi) = h(x,t)$ into (4) and integrating once with respect to ξ, we obtain

$$-Vh_o + Ch_o^3 h_{o\xi\xi\xi} - \left(B + \frac{N}{2\beta^3}\right) h_o^3 h_{o\xi} + U(h_o^3)_\xi = d\,, \tag{13}$$

where d is the constant of integration. The approximation given by (11) reduces the problem to the Newtonian case with a modified Bond number (see [4]). Applying the BCs given in (12), we find that $d = -b(1+b)$ and $V = U(1+b+b^2)$.

4.3 Three-dimensional equation in the limit of small wavenumber

Now moving onto the more interesting phenomena of analyzing instabilities at the contact line, we perform LSA on the 3D model (3) with (11) in the direction transverse to the flow. Here, we consider the constant-flux-driven flow where the volume of the fluid is infinite and is constantly being injected into the film and look for solutions in terms of small wave numbers [2].

Initially, the flow in the transverse direction is fairly stable, however for longer times, fingering instabilities begin to develop under sufficient conditions [4]. Hence it is of interest to apply small perturbations to the leading order equation given by (13) and analyze under what conditions these dynamics occur. Considering only the semi-infinite domain, we write the solution in the form

$$h(x,y,t) = h_o(\xi) + \epsilon\phi(\xi)e^{\sigma t + iky} + O(\epsilon^2)\,, \tag{14}$$

and substitute into (3), we find the O(ϵ) equation

$$\begin{aligned} -\sigma\phi \;=\; & -Vg_\xi + C\left[k^4 h_o^3\phi - k^2\left((h_o^3\phi_\xi)_\xi + h_o^3\phi_{\xi\xi}\right) + (h_o^3\phi_{\xi\xi\xi} + 3h_o^3 h_{o\xi\xi\xi}\phi)_\xi\right] \\ & + \left(B + \frac{N}{2\beta^3}\right)\left(k^2 h_o^3\phi - (h_o^3\phi_\xi + 3h_o^2 h_{o\xi}\phi)_\xi\right) + 3U(h_o^2\phi)_\xi\ . \end{aligned} \tag{15}$$

Since (15) contains only even powers of k, the solution should also depend only on even powers. Hence, in the limit of a small wavenumber, we can apply the following asymptotic expansion:

$$\phi = \phi_o + k^2\phi_1 + O(k^4)\,, \qquad \sigma = \sigma_o + k^2\sigma_1 + O(k^4)\,. \tag{16}$$

Substituting (16) into (15), the leading order equation is

$$-\sigma_o\phi_o = \left[-V\phi + C(h_o^3\phi_{o\xi\xi\xi} + 3h_o^2h_{o\xi\xi\xi}\phi_o - \left(B + \frac{N}{2\beta^3}\right)(h_o^3\phi_{o\xi} + 3h_o^2h_{o\xi}\phi_o)_\xi + 3Uh_o^2\phi_o \right]_\xi . \tag{17}$$

In Kondic (2003) [4], the modified position of the contact line by the perturbation in (14) was considered and the boundary conditions were accordingly linearized and it was concluded that $\phi_o(\xi) = h_{o\xi}(\xi)$. Hence, substituting this expression into (17),

$$-\sigma_o h_{o\xi} = \left[-Vh_o + Ch_o^3h_{o\xi\xi\xi} - \left(B + \frac{N}{2\beta^3}\right)h_o^3h_{o\xi} + U(h_o^3)_\xi \right]_{\xi\xi} , \tag{18}$$

we see that this is the right hand side is the leading order equation and hence $\sigma_o = 0$.

The $O(k^2)$ is given below:

$$\begin{aligned} -\sigma_1 h_{o\xi} &= -V\phi_{1\xi} + C\Big(-(h_o^3h_{o\xi\xi})_\xi - h_o^3h_{o\xi\xi\xi} + (h_o^3\phi_{1\xi\xi\xi})_\xi + 3(ho^2h_{o\xi\xi\xi}\phi_1)_\xi \Big) \\ &\quad + \left(B + \frac{N}{2\beta^3}\right)\Big(h_o^3h_{o\xi} - (h_o^3\phi_1)_{\xi\xi}\Big) + 3U(h_{o\xi}\phi_1)_\xi \ . \end{aligned} \tag{19}$$

Integrating (19) and applying the BCs, we obtain

$$\sigma_1(k) = \frac{1}{1-b}\int_{-\infty}^{0} Ch_o^3h_{o\xi\xi\xi} - \left(B + \frac{N}{2\beta^3}\right)h_o^3h_{o\xi}\,\mathrm{d}\xi . \tag{20}$$

Therefore, we are able to obtain an approximate growth-rate as expressed in (16):

$$\sigma \approx \frac{k^2}{1-b}\int_{-\infty}^{0} Ch_o^3h_{o\xi\xi\xi} - \left(B + \frac{N}{2\beta^3}\right)h_o^3h_{o\xi}\,\mathrm{d}\xi . \tag{21}$$

Analysis was also performed in the case where β was not assumed to be much larger than h so that the approximation in (11) was not applied. The result $\phi(\xi) = h_{0\xi}(\xi)$ still holds and a similar, yet much more complicated expression is obtained for $\sigma(k)$. Future work consists of analyzing the expression for $\sigma(k)$ determining the sign to predict under what conditions instabilities will occur.

5 Conclusion

In essence, it is quite difficult to predict under what conditions the interesting dynamics and instabilities at the contact line develop. Also in experiments conducted by Dupiano, Naughton, Pineda, and Shah, instabilities at the contact line were not observed making comparisons difficult. Future work is being performed to predict more about the fluid dynamics of NLCs.

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