On finite exponential moments for branching processes and busy periods for queues

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Abstract

Using a known fact that a Galton–Watson branching process can be represented as an embedded random walk, together with a result from Heyde [1964], we first derive finite exponential moment results for the total number of descendents of an individual. We use this basic and simple result to prove analogous results for the population size at time t and the total number of descendents by time t in an age-dependent branching process. This has applications in justifying the interchange of expectation and derivative operators in simulation-based derivative estimation for generalized semi-Markov processes. Next, using the result from Heyde [1964], we show that in a stable $GI/GI/1$ queue, the length of a busy period and the number of customers served in a busy period have finite exponential moments if and only if the service time does.

Keywords Branching process, busy period, decoupling, random walk, single-server queue

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1 Introduction

Consider a Markov chain that has the form

$$X_{n+1} = \sum_{j=1}^{X_n} Z_{n,j}, \ \ X_0 = 1 \tag{1}$$

with $\{Z_{n,j} : n \geq 0, \ j \geq 1\}$ i.i.d. (generically denoted by Z), non-negative and discrete. For non-triviality we assume that $E(Z_{n,j}) > 0$. This is usually called the *Galton–Watson* (GW) process (or *Bienayme*–Galton–Watson process). $Z_{n,j}$ denotes the number of progeny of the j^{th} individual from the n^{th} generation. Let

$$K_n \stackrel{\text{def}}{=} \sum_{i=0}^{n} X_i$$

and

$$K \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} X_n.$$

Then $K_n - 1$ (resp., $K - 1$) may be interpreted as the total number of descendents of an individual by time n (resp., ∞). It is well known that if $E(Z) < 1$ then $P(K < \infty) = 1$ (this also holds when $E(Z) = 1$ and $Var(Z) > 0$).

In the GW model, each individual lives for exactly 1 unit of time, but by allowing a general random lifetime Y, one obtains a more general *age-dependent branching process* (see, e.g., Athreya and Ney [1972]). Let $W(t)$ denote the total number of living individuals at time t, and $V(t) - 1$ the total number of descendents by time t.

It is of intrinsic interest to determine conditions under which K, X_n, K_n, $W(t)$ and $V(t)$ have finite moment generating functions in neighborhoods of the origin. In particular, necessary and sufficient conditions ensuring that there exists an $s_0 > 0$ such that $E(e^{sK}) < \infty$, $0 < s \leq s_0$, are developed (similarly for K_n, X_n, $W(t)$ and $V(t)$).

In Section 2 we first focus on K. By using a known embedded Markov chain technique (e.g., Quine and Szczotka [1994]), together with a result of Heyde [1964], we show that if $E(Z) < 1$

then K has finite moment generating function in a neighborhood of the origin if and only if Z does. (We assume this preliminary result to be known, but have not seen it stated in the literature.)

A simple result that seems intuitive, but for which we have not found any formal proof in the literature, is the following: if $E(Z) < \infty$ then for any $n \geq 0$, X_n and K_n have finite moment generating functions in neighborhoods of the origin if and only if Z does. We do not claim any originality for this result, but for the sake of completeness we give a simple formal proof here.

In Section 3, we use the results about the GW process, to show similar results for the age-dependent processes $W(t)$ and $V(t)$. This work on finite exponential moments of age-dependent branching processes is motivated by the problem considered in Nakayama and Shahabuddin (1998). To justify the interchange of derivative and expectation in estimating via simulation the derivative of a performance measure in generalized semi-Markov processes (GSMPs), one needs a condition of the type: $E(e^{sN(t)}) < \infty$ for all s in some neighborhood of zero, where $N(t)$ is the number of transitions of the GSMP in $[0, t]$. This kind of condition is proved by first constructing an age-dependent branching process $V(t)$ that provides upper bounds for $N(t)$ (and then referring to the results in the present paper).

In our final section, we investigate finiteness of moments in another setting. Using the result from Heyde [1964] alluded to earlier, together with a decoupling inequality of De La Peña [1994], we show that the number of customers served in a busy period and the length of a busy period in a stable $GI/GI/1$ queue have finite moment generating functions in neighborhoods of zero if and only if the service-time distribution does.

2 The case of the Galton–Watson Process

Consider the branching process (1) described in the Introduction.

Theorem 2.1 *If $E(Z) < 1$, then there exists an $s_0 > 0$ such that $E(e^{sK}) < \infty,\ 0 < s \le s_0$ if and only if there exists an $s_1 > 0$ such that $E(e^{sZ}) < \infty,\ 0 < s \le s_1$.*

To prove the Theorem, we utilize an embedded random-walk technique used previously by Quine and Szczotka [1994] (the essential idea going back to Section 6 in Harris [1952], and also used by Lindvall [1976]). An equivalent representation of (1) is $X_0 = 1$, $X_1 = Z_1$, and

$$X_{n+1} = \sum_{j=K_{n-1}+1}^{K_{n-1}+X_n} Z_j 1_{\{X_n \ge 1\}} \quad n = 1, 2, \ldots, \tag{2}$$

where 1_A is the indicator random variable of event A and the Z_i's, $i \ge 1$, are i.i.d. and have the same distribution as Z. Then, as is shown in Quine and Szczotka [1994] for example, K is the first strictly descending ladder epoch of a random walk with increments $\Delta_n \equiv Z_n - 1$:

$$K = \min\{m \ge 1 : Z_1 - 1 + \cdots + Z_m - 1 < 0\}.$$

(Note further that if we denote the Δ_n's generically by Δ, then $E(\Delta) < 0$ because $E(Z) < 1$ by assumption. Thus the random walk has negative drift which is why $P(K < \infty) = 1$.)

The proof of Theorem 2.1 then follows by letting $\tau = K$ in the following Theorem 2.2 due to Heyde [1964] and noting that $E(e^{s\{Z-1\}^+}) < \infty$ if and only if $E(e^{sZ}) < \infty$.

Theorem 2.2 *[Heyde [1964], Theorem 1] Let $\{\Delta_n : \ n \ge 1\}$ (generically denoted by Δ) be i.i.d. with $E(\Delta) < 0$. Define the first strictly descending ladder epoch, τ, by*

$$\tau \stackrel{\text{def}}{=} \min\{m \ge 1 : \Delta_1 + \cdots + \Delta_m < 0\}. \tag{3}$$

Then there exists an $s_0 > 0$ such that $E(e^{s\tau}) < \infty,\ 0 < s \le s_0$ if and only if there exists an $s_1 > 0$ such that $E(e^{s\{\Delta\}^+}) < \infty,\ 0 < s \le s_1$.

(In Heyde [1964] results are stated in terms of positive drift and strictly ascending ladder epochs. Theorem 2.2 is also stated as Theorem III.3.2 in Gut [1988].) That the result does not hold in general when $E(Z) = 1$ can be seen by considering the simple symetric random walk with $P(Z = 2) = P(Z = 0) = 0.5$, for which the first ladder height has infinite mean.

In the next section we will be using similar finite moment generating function results for X_n and K_n, $n \geq 1$. These are stated in Lemma 2.1. Note that this lemma does not need the condition that $E(Z) < 1$. Although this result seems intuitively obvious, and has been alluded to in the literature (see, e.g., Selivanov [1969]) nowhere did we find a formal proof. For the sake of completeness we give a simple proof.

Lemma 2.1 *For all $n \geq 2$, there exists an $s_n > 0$ (resp., $\tilde{s}_n$) such that $E(e^{sX_n}) < \infty$, $0 < s \leq s_n$ (resp., $E(e^{sK_n}) < \infty$, $0 < s \leq \tilde{s}_n$) if and only if there exists an $s_1 > 0$ such that $E(e^{sZ}) \equiv E(e^{sX_1}) \equiv E(e^{s(K_1-1)}) < \infty$, $0 < s \leq s_1$.*

Proof : We work in terms of generating functions for discrete random variables. Let $\phi(s) = E(s^Z)$ and $\phi_n(s) = E(s^{X_n})$ for $s \geq 0$ (note that for the case of defining $\phi(0)$ and $\phi_n(0)$ we define $0^0 \equiv 1$). Then the above lemma is equivalent to the following: for all $n \geq 2$, there exists $s'_n > 1$ (resp., $\tilde{s}'_n$) such that $\phi_n(s) < \infty$, $1 < s \leq s'_n$, (resp., $E(s^{K_n}) < \infty$, $1 < s \leq \tilde{s}'_n$) if and only if there exists an $s'_1 > 1$ such that $\phi(s) < \infty$, $1 < s \leq s'_1$, (i.e., the above Lemma will hold with $s_n = \ln(s'_n)$ for $n \geq 1$).

We will use a well known fact from branching process theory that $\phi_n(s) = \phi^{(n)}(s)$ for $s > 0$ (see e.g., Athreya and Ney (1972), Pg 2), where the $\phi^{(n)}(s)$ is the n-fold composition of the function $\phi(s)$, i.e., $\phi^{(1)}(s) \equiv \phi(s)$, $\phi^{(2)}(s) \equiv \phi(\phi(s))$ and so on.

First we prove the sufficiency using induction. Note that since $Z \geq 0$, $\phi(s)$ is continuous and non-decreasing for $1 \leq s \leq s'_1$ with $\phi(1) = 1$. Assume that there exists $s'_{n-1} > 1$, such that $\phi_{n-1}(s) < \infty$ for $1 < s \leq s'_{n-1}$. Since $X_{n-1} \geq 0$, $\phi_{n-1}(s)$ is continuous and non-decreasing for $1 < s < s'_{n-1}$ with $\phi_{n-1}(1) = 1$. Hence there exists a $s'_n > 1$ such that $\phi_{n-1}(s'_n) \leq s'_1$, and therefore $\phi_n(s'_n) = \phi(\phi_{n-1}(s'_n)) \leq \phi(s'_1) < \infty$. Then the non-decreasing nature of $\phi_n(s)$ over the interval $1 < s \leq s'_n$ implies that $\phi_n(s) < \infty$, for s in that interval.

We use induction to show finiteness of $E(s^{K_n})$, knowing that $E(s^{K_1}) < \infty$ for $1 < s \leq s_1'$. Assuming the existence of $\tilde{s}_{n-1}'$ such that $E(s^{K_{n-1}}) < \infty$ for $1 < s \leq \tilde{s}_{n-1}'$ and $n \geq 2$, it follows from the Schwarz Inequality (see, e.g., Billingsley [1986], Pg 283) that for $s > 1$,

$$E(s^{K_n}) = E(s^{K_{n-1}+X_n}) \leq \sqrt{E(s^{2K_{n-1}})E(s^{2X_n})}.$$

Therefore choosing $\tilde{s}_n' = \sqrt{\min\{\tilde{s}_{n-1}', s_n'\}}$ we are done.

The necessity of the condition for the case of K_n follows immediately from the fact that $K_n \geq X_1$ a.s. For X_n, we start by noting that if $\phi(s) = \infty$ for all $s > 1$, then $\phi_2(s) = \phi(\phi(s)) = \phi(\infty) = \infty$ for all $s > 1$, and the same is true for $\phi_n(s)$ by induction. ■

3 Age-Dependent Branching Process

An age-dependent branching process starts with one individual at time $t = 0$. After a random lifetime $Y \geq 0$, with cumulative distribution function (cdf) $G(x) = P(Y \leq x)$, the individual gives birth to R individuals and then dies at that instant. Here R is a non-negative discrete random variable with a general distribution (playing the role of Z in the GW model). Each individual thus generated then behaves identically to the first one, with the lifetimes of the individuals and number of progeny generated by the individuals, constituting i.i.d. sequences, each sequence being independent of the other. The whole process proceeds similarly producing future generations. Note that a Galton-Watson process is the special case when $P(Y = c) = 1$, for some constant $c > 0$. Let $W(t)$ be the (left-continuous) stochastic process denoting the number of individuals alive at time $t-$, and $V(t)$ be the (left-continuous) stochastic process denoting the total number of individuals that have been alive (even for a time duration of 0) during $[0, t)$. As convention, we take $W(0-) = V(0-) = 1$. Note that since we allow $G(0) > 0$, there may be instantaneous, multiple transitions at a given time t.

We will use the results of the last section to prove the following theorem:

Theorem 3.1 *For non-triviality, assume that $E(Y) > 0$ and $E(R) > 0$. If $G(0)E(R) < 1$ and there exists an $s_1 > 0$ such that $E(e^{sR}) < \infty,\ 0 < s \le s_1$, then for all $t > 0$ there exists an $s_0 > 0$ such that $E(e^{sW(t)}) \le E(e^{sV(t)}) < \infty,\ 0 < s \le s_0$.*

To the best of the authors' knowledge, no results of this type have been shown in the existing branching process literature. There are some results for finiteness of $E(\phi(W(t)))$ for a class of functions $\phi(\cdot)$ (Athreya and Ney (1972), Pg 153). However the exponential function does not belong to this class of functions and the methods of proof used there are not applicable here.

Proof : Note that for $E(R) < 1$ this result is immediate from Theorem 2.1: In this case, $V(t) \le V(\infty)$, the total number of individuals generated until extinction, which would be the same as when $P(Y = c) = 1$, namely the GW process. We are thus concerned with the case where $E(R) \ge 1$. Since $G(0) < 1/E(R)$, there exists a $\delta > 0$, such that $G(\delta) < 1/E(R)$.

The main idea of the proof is to construct a new age-dependent branching process, such that the $(V(t) : t \ge 0)$ component of the original process, is bounded by that of the new process. The new age-dependent branching process is constructed such that it has death/birth transitions only at discrete times, and is thus more tractable using GW branching process theory.

Construction of the bounding, age-dependent branching process: Let $Y_1, Y_2, \ldots$ be the i.i.d. sequence of lifetimes used in the original process. Consider a modified age-dependent branching process where the Y_j's in the original process are now changed to $\tilde{Y}_j$'s defined as follows: $\tilde{Y}_j = 0$ if $0 \le Y_j \le \delta$ and $\tilde{Y}_j = \delta$ otherwise. Let $\tilde{W}(t)$ be the total number of individuals in this modified branching process at time $t-$ and $\tilde{V}(t)$ be the total number of individuals that have been alive (even for a time duration 0) in time interval $[0, t)$. Since $W(t) \le V(t) \le \tilde{V}(t)$ w.p. 1, we only need to show finiteness of exponential moments for $\tilde{V}(t)$.

We will now cast this in the GW process framework of Section 1. Clearly $\tilde{W}(t)$ only changes at times $n\delta$, $n = 1, 2, \ldots$. In fact, the process $(\tilde{W}(t), \tilde{V}(t))$ is almost like a GW process, except for the instantaneous transitions at time $n\delta$. In particular the $\tilde{V}(t)$ process also has to keep track of the generations with zero lifetimes. To see this more clearly, define $X_n \equiv \tilde{W}(n\delta)$, $n \geq 0$, i.e., the total number of individuals at time $n\delta$ that have been alive for time δ (recall that $\tilde{W}_n$ is left-continuous). Then X_n is a GW process defined in Section 1 with the progeny random variable Z defined in the following way: Z is the total number of individuals with lifetime δ, that an individual with lifetime δ generates.

Let us now take a closer look at this progeny random variable Z. The attempt will be to see whether Z has finite exponential moments, so that the K_n corresponding to the GW process X_n has finite exponential moment for any n. By letting $n = \lceil t/\delta \rceil$ for any given t, this would have implied that the process $\tilde{V}(t)$ has finite exponential moments, *had* $\tilde{V}(t)$ *not taken into account the generations with zero lifetimes*. Obviously, this is not the case. Nevertheless, we will first proceed with proving that K_n has finite exponential moments, and deal with $\tilde{V}(t)$ later.

Consider the sequence of *instantaneous* descendants arising from an individual, say P, that has been alive for time δ in the modified age-dependent branching process. First this individual will generate R individuals. These are termed as *instantaneous* first-generation descendants arising from P (these are not actually instantaneous, since P has been alive for time δ, but we will just call it so for simplicity of notation). Each of these individuals will either have a lifetime of zero or have a lifetime of δ. The ones with a lifetime of zero will generate more individuals at that very instant. These generated individuals are termed *instantaneous* second-generation descendants arising from P. Define $X'_0 = 1$, and for $l \geq 1$, define X'_l to be the number of *instantaneous* l^{th}-generation descendants arising from P. The $\{X'_l : l \geq 0\}$ is itself a slightly modified GW process given by $X'_0 = 1$, $X'_1 \stackrel{D}{=} R$ (the symbol " $\stackrel{D}{=}$ " means "has the same probability distribution as"), and $X'_{l+1} = \sum_{j=1}^{X'_l} R'_{l,j}$, for $l \geq 1$ where $\{R'_{l,j}\}$ (generically denoted by R') are i.i.d., non-negative and discrete and given by the following: $R' \stackrel{D}{=} R$ with probability $G(\delta)$ and $R' = 0$ with probability $1 - G(\delta)$. Clearly $E(e^{sR}) < \infty$ if and only if $E(e^{sR'}) < \infty$.

Let $K' = \sum_{l=0}^{\infty} X'_l$ be the total number of instantaneous descendants arising from P. Since $E(R') = G(\delta)E(R) < 1$, using Theorem 2.1 (it is trivial to show that this theorem will hold for the slightly modified GW process described above, where individuals of the first generation are generated differently from the individuals of other generations; one can just neglect the distribution of X'_1) we get that there exists s_0 such that $E(e^{sK'}) < \infty$ for $0 < s \leq s_0$.

Note that the Z corresponding to X_n is the total number of instantaneous descendants (arising from an individual with life time δ) that ended up having a lifetime greater than 0. Hence Z is bounded by K' and so $E(e^{sZ}) < \infty$ for $0 < s \leq s_0$. From Lemma 2.1, there exists $\tilde{s}_n$, such that $E(e^{sK_n}) < \infty$ for $0 < s \leq \tilde{s}_n$.

However, as mentioned before, this does not suffice for proving that $\tilde{V}(t)$ has finite exponential moments. To prove this, construct a modified GW process $\tilde{X}_n$ by adding some more branches to X_n in the following manner. Recall that in X_n a typical individual has Z progeny where the random variable Z has been defined before. In the GW process $\tilde{X}_n$, we increase the number of progeny of this individual from Z to K'; recall that K' is the total number of instantaneous descendants arising from an individual who has been alive for time δ, in the modified age-dependent branching process. We do this for all individuals in X_n. Each new progeny thus generated starts a branching process with the progeny distribution being that of K'. Hence the $\tilde{X}_n$ is a GW process with progeny random variable $\tilde{Z} = K'$. Note that

$$\tilde{V}(t) \leq \sum_{i=0}^{n} \tilde{X}_i \equiv \tilde{K}_n, \tag{4}$$

where $n = \lceil t/\delta \rceil$. This is because the $\tilde{X}_i$ now includes individuals produced in the modified age-dependent branching process that have zero lifetimes. Using Lemma 2.1 and the fact shown before that $E(e^{sK'}) < \infty$ for $0 < s \leq s_0$, we get that $\tilde{K}_n$ has finite exponential moments. Using (4) we get that $\tilde{V}(t)$ has finite exponential moments. ■

The following theorem is a partial converse to Theorem 3.1.

Theorem 3.2 *Let $t > 0$ be such that $P(Y \leq t) > 0$. If $E(e^{sR}) = \infty$ for some $s > 0$, then $E(e^{sV(t)}) \geq E(e^{sW(t)}) = \infty$.*

The proof is detailed, but the main idea of the proof is as follows. The first step is to find a positive constant b so that Y lies in an appropriately chosen, small neighborhood of b with positive probability p. Then one lower bounds the probability of a death, or equivalently a birth of size R, happening within the interval $(t-b, t]$. Out of these R births, let $\tilde{R}$ denote those who have lifetimes in the above mentioned neighborhood of b, and thus will be alive at time t. It is easy to see that $\tilde{R}$ is a Binomial(R, p), and that it has infinite exponential moments if R has the same. However, $\tilde{R}$ is also a lower bound for $W(t)$ conditional on the death happening in the interval $(t-b, t]$, and so the infinite exponential moment property is also imparted to this $W(t)$.

Proof : Since $E(Y) > 0$, there exists a $b > 0$ such that for all small enough $\delta > 0$, $P(b - \delta < Y \leq b) > 0$. For any t, define $n_t = \lfloor t/b \rfloor$. Choose δ such that

$$t < (n_t + 1)b - (n_t + 1)\delta, \tag{5}$$

and define

$$p \equiv P(b - \delta < Y \leq b) > 0. \tag{6}$$

We consider first the case that $n_t > 0$. Let $\mathcal{A}$ be the event of at least one death (and corresponding progeny generation) in the interval $(n_t b - n_t\delta, n_t b]$. We first determine a lower bound on $P(\mathcal{A})$. Let $\mathcal{A}'$ be the event that the branching process is alive for at least n_t generations (the starting individual is defined as the first generation; even if an individual is alive for only time 0, it is still counted as one generation). Event $\mathcal{A}'$ occurs if and only if a successive sequence of $n_t - 1$ individuals, that includes the starting individual, produce non-zero progeny. Hence $P(\mathcal{A}') = (P(R > 0))^{n_t - 1}$. Let Y_1, Y_2, $\ldots Y_{n_t}$ be the sequence of lifetimes in one of the branches of the sample tree which is alive for at least n_t generations. Then $\cap_{i=1}^{n_t}\{b - \delta < Y_i \leq b\} \cap \mathcal{A}' \subset \mathcal{A}$.

Since the Y_i's are independent of one another, and of any other random variable in the system, we obtain $P(\mathcal{A}) \geq p^{n_t}(P(R > 0))^{n_t - 1} > 0$.

Let R be the number of progeny corresponding to a death in the interval $(n_t b - n_t \delta, n_t b]$ and let $\tilde{R}$ be the number of these that have a lifetime in the interval $(b - \delta, b]$. From (5) we see that all these $\tilde{R}$ will be alive at time t, and thus $W(t) \geq \tilde{R}$, conditional on the event $\mathcal{A}$ happening. Also, $\tilde{R}$ is Binomial(R, p) and independent of $\mathcal{A}$, since R is independent of $\mathcal{A}$. Now

$$\begin{aligned} E(e^{sW(t)}) &\geq E(e^{sW(t)}|\mathcal{A})P(\mathcal{A}) \\ &\geq E(e^{s\tilde{R}}|\mathcal{A})P(\mathcal{A}) \\ &= E(e^{s\tilde{R}})P(\mathcal{A}) \\ &= E((pe^s + (1-p))^R)P(\mathcal{A}) \\ &\geq pE(e^{sR})P(\mathcal{A}) \end{aligned} \tag{7}$$

The result for $W(t)$ follows from the fact that p and $P(\mathcal{A})$ are positive and $E(e^{sR}) = \infty$.

Now consider the case that $n_t = 0$. From (5), the δ in this case is chosen such that $t < b - \delta$. Since we assume that $P(Y \leq t) > 0$, there is a positive probability that the first death happens in the interval $[0, t]$. Again, letting R to be the number of progeny corresponding to this death and $\tilde{R}$ to be the number of these that have lifetimes in the interval $(b - \delta, b]$, we get that $W(t) \geq \tilde{R}$. We can then use the same argument as (7) where the event $\mathcal{A}$ is replaced by the event that the first death happens in the time interval $[0, t]$.

Finally, since $V(t) \geq W(t)$, we conclude that $E(e^{sV(t)}) \geq E(e^{sW(t)}) = \infty$. ■

4 Busy Periods in GI/GI/1 Queues

It is known that for the $M/GI/1$ queue, the length of a busy period has finite kth moment if and only if the service time does (see, e.g., Wolff [1989]) [1]. In this section, we prove further results in this direction. In particular, we show that for the steady-state $GI/GI/1$ queue, the length of a busy period and the number of customers served in a busy period have finite moment generating functions in neighborhoods of zero if and only if the service-time distribution does.

A $GI/GI/1$ queue has by definition an i.i.d. sequence of interarrival times $\{T_n : n \geq 0\}$ and i.i.d. service times $\{S_n : n \geq 0\}$ with the two sequences independent. $T_n,\ n \geq 0$, is the length of time between the nth and $(n+1)$th customer. We assume that the system is empty at time $t = 0$, at which point the initial customer arrives bringing service time S_0. The busy period B is then defined as the length of time until the system becomes idle again, and we let K denote the number of customers served in the busy period. B is of length at least S_0 and K is of length at least 1. With $\Delta_n = S_{n-1} - T_{n-1},\ n \geq 1$, and $\Delta = S - T$ denoting a generic such difference, it is well known that K is the first strictly descending ladder epoch in a random walk with increments Δ_n (see, e.g., Chapter 9 in Wolff [1989]). Noting then that $E(e^{s\{S-T\}^+}) < \infty$ if and only if $E(e^{sS}) < \infty$ (because S and T are assumed independent), an application of Theorem 2.2 yields:

Proposition 4.1 *For a $GI/GI/1$ queue with generic interarrival time T and generic service time S: If $E(S - T) < 0$ then there exists an $s_0 > 0$ such that $E(e^{sK}) < \infty,\ 0 < s \leq s_0$ if and only if there exists an $s_1 > 0$ such that $E(e^{sS}) < \infty,\ 0 < s \leq s_1$.*

Let the random variable B denote the duration of a generic busy period in a $GI/GI/1$ queue. We next prove:

[1] This is different from finite moment results for the steady-state delay, where finiteness of one higher moment of the service time is needed (Kiefer and Wolfowitz [1956]).

Proposition 4.2 *For a $GI/GI/1$ queue with generic interarrival time T and generic service time S, if $E(S-T)<0$ then there exists an $s_0>0$ such that $E(e^{sB})<\infty,\ 0<s\leq s_0$ if and only if there exists an $s_1>0$ such that $E(e^{sS})<\infty,\ 0<s\leq s_1$.*

Proof : Since

$$B=\sum_{j=1}^{K} S_j, \tag{8}$$

$B\geq S_1$ and necessity follows. To prove sufficiency, observe that K is a stopping time w.r.t. $\{(S_n,T_n)\}$. A decoupling inequality of De La Peña [1994], p.201, allows us to compare $\sum_{j=1}^{K} S_j$ with $\sum_{j=1}^{K}\tilde{S}_j$, where $\{\tilde{S}_n\}$ has the same distribution as $\{S_n\}$ but is taken as independent of K, yielding

$$E\left(e^{s\sum_{j=1}^{n} S_j I\{K\geq j\}}\right)\leq\sqrt{E\left(e^{2s\sum_{j=1}^{n}\tilde{S}_j I\{K\geq j\}}\right)}.$$

Since in each expectation above, the argument is nonnegative and nondecreasing in n, we can apply the monotone convergence theorem (as $n\longrightarrow\infty$) to each side and conclude that

$$E(e^{sB})\leq\sqrt{E\left(e^{2s\sum_{j=1}^{K}\tilde{S}_j}\right)}=\sqrt{E\left(\{E(e^{2sS})\}^K\right)},$$

where the equality comes from the independence of K and $\tilde{S}_j$. By assumption, $E(e^{sS})<\infty$ for all s in some interval $0\leq s\leq s_1$ with s_1 positive, and then it follows from basic properties of moment generating functions that $E(e^{sS})$ is increasing and continuous on that interval. Writing $E(x^K)=E(e^{(\ln x)K})$, and appealing to Proposition 4.1 and the continuity of $\ln x$ in a neighbourhood of $x=1$, shows the existence of some sufficiently small positive s_0 such that

$$E(e^{sB})\leq\sqrt{E\left(\{E(e^{2sS})\}^K\right)}<\infty,\ 0<s\leq s_0.$$

■

Comments for Section 4

1. The special case of the $M/GI/1$ queue can also be handled by using a different approach, i.e., the theory developed in Pakes [1996].

2. Using the Cauchy-Schwarz inequality together with Equation (8) and Proposition 4.1, one can obtain Proposition 4.2 more directly; however, the proof of the inequality of De La Peña [1994] uses the Cauchy-Schwarz inequality anyway. Alternatively one can avoid using Cauchy-Schwarz to prove Proposition 4.2 by taking advantage of the well-known fact that

$$(\psi(t))^{-n} e^{t \sum_{k=1}^{n} S_k}, \ n \geq 1,$$

forms a martingale, where $\psi(t) \stackrel{\text{def}}{=} E(e^{tS})$ is the moment generating function of S.

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