# ASYMPTOTICS OF LIKELIHOOD RATIO DERIVATIVE ESTIMATORS IN SIMULATIONS OF HIGHLY RELIABLE MARKOVIAN SYSTEMS

MARVIN K. NAKAYAMA
*Computer and Information Science Department*
*New Jersey Institute of Technology*
*Newark, NJ 07102*

We discuss the estimation of derivatives of a performance measure using the likelihood ratio method in simulations of highly reliable Markovian systems. We compare the difficulties of estimating the performance measure and of estimating its partial derivatives with respect to component failure rates as the component failure rates tend to 0 and the component repair rates remain fixed. We first consider the case when the quantities are estimated using naive simulation; i.e., when no variance reduction technique is used. In particular, we prove that in the limit, some of the partial derivatives can be estimated as accurately as the performance measure itself. This result is of particular interest in light of the somewhat pessimistic empirical results others have obtained when applying the likelihood ratio method to other types of systems. However, the result only holds for certain partial derivatives of the performance measure when using naive simulation. More specifically, we can estimate a certain partial derivative with the same relative accuracy as the performance measure if the partial derivative is associated with a component either having one of the largest failure rates or whose failure can trigger a failure transition on one of the "most likely paths to failure." Also, we develop a simple criterion to determine which partial derivatives will satisfy either of these properties. In particular, we can identify these derivatives using a sensitivity measure which can be calculated for each type of component.

We also examine the limiting behavior of the estimates of the performance measure and its derivatives which are obtained when an importance sampling scheme known as balanced failure biasing is used. In particular, we show that the estimates of all derivatives can be improved. In contrast to the situation that arose when using naive simulation, we prove that in the limit, *all* derivatives can be estimated as accurately as the performance measure when using balanced failure biasing.

Finally, we formalize the notion of a "most likely path to failure" in the setting of highly reliable Markovian systems. We accomplish this by proving a conditional limit theorem on the distribution of the sample paths leading to a system failure, given that a system failure occurs before the system returns to the state with all components operational. We use this result to establish our other results. (SIMULATION; GRADIENT ESTIMATION; LIKELIHOOD RATIOS; HIGHLY RELIABLE SYSTEMS; IMPORTANCE SAMPLING).

# 1 Introduction

Large and complex systems arise in many technological areas, such as computer systems, communications networks, transaction processing systems, and power systems. These systems often need to be highly reliable, and a designer of such a system is confronted with a multitude of choices for its layout. For example, how much redundancy should be built into the system? How does the failure rate of a given component affect the overall system performance? Thus, a need arises for methodologies that can be used to explore the tradeoffs of the different designs.

Information on the partial derivatives of the system performance measures with respect to various parameters of the system, such as the components' failure and repair rates, can be particularly useful in this regard. Using the partial derivatives, the designer is able both to validate the model and to identify critical components. For example, the correctness of a mathematical formulation may be in question if increasing the failure rate of a component resulted in a higher steady-state availability of the system. Also, the designer with a limited budget may want to concentrate his or her efforts on improving the performance of the components corresponding to the largest partial derivatives to gain the greatest improvement in the overall system performance.

In this paper we consider estimating the partial derivatives of a performance measure with respect to various input parameters via simulation. Simulation is an appropriate tool for estimating performance measures and their derivatives for highly reliable systems. For Markovian systems, analytic (i.e., non-simulation) methods are available for determining the quantities of interest. However, these methods are often not practical due to the size and complexity of many reliability systems. For example, a system consisting of $K$ different components, where each component may be either in a functioning or failed state, will lead to a model having $2^K$ states. In non-Markovian settings, analytic methods may not exist or may be difficult to implement.

The technique we use to estimate the derivatives is the likelihood ratio method, which has been previously studied in numerous papers including Glynn (1986) and Reiman and Weiss (1989). The methodology is applicable to a wide spectrum of stochastic simulations (see Glynn 1990), and many aspects of the approach have been examined; e.g., see Glasserman (1990b), L'Ecuyer et al. (1989), and L'Ecuyer (1990). The technique has also been called the "score function" method by Rubinstein (1986, 1989). An alternative approach for estimating derivatives is infinitesimal perturbation analysis, which was introduced by Ho and Cao (1983) and has been extensively investigated; e.g., see Suri (1989), Suri and Zazanis (1988), Glasserman (1990a, 1992), Ho (1987), Heidelberger et al. (1988), and L'Ecuyer (1990).

Previous theoretical and empirical work has resulted in a somewhat pessimistic view of the likelihood ratio method. Both Reiman and Weiss (1989) and Glynn (1987) argue that as the length of an observation grows to infinity, the variance of the derivative estimator grows linearly in the length of the observation. Hence, in applications where observations tend to be fairly long, the method yields noisy derivative estimates. For example, Reiman and Weiss (1989) found that when experimenting with a queueing model having moderate traffic intensity, the relative standard deviations of the derivative estimates were several times larger than that of the estimate of the performance measure. Similar results have been obtained by others, including L'Ecuyer et al. (1989) and Glasserman (1992). However, we now show that when the likelihood ratio method is applied in the proper setting, it is no more difficult to estimate certain partial derivatives of a performance measure than it is to estimate the performance measure itself.

We consider the performance measure mean time to failure (MTTF) in the context of highly reliable Markovian systems. The main motivation for analytically studying these types of systems is the ongoing work on the state-of-the-art software package SAVE (System AVailability Estimator) at IBM; see Goyal and Lavenberg (1987). The SAVE package computes a variety of performance measures of highly reliable Markovian system using one of two methods: numerical (i.e., non-simulation) and simulation. Due to the large state spaces of these types of systems, the numerical method truncates the state space and gives error bounds on the resulting estimates. When using the simulation method, we no longer have to worry about the size of the state space since the entire generator matrix does not have to be stored; e.g., see Goyal et al. (1992). However, we now encounter a different problem. Because of the rareness of system failures in the types of systems SAVE considers, the resulting performance measure estimators are typically poor when using naive simulation (i.e., without using any variance reduction techniques). Hence, we must use variance reduction techniques to obtain efficient estimators. In particular, SAVE incorporates importance sampling; see Glynn and Iglehart (1989) for an overview of this method.

The basic idea of importance sampling in simulations of highly reliable Markovian systems is to change the dynamics of the system (i.e., the underlying probability measures) so that the system fails more frequently. Since the system now being simulated is different than the original one, we must multiply the estimator by a correction factor known as the likelihood ratio to obtain unbiased estimates. A number of importance sampling methodologies known as "failure biasing" schemes have been proposed for simulating highly reliable Markovian systems (e.g., see Lewis and Böhm 1984, Shahabuddin et al. 1988, and Goyal et al. 1992). The basic idea of these methods is as follows. Consider any state from which there are both repair transitions (i.e., some components are repaired) and failure transitions (i.e., some components fail) possible. Under the original dynamics of the system, it is much more likely that a repair occurs. In failure biasing we increase the total probability of a failure transition and correspondingly decrease the total probability of a repair transition so that they are of the same order of magnitude. In practice, failure biasing often reduces the variance by orders of magnitude. We can also apply importance sampling to the estimation of derivatives, and Nakayama, Goyal, and Glynn (1990) showed empirically that there is a comparable variance reduction for the derivative estimators. The goal of this paper is to explain analytically the experimental results of Nakayama, Goyal, and Glynn (1990).

To do this, we use the mathematical framework of highly reliable Markovian systems developed by Shahabuddin (1991), which we now describe. It turns out (e.g., see Shahabuddin et al. 1988 and

Goyal et al. 1992) that the MTTF can be expressed as a ratio of two expectations and estimated using regenerative simulation; see Crane and Iglehart (1974) for more details on the regenerative method. We can estimate each of the expectations using independent regenerative cycles, and Goyal et al. (1992) showed that the resulting ratio estimator satisfies a central limit theorem. Using a matrix algebraic approach of analysis, Shahabuddin (1991) derived asymptotic expressions for this estimator and proved that when using naive simulation, the relative error (defined as the standard deviation of an estimator over its mean) of the ratio estimator of the MTTF diverges to infinity as the failure rates of the components tend to zero and the repair rates remain fixed. Hence, it becomes more difficult to estimate the MTTF via naive simulation as system failures become rare. In practice this means that to obtain an estimator with a confidence interval having a fixed relative width, we must increase the number of samples (or regenerative cycles) as system failures become less frequent. This is the main problem with rare event simulation.

Shahabuddin (1991) proposed the "balanced failure biasing" method of importance sampling and analytically examined the behavior of the estimator of the denominator of the MTTF obtained using this technique. In particular, Shahabuddin proved that the estimates the denominator of the MTTF obtained using balanced failure biasing are asymptotically stable; i.e., the relative errors remain bounded as the failure rates go to zero and the repair rates remain fixed. Consequently, we obtain good estimates of the denominator of the MTTF regardless of the rareness of system failures when balanced failure biasing is employed. Thus, to obtain an estimator with a given relative width for its confidence interval, we only need to simulate a fixed number of regenerative cycles, where the number of cycles does not depend on how rarely the system fails (under the original probability measure).

Furthermore, Shahabuddin (1991) proved that the numerator of the ratio expression for the MTTF is easy to estimate via naive simulation (i.e., its relative error is bounded) and the main contribution to the variance of the ratio estimator obtained using naive simulation stems from the estimate of the denominator, which has unbounded relative error in the limit. This is to be expected, as the term in the denominator is the probability of a rare event, which tends to zero as the component failure rates vanish. Therefore, we now focus only on the denominator of the ratio formula for the MTTF.

In this paper we consider the estimation of the partial derivatives with respect to the components' failure rates of the denominator term in the ratio formula for the MTTF. First, we derive asymptotic expressions for the partial derivatives and the variances of their estimators when using naive simulation; see Theorem 3 in Section 4.1. We then use this result to show that when using naive simulation, the relative errors of the partial derivative estimates also diverge to infinity in the limit, and so the derivatives are difficult to estimate when system failures are rare; see Corollary 4 in Section 4.2. However, we prove that the ratio of the relative error of the estimates of certain partial derivatives of the denominator over the relative error of the denominator's estimator remains bounded as the failure rates of the components go to zero and the repair rates remain fixed when importance sampling is not employed. Thus, as system failures become rare, it is no more difficult to estimate via naive simulation certain derivatives than it is to estimate the denominator term itself. However, the result only holds for derivatives with respect to failure rates of certain components. More specifically, this result is valid for partial derivatives that are associated either with a component having one of the largest failure rates or with a component whose failure can trigger a failure transition on one of the "most likely paths to failure"; see Corollary 5 in Section 4.2. Since the second condition just given is somewhat difficult to verify for large models, we develop a simple criterion to determine which derivatives can be estimated with the same relative accuracy as the performance measure. Specifically, we can identify these derivatives using a sensitivity measure which can be calculated for each type of component; see Corollaries 6 and 7 in Section 4.2.

We also show that importance sampling can be applied to obtain efficient derivative estimators. In particular, we prove that when balanced failure biasing is applied, the relative error of the estimate of the partial derivative with respect to *any* of the component failure rates remains bounded as the failure rates go to zero and the repair rates remain fixed (see Corollary 10). In addition, it turns out that all of the derivatives we consider are equally easy to estimate in the limit when balanced failure biasing is employed (see Corollary 11), which is in sharp contrast to the situation that arose by using naive simulation.

Finally, while Shahabuddin's (1991) matrix algebraic approach can be used to investigate the limiting behavior of the derivative estimators, the resulting analysis is quite tedious. Hence, we develop a new, simpler methodology based on the notion of "most likely paths" to system failure. To do this, we first prove a result (Theorem 2) which formalizes the idea of "most likely paths" into a mathematically rigorous statement. Our theorem is a conditional limit theorem for the distribution of sample paths which lead to system failure, given that a system failure occurs before the system returns to the fully operational state. We then apply the result to study the asymptotic properties of the gradient estimates.

Several authors have applied the concept of most likely paths to failure in a different manner to study other aspects of highly reliable systems. Gnedenko and Solovyev (1975) develop the same idea (which they call the "main event") and use it to analyze the limiting distribution of the (appropriately normalized) time to system failure; also see Gertsbakh (1984). Also, Shahabuddin (1990, Section 2.5.1) employs the notion to develop some approximate bounds for the variance of the estimator of the denominator of the MTTF obtained using balanced failure biasing. All of these authors, however, work with models that have some simplifying assumptions which we do not use.

The rest of the paper is organized as follows. In Section 2 we describe the model of highly reliable Markovian systems developed by Shahabuddin (1991) and appropriately modify it for our context. Section 3 reviews the work of Shahabuddin (1991) on the estimation of the denominator in the ratio expression for of the MTTF using naive simulation. Also, we extend Shahabuddin's results by analyzing the paths to system failure and develop the most likely paths to failure theorem. In

Section 4 we briefly describe the likelihood ratio method for estimating derivatives. In addition we discuss the difficulty of estimating the partial derivatives of the denominator term of the MTTF with respect to the component failure rates using naive simulation and make comparisons with the difficulty in estimating the denominator itself. In Section 5 we discuss Shahabuddin's work on the estimation of the denominator of the MTTF using the balanced failure biasing method of importance sampling. In Section 6 we consider the estimation of the derivatives when applying balanced failure biasing. Concluding remarks and directions for future research are included in Section 8. Section 9 contains most of the proofs.

## 2 Markovian Models of Highly Reliable Systems

The following is a description of Shahabuddin's (1991) model of a highly reliable Markovian system, which we have slightly modified for our context. We assume the system is composed of $C$ different types of components, with $n_i$ components of type $i$, where $0 < C < \infty$ and $0 < n_i < \infty$, $1 \leq i \leq C$. As time progresses, the components randomly fail and get repaired, where the repair discipline is arbitrary. We denote the state space of the system by $E$, where $|E| < \infty$. Our analysis is independent of the actual form of $E$. Note that a state $x \in E$ keeps track of the number of the failed components of each type as well as any necessary information about the queueing at the repair facility. We define $n_i(x)$ to be the number of components of type $i$ that are operational in state $x$. The system starts out in the state in which all components are operational; we label this state as state 0. We decompose the state space $E = U \cup F$, where $U$ is the set of states for which the system is considered operational, and $F$ is the set of failed states. We assume that

$$\text{if } x \in U \text{ and } y \in E \text{ with } n_i(y) \geq n_i(x) \text{ for all } i, \text{ then } y \in U. \tag{1}$$

The lifetimes of components of type $i$ are exponentially distributed with rate $\lambda_i$, where $0 < \lambda_i < \bar{\lambda}_i$. We say that a transition $(x, y)$ is a *failure transition* if $n_j(y) \leq n_j(x)$ for all $j$ with $n_i(y) < n_i(x)$ for some $i$. If $(x, y)$ is a failure transition, then we use the notation "$y \succ x$." To accommodate component interactions, we allow for the possibility of *component failure propagation*; i.e., the failure of one component causes other components also to fail instantaneously with some given probability. More precisely, suppose that the system is currently in state $x$ and a component of type $i$ fails. Let $y \in E$ be some state such that $n_j(y) \leq n_j(x)$ for all $j$ with $n_i(y) < n_i(x)$. Then, after the component of type $i$ fails, the system immediately enters state $y$ with probability $p(y; x, i) \geq 0$. In this situation, the failure of the component of type $i$ caused $n_j(x) - n_j(y)$ components of type $j$, $j \neq i$, to fail also, and we say that the failure of the component of type $i$ *triggered* the failure transition $(x, y)$. In addition, if $n_i(x) - n_i(y) \geq 2$, then the component of type $i$ that originally failed caused other components of type $i$ to fail. For example, in a computer system, the failure of a processor might "contaminate" the database, thus causing the database to fail also.

Once a component fails, it is sent to a repair facility having a fixed number of repairmen. A transition $(x, y)$ is called a *repair transition* if $n_j(y) \geq n_j(x)$ for all $j$ with $n_i(y) > n_i(x)$ for some $i$.

If $(x, y)$ is a repair transition, then we use the notation "$y \prec x$." A repair transition $(x, y)$ occurs with exponential rate $\mu(x, y) \geq 0$; thus, we allow for the repair of some failed component to depend on the state of the system. For example, if a processor has a power supply and both are failed, the repairperson cannot fix the processor until its power supply is repaired. Furthermore, we allow for the possibility of the repair facility completing service on more than one component at a given instance. This may happen when the system contains a module composed of a certain number of sub-components and the entire module is replaced when any of the sub-components fail. On a repair transition $(x, y)$, $n_i(y) - n_i(x)$ components of type $i$, $1 \leq i \leq C$, complete repair.

We assume that a single transition cannot consist of some number of components failing and other components simultaneously completing repair. For example, this may occur if a repairperson, when replacing a failed component, accidentally breaks another component. However, we do not allow this in our model. Thus, because of the structure of $F$ given in (1), the only way the system can reach a failed state is through a failure transition.

Let $Y = \{Y(s) : s \geq 0\}$ be the continuous-time Markov chain (CTMC) defined on the state space $E$ arising from our model of the highly reliable system. We denote the infinitesimal generator matrix of $Y$ by $Q(\theta) = \{q(\theta, x, y) : x, y \in E\}$, where the parameter $\theta$ is given by $\theta = (\lambda, \mu)$ with $\lambda = (\lambda_1, \ldots, \lambda_C)$ and $\mu = \{\mu(x, y) : y \prec x\}$. Thus, for all $x \neq y$,

$$q(\theta, x, y) = \begin{cases} \sum_{i=1}^{C} n_i(x) \lambda_i p(y; x, i) & \text{if } y \succ x \\ \mu(x, y) & \text{if } y \prec x \\ 0 & \text{otherwise} \end{cases}, \tag{2}$$

and $q(\theta, x, x) = -\sum_{y \neq x} q(\theta, x, y)$. As mentioned in the introduction, we will be estimating derivatives with respect to failure rates of the components. Thus, we allow the vector of component failure rates $\lambda$ to vary in some set $\Lambda = \{\lambda : 0 < \lambda < \bar{\lambda}\}$, where $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_C)$. (Since we do not consider derivatives with respect to the repair rates, we assume that $\mu$ is fixed at $\mu = \mu_0$. For work on derivatives with respect to $\mu$, see Nakayama 1991.) By defining the generator matrix in this manner, $Q(\theta)$ is continuously differentiable in $\lambda \in \Lambda$. We employ the notation $P_\theta$ and $E_\theta$ to represent the probability measure and expectation, respectively, induced by the generator matrix $Q(\theta)$ for some fixed value of $\theta$.

We use regenerative simulation to estimate the mean time to failure in the following manner. For any set of states $A \subset E$, define $T_A = \inf\{s > 0 : Y_s \in A,\ Y_{s^-} \notin A\}$. It can be shown (e.g., see Goyal et al. 1992) that the mean time to failure (given that the system starts in state 0) is expressable as the following ratio of expectations:

$$\eta(\theta) \equiv E_\theta[T_F] = \frac{E_\theta[\min\{T_F, T_0\}]}{E_\theta[1\{T_F < T_0\}]}, \tag{3}$$

where $1\{\,\cdot\,\}$ denotes the indicator function of the event $\{\,\cdot\,\}$. We then estimate both the numerator and denominator in (3) using regenerative simulation with state 0 as the regenerative state (i.e., the successive times at which $Y$ enters state 0 form a sequence of regeneration points); see Crane and

Iglehart (1974) for more details on the regenerative method of simulation. The advantage of using the ratio representation is that we can independently estimate the numerator and denominator using different simulation methods. This technique is known as measure specific importance sampling; see Goyal et al. (1992).

Let $X = \{X_n : n \geq 0\}$ be the embedded discrete-time Markov chain (DTMC) of $Y$. Recall that $X$ has transition matrix $P(\theta)$ defined by $P(\theta, x, y) = q(\theta, x, y)/q(\theta, x)$ for $x \neq y$, where $q(\theta, x) = -q(\theta, x, x)$, and $P(\theta, x, x) = 0$. We define

$$\Gamma(\theta) \equiv \{(x, y) : P(\theta, x, y) > 0\},$$

which is the set of possible transitions that $X$ can take under parameter value $\theta$. Note that $\Gamma(\theta)$ is independent of $\theta$ for $\theta \in \Theta$, where $\Theta = \{(\lambda, \mu) : \lambda \in \Lambda,\ \mu = \mu_0\}$. Consequently, we write

$$\Gamma(\theta) = \Gamma.$$

We can now apply conditional Monte Carlo (or discrete time conversion) to (3) to obtain another estimator of the MTTF. For any set of states $A \subset E$, we first define $\tau_A = \inf\{n > 0 : X_n \in A\}$ and $\alpha_A(\theta) = \sum_{k=0}^{\tau_A - 1} 1/q(\theta, X_k)$. Then we apply conditional Monte Carlo by conditioning on $X$; i.e.,

$$E_\theta[\min\{T_F, T_0\}] = E_\theta[E_\theta[\min\{T_F, T_0\} \mid X]] = E_\theta[\min\{\alpha_F, \alpha_0\}]$$

and

$$E_\theta[1\{T_F < T_0\}] = E_\theta[E_\theta[1\{T_F < T_0\} \mid X]] = E_\theta[1\{\tau_F, \tau_0\}].$$

Thus, we obtain the following expression for the MTTF:

$$\eta(\theta) = \frac{E_\theta[\min\{\alpha_F(\theta), \alpha_0(\theta)\}]}{E_\theta[1\{\tau_F < \tau_0\}]}. \tag{4}$$

Note that we can estimate the MTTF by simulating only the embedded DTMC $X$. For more details on conditional Monte Carlo (or discrete-time conversion), see Hordijk, Iglehart, and Schassberger (1976) or Fox and Glynn (1986).

The type of highly reliable system we examine is one which consists of highly reliable components; i.e., the components' repair rates are much larger than the components' failure rates. We use the following approach to model highly reliable components. We assume that the failure rate of components of type $i$ is given by $\lambda_i = \lambda_i(\epsilon)$, where

$$\lambda_i(\epsilon) = \tilde{\lambda}_i \epsilon^{b_i}, \tag{5}$$

with $\tilde{\lambda}_i > 0$, $b_i \geq 1$, and $\epsilon > 0$. For the sake of simplicity, we will assume that the $b_i$ are integer valued, which we can do (essentially) without loss of generality. By letting $\epsilon \to 0$ and fixing the repair rate parameters $\mu$, the component repair rates are much larger than the failure rates. (This modeling technique has been used by others, including Gnedenko and Solovyev 1975, Gertsbakh 1984, Katehakis and Derman 1989, and Shahabuddin 1991. Gnedenko and Solovyev 1975, Gertsbakh 1984, and Katehakis and Derman 1989 assumed that all of the $b_i = 1$, whereas Shahabuddin 1991 allowed the $b_i \geq 1$ to be different.) We assume that the probability mass function $p(\,\cdot\,; x, i)$ used in defining the component failure propagation is independent of $\epsilon$ for all $x$ and $i$.

Let $b_0 = \min\{b_i : 1 \leq i \leq C\}$. If $b_i = b_0$ for all $1 \leq i \leq C$, then we say the system is *balanced*; otherwise, the system is called *unbalanced*.

When the failure rates are expressed as functions of $\epsilon$, we denote our parameter $\theta$ by $\theta(\epsilon) = (\lambda(\epsilon), \mu)$, where $\lambda(\epsilon) = (\lambda_1(\epsilon), \ldots, \lambda_C(\epsilon))$. Thus, the transition rate of entering state $y$ from state $x$, $x \neq y$, can be expressed as

$$q(\theta(\epsilon), x, y) = \begin{cases} \sum_{k=d_1(x,y)}^{d_2(x,y)} c_k(x,y)\epsilon^k & \text{if } y \succ x, \\ \mu(x,y) & \text{if } y \prec x, \\ 0 & \text{otherwise} \end{cases}, \tag{6}$$

where $d_1(x,y) \geq 1$ and $d_2(x,y) \geq d_1(x,y)$ are integer-valued, $c_k(x,y) \geq 0$ and $c_{d_1(x,y)}(x,y) > 0$, and $\epsilon > 0$. (Throughout this paper, we use the variables $c$ and $d$ in many different expressions and contexts; i.e., we employ $c(x,y)$, $d(x,y)$, $c_i$, $d_i$, $c_i'$, $d_i$, etc. repeatedly. The values of these variables change from one usage to the next unless otherwise specified.) Hence, all failure transitions occur at a rate which is some power of $\epsilon$ and all repair transitions have rate which is independent of $\epsilon$. Also, observe that there exists a state $y \in E$ such that $(0, y) \in \Gamma$ and $q(\theta(\epsilon), 0, y) = c(0, y)\epsilon^{b_0}$, where $c(0, y) > 0$. This implies that the total rate out of state 0, which is given by $q(\theta(\epsilon), 0) = \sum_{(0,z)\in\Gamma} q(\theta(\epsilon), 0, z)$, is of the order $\epsilon^{b_0}$. Finally, when we have $\theta = \theta(\epsilon)$, it is easily seen that $\Gamma(\theta(\epsilon)) = \Gamma$; i.e., $\Gamma(\theta(\epsilon))$ is independent of $\theta(\epsilon)$ for all $\epsilon > 0$.

To prove our results, we need to assume that the Markovian model of the highly reliable system satisfies some conditions, which were developed by Shahabuddin (1991). Before describing the assumptions, we make some definitions. Let $c$ be some constant. A function $f$ is said to be $o(\epsilon^c)$ if $f(\epsilon)/\epsilon^c \to 0$ as $\epsilon \to 0$. Also, we use the notation $f(\epsilon) \sim \epsilon^d$ if we can express $f(\epsilon) = c_1\epsilon^d + o(\epsilon^d)$, where $c_1 \neq 0$ is independent of $\epsilon$. Then we assume the following:

**A1** *The CTMC $Y$ is irreducible over $E$.*

**A2** *For each state $x \in E$ with $x \neq 0$, there exists a state $y \in E$ (which depends on $x$) such that $(x, y) \in \Gamma$ and $y \prec x$.*

**A3** *For all states $z \in F$ such that $(0, z) \in \Gamma$, $q(\theta(\epsilon), 0, z) = o(\epsilon^{b_0})$.*

Assumption A1 ensures that there are no deadlocks. For example, this may occur if two components are failed, but to be repaired, they both depend on the other being operational.

Assumption A2 states that there is at least one repair transition possible from each state $x \neq 0$. Hence, for $x \neq 0$, $q(\theta(\epsilon), x) = c(x) + o(1)$, where $c(x) > 0$. This implies that all failure transitions $(x, y)$ with $x \neq 0$ have transition probability $P(\theta(\epsilon), x, y) \sim \epsilon^{d(x,y)}$, where $d(x, y) \geq b_0$.

Assumption A3 guarantees that transitions which take the system from state 0 to a failed state have transition rates that are much smaller than $\epsilon^{b_0}$, which is the magnitude of the largest transition rate from state 0. Therefore, if $z \in F$ and $(0, z) \in \Gamma$, then $(0, z)$ will have a transition probability which is $o(1)$. This ensures that system failures are rare events for the embedded DTMC when $\epsilon$ is small.

From these assumptions the elements of the transition matrix have a certain form. Consider $(x, y) \in \Gamma$ and fix $\mu$. Then, as $\epsilon \to 0$,

$$P(\theta(\epsilon), x, y) = \begin{cases} c(x,y) + o(1) & \text{if } x \neq 0 \text{ and } y \prec x \\ c(x,y)\epsilon^{d_1(x,y)} + o(\epsilon^{d_1(x,y)}) & \text{if } x \neq 0 \text{ and } y \succ x \\ c(x,y)\epsilon^{d_2(x,y)} + o(\epsilon^{d_2(x,y)}) & \text{if } x = 0 \text{ and } y \succ x \end{cases}, \tag{7}$$

where $c(x, y) > 0$, $d_1(x, y) \geq b_0$, and $d_2(x, y) \geq 0$ are independent of $\epsilon$.

We now give a simple example of a system having the structure described in this section. (We will return to this example later when developing our results.)

**Example 1** Consider a system consisting of three types of components, where the first two types of component have a redundancy of 2 (i.e., $n_1 = n_2 = 2$) and failure rates $\lambda_1(\epsilon) = \lambda_2(\epsilon) = \epsilon$ (i.e., $b_1 = b_2 = 1$), and the third type of component has a redundancy of 1 (i.e., $n_3 = 1$) and failure rate $\lambda_3(\epsilon) = \epsilon^3$ (i.e., $b_3 = 3$). Thus, $b_0 = 1$. There is a single repairperson who repairs all components at rate 1, and the repair discipline is processor sharing. The system fails when all of the components of any type are failed. It is sufficient to define the state of the system to be $\langle x_1, x_2, x_3 \rangle$, where $x_i$ is the number of failed components of type $i$. Figure 1 contains the transition probability diagram of the model, where we have combined all states having the component of type 3 failed into a single state $\langle x_1, x_2, 1 \rangle$. Therefore, for this example, $U = \{\langle 0,0,0 \rangle, \langle 0,1,0 \rangle, \langle 1,0,0 \rangle, \langle 1,1,0 \rangle\}$ and $F = \{\langle 0,2,0 \rangle, \langle 2,0,0 \rangle, \langle 1,2,0 \rangle, \langle 2,1,0 \rangle, \langle 2,2,0 \rangle, \langle x_1, x_2, 1 \rangle\}$. ■

# 3 Estimating the Performance Measure Using Naive Simulation

In this section we first review some of the results of Shahabuddin (1991) on the difficulty of estimating the MTTF using the ratio formula given in (4) when using naive simulation. We then extend this result by proving a theorem about the behavior of the different sample paths which lead to system failures.

Recall that we defined the relative error of an estimator as the ratio of its standard deviation over its mean. (By an "estimator," we mean a random variable under some probability measure.) Shahabuddin showed that the numerator in (4) is easy to estimate when using naive simulation; i.e., if we consider the random variable $\min\{\alpha_F, \alpha_0\}$ under the original probability measure $P_{\theta(\epsilon)}$, then its relative error remains bounded as $\epsilon \to 0$. In practice this means that we only need a fixed number of samples (or regenerative cycles) of the numerator in (4) to obtain a confidence interval of a given width, independent of how small $\epsilon$ is.

Consequently, we only consider the estimation of $\gamma(\theta(\epsilon)) \equiv E_{\theta(\epsilon)}[1\{\tau_F < \tau_0\}]$, which is the denominator term in (4). Let $\hat{\gamma}(\theta(\epsilon))$ denote the estimator of $\gamma(\theta(\epsilon))$ obtained using naive simulation, and let $RE(\hat{\gamma}(\theta(\epsilon)))$ denote its relative error. The following proposition, due to Shahabuddin (1991), shows the difficulty of estimating $\gamma(\theta(\epsilon))$ using naive simulation.

**Proposition 1 (Shahabuddin)** *Consider a model of any highly reliable Markovian system (as described in Section 2) which satisfies Assumptions A1–A3. For all $\epsilon$ sufficiently small, there exists $r \geq 1$ and $a_0 > 0$ (which depend on the model) such that*

**(i)** $\gamma(\theta(\epsilon)) \equiv E_{\theta(\epsilon)}[1\{\tau_F < \tau_0\}] = a_0\epsilon^r + o(\epsilon^r)$.

*Also, using naive simulation,*

**(ii)** $\sigma^2(\hat{\gamma}(\theta(\epsilon))) \equiv Var_{\theta(\epsilon)}[1\{\tau_F < \tau_0\}] = \gamma(\theta(\epsilon)) - \gamma^2(\theta(\epsilon)) \approx \gamma(\theta(\epsilon))$,

**(iii)** $RE(\hat{\gamma}(\theta(\epsilon))) = \frac{\sigma(\hat{\gamma}(\theta(\epsilon)))}{\gamma(\theta(\epsilon))} = \frac{\sqrt{a_0\epsilon^r + o(\epsilon^r)}}{(a_0\epsilon^r + o(\epsilon^r))} = \epsilon^{-r/2}\left(a_0^{-1/2} + o(1)\right) \rightarrow \infty$ *as* $\epsilon \rightarrow 0$.

Proposition 11 gives an asymptotic expression for the denominator term in (4). It should be noted that the expression is independent of the simulation method used to estimate it. However, the variance of the estimate of $\gamma(\theta(\epsilon))$ depends on the simulation technique employed, and 2 gives an asymptotic expression for this quantity when using naive simulation. Part 3 shows that using naive simulation, the relative error of the estimator of the denominator term grows without bound as the failure rates go to zero and the repair rates remain fixed. Thus, it becomes harder to estimate the performance measure $\gamma(\theta(\epsilon))$ as the system failures become rarer. In practice this means that to obtain a confidence interval of some fixed relative width, the number of samples (or regenerative cycles) must increase as system failures become less frequent. (Actually, when performing a relative error analysis, the amount of time required to generate each of the cycles also should be taken into account. This amounts to multiplying the variance factor by the mean number of computer operations per cycle; see Glynn and Whitt 1992. However, in our case, we can easily show that the mean length of a cycle remains bounded as $\epsilon \rightarrow 0$, so all of our results remain valid even when this is taken into account.)

Now we want to analyze more closely the event of a system failure. We do so by examining the sample paths of the embedded DTMC (i.e., a sequence of component failures and repairs) which lead to system failure before returning to state 0. Typically, there are many sample paths of this type. However, it turns out that for small $\epsilon$, system failures usually occur in only a relatively small number of possible ways; i.e., the probability of the sample paths in this set have much larger probability than the other paths. We call this set of sample paths the "most likely paths to system failure."

Formally, set $\tau = \min\{\tau_0, \tau_F\}$, and define

$$\Delta = \{\ (x_0, \ldots, x_n)\ :\ n \geq 1,\ x_0 = 0,\ x_n \in F,\ x_k \notin \{0, F\} \text{ and } (x_{k-1}, x_k) \in \Gamma \text{ for } 1 \leq k < n\ \},$$

which is the set of sample paths for which $\tau_F < \tau_0$ and is independent of $\epsilon$. Let "$\Rightarrow$" denote weak convergence; see p. 192 of Billingsley (1986). Then we have the following conditional limit theorem; see Section 9 for the proof.

**Theorem 2** *Consider a model of any highly reliable Markovian system (as described in Section 2) which satisfies Assumptions A1–A3. Then*

$$P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) \in \cdot \mid \tau_F < \tau_0\} \Rightarrow P_0\{(X_0, \ldots, X_\tau) \in \cdot\ \},$$

*as* $\epsilon \to 0$*, where* $P_0$ *is some limiting probability measure on* $\Delta$*.*

We now make some remarks about Theorem 2.

**(i)** We say that a sample path $(x_0, \ldots, x_n) \in \Delta$ has a *largest asymptotic probability* if

$$\liminf_{\epsilon \to 0} \frac{P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_n)\}}{P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) = (y_0, \ldots, y_m)\}} > 0$$

for all paths $(y_0, \ldots, y_m) \in \Delta$. It turns out that the set of paths $(x_0, \ldots, x_n) \in \Delta$ for which $P_0\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_n)\} > 0$ is exactly the paths with the largest asymptotic probability. Hence, in the limit, these are the most likely paths to system failure.

**(ii)** We can explicitly state the exact form of the limiting distribution $P_0$ as follows. Let $\Delta_r$ be the set of paths which have a largest asymptotic probability. Then, it turns out that

$$P_0\{(X_0, \ldots, X_\tau) \in \cdot\ \} = \lim_{\epsilon \to 0} P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) \in \cdot \mid (X_0, \ldots, X_\tau) \in \Delta_r\};$$

see the proof of Theorem 2. Thus, for small $\epsilon$, given that the system fails, with very high probability, it fails by taking one of the paths having a largest asymptotic probability (i.e., one of the paths in $\Delta_r$). Also, the particular path in $\Delta_r$ that is taken is approximately determined by the conditional distribution of selecting a path in $\Delta_r$, given that paths are only chosen from $\Delta_r$.

**(iii)** Theorem 2 (and Theorem 13 in Section 9) forms the basis of our method for establishing certain properties about derivative estimators. To prove our results, we need to examine the behavior of certain random variables defined over sample paths leading to system failure before regenerating; i.e., paths for which $\tau_F < \tau_0$. Using Theorems 2 and 13, we can show that it is sufficient to consider only the most likely paths to system failures to determine asymptotic expressions for the derivative estimators and their variances; see Section 9 for further details.

**(iv)** As we mentioned in Section 1, several authors have applied the concept of most likely paths to failure in a different manner to study other aspects of highly reliable systems. For example, see Gnedenko and Solovyev (1975), Gertsbakh (1984), and Shahabuddin (1990, Section 2.5.1).

However, all of these authors consider only balanced systems with no failure propagation. We should note, though, that Gnedenko and Solovyev (1975) and Gertsbakh (1984) assume the repair times have general distribution.

**(v)** Many authors have established conditional limit theorems similar to Theorem 2 in the context of queueing systems. For example, Anantharam (1990) proved conditional limit theorems for the case of large delays in GI/G/1 queues. Also, see Asmussen (1982), Anantharam, Heidelberger, and Tsoucas (1990), and Csiszár, Cover, and Choi (1987), among others.

**(vi)** In our proof of Theorem 2, we essentially re-establish Proposition 1(i) but using a method different than the one originally used by Shahabuddin (1991). We include our proof since it incorporates many ideas that we will need later to prove results about the derivative estimators.

**(vii)** We can easily show that Theorem 2 holds for more general models than the one described in Section 2. Specifically, we could have allowed the components to have state-dependent failure rates and the component failure propagation probabilities to depend on $\epsilon$. However, we need the additional structure in Section 2 to obtain results on derivative estimators.

**(viii)** We can calculate the constant $a_0$ in Proposition 11 in terms of the limiting probability measure $P_0$ from Theorem 2. We show how to do this after the proof of Theorem 2 in Section 9.

**(ix)** It is important that we conditioned on the event $\{\tau_F < \tau_0\}$ in Theorem 2. If we had not done this, the limiting distribution of sample paths would not include any paths to system failure. This can be seen from the following reasoning. Consider some path to system failure $(x_0, \ldots, x_n) \in \Delta$. If $n = 1$, then $x_1 \in F$ and $P(\theta(\epsilon), x_0, x_1) = o(1)$ as $\epsilon \to 0$ by Assumption A3. If $n > 1$, then $(x_{n-1}, x_n)$ is a failure transition with $x_{n-1} \neq 0$, and Assumption A2 implies $P(\theta(\epsilon), x_{n-1}, x_n) = o(1)$ as $\epsilon \to 0$. Hence, in both cases, the probability of the sample path leading to a system failure is $P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_n)\} = o(1)$ as $\epsilon \to 0$. Thus, $P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_n)\} \to 0$ as $\epsilon \to 0$, and so in the limit, all paths leading to system failure have probability 0.

To gain a better understanding of Proposition 1 and Theorem 2, we now examine how the results apply to our previous example.

**Example 1 (continued)** The set of paths for which $\tau_F < \tau_0$ is given by

$$\begin{aligned}\Delta &= \{(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle), (\langle 0,0,0\rangle, \langle 0,1,0\rangle, \langle 0,2,0\rangle), (\langle 0,0,0\rangle, \langle 0,0,1\rangle), \\ &\quad (\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 1,1,0\rangle, \langle 2,1,0\rangle), (\langle 0,0,0\rangle, \langle 0,1,0\rangle, \langle 1,1,0\rangle, \langle 2,1,0\rangle), \ldots\}.\end{aligned}$$

Note that

$$P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) = (\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle)\} = (1/2)\epsilon + o(\epsilon)$$

$$
\begin{aligned}
P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (\langle 0,0,0\rangle, \langle 0,1,0\rangle, \langle 0,2,0\rangle)\} &= (1/2)\epsilon + o(\epsilon) \\
P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (\langle 0,0,0\rangle, \langle 0,0,1\rangle)\} &= (1/4)\epsilon^2 + o(\epsilon^2) \\
P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 1,1,0\rangle, \langle 2,1,0\rangle)\} &= \epsilon^2 + o(\epsilon^2) \\
P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (\langle 0,0,0\rangle, \langle 0,1,0\rangle, \langle 1,1,0\rangle, \langle 2,1,0\rangle)\} &= \epsilon^2 + o(\epsilon^2) \\
&\vdots
\end{aligned}
$$

It is easy to see that $(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle)$ and $(\langle 0,0,0\rangle, \langle 0,1,0\rangle, \langle 0,2,0\rangle)$ are the only paths to failure that have probability of the order $\epsilon$. Also, we can show that the sum of the probabilities of all other paths is $o(\epsilon)$. (This is shown in the proof of Theorem 2.) Thus, $\gamma(\theta(\epsilon)) = \epsilon + o(\epsilon)$ and $r = 1$. Also, $\sigma^2(\hat{\gamma}(\theta(\epsilon))) = \epsilon + o(\epsilon)$ and $RE(\hat{\gamma}(\theta(\epsilon))) = \epsilon^{-1/2} + o(\epsilon^{-1/2})$. Finally,

$$
\Delta_1 = \{(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle), (\langle 0,0,0\rangle, \langle 0,1,0\rangle, \langle 0,2,0\rangle)\}
$$

is the set of most likely paths to system failure. ∎

## 4 Estimating Derivatives Using Naive Simulation

Our goal is to estimate via naive simulation the partial derivative of $\eta(\lambda, \mu)$ with respect to the failure rate $\lambda_i$ of component type $i$, where $\eta(\lambda, \mu)$ is defined in (4). (Nakayama 1991 considers the estimation of derivatives with respect to the rarity parameter $\epsilon$ and with respect to the repair rate parameter $\mu$ when $\mu(x,y) = c(x,y)\mu$ for some $c(x,y) \geq 0$.) Applying the ratio rule of differentiation yields

$$
\partial_{\lambda_i}\eta(\lambda,\mu) = \frac{E_{(\lambda,\mu)}[1\{\tau_F < \tau_0\}]\ \partial_{\lambda_i} E_{(\lambda,\mu)}[\alpha(\lambda,\mu)] - E_{(\lambda,\mu)}[\alpha(\lambda,\mu)]\ \partial_{\lambda_i} E_{(\lambda,\mu)}[1\{\tau_F < \tau_0\}]}{E^2_{(\lambda,\mu)}[1\{\tau_F < \tau_0\}]},
$$

where $\alpha(\lambda,\mu) = \min\{\alpha_F(\lambda,\mu), \alpha_0(\lambda,\mu)\}$ and we use the notation $\partial_{\lambda_i} A(\lambda,\mu) \equiv \frac{\partial}{\partial \lambda_i} A(\lambda,\mu)$ for some function $A(\lambda,\mu)$. As previously mentioned, we only focus on estimating the derivative of the performance measure $\gamma(\lambda,\mu) = E_{(\lambda,\mu)}[1\{\tau_F < \tau_0\}]$. (We can evaluate $\partial_{\lambda_i} E_{(\lambda,\mu)}[\alpha(\lambda,\mu)]$ in a similar manner.) To compute $\partial_{\lambda_i}\gamma(\lambda,\mu)$, we would like to interchange the order of the derivative and expectation operators by appealing to the dominated convergence theorem. However, the probability measure used to evaluate the expectation depends on $\lambda_i$, and so the theorem cannot be applied directly.

To rectify the situation, note that

$$
\begin{aligned}
E_\theta[1\{\tau_F < \tau_0\}] &= \int 1\{\tau_F < \tau_0\}(\omega)\ dP_\theta(\omega) \ = \int 1\{\tau_F < \tau_0\}(\omega)\ \frac{dP_\theta}{dP_{\theta_0}}(\omega)\ dP_{\theta_0}(\omega) \\
&= \int 1\{\tau_F < \tau_0\}(\omega)\ L(\theta,\omega)\ dP_{\theta_0}(\omega) \ = \ E_{\theta_0}[1\{\tau_F < \tau_0\}\ L(\theta)], \qquad (8)
\end{aligned}
$$

where $\theta_0 \in \Theta$ is some fixed value of the parameter $\theta$ and $L(\theta,\omega) = \frac{dP_\theta}{dP_{\theta_0}}(\omega)$. $L(\theta)$ is known as the Radon-Nykodym derivative of $P_\theta$ with respect to $P_{\theta_0}$, or simply the "likelihood ratio." (Since

$\theta \in \Theta$, $P_\theta$ and $P_{\theta_0}$ are mutually absolutely continuous, so (8) is valid.) Since $\theta_0$ is fixed, the expectation on the right hand side of (8) has a probability measure which is independent of the parameter $\lambda_i$.

We can now easily differentiate the right hand side of (8), assuming that the derivative and expectation operators can be interchanged. (Glynn 1986 proved the validity of the interchange for DTMCs under regularity conditions which can easily be shown to hold in our context.) Since we are working with DTMCs, $dP_\theta(\omega) = \prod_{k=0}^{\tau(\omega)-1} P(\theta, X_k(\omega), X_{k+1}(\omega))$, and so

$$\partial_{\lambda_i} E_\theta[1\{\tau_F < \tau_0\}] = E_{\theta_0}[1\{\tau_F < \tau_0\}\ \partial_{\lambda_i} L(\theta)],$$

where

$$\partial_{\lambda_i} L(\theta) = \left[\sum_{k=0}^{\tau-1} \frac{\partial_{\lambda_i} P(\theta, X_k, X_{k+1})}{P(\theta_0, X_k, X_{k+1})}\right] L(\theta).$$

The expressions simplify when we evaluate $\partial_{\lambda_i} E_\theta[1\{\tau_F < \tau_0\}]$ at the point $\theta = \theta_0$ since $L(\theta_0) = 1$. This technique is known as the "likelihood ratio" method for computing derivatives; see Glynn (1986) and Reiman and Weiss (1989) for further details.

After computing the partial derivatives, we set $\lambda_j = \lambda_j(\epsilon) \equiv \tilde{\lambda}_j \epsilon^{b_j}$ for all component types $j$, as in (5), and evaluate the partial derivatives at $\theta(\epsilon) = (\lambda(\epsilon), \mu)$. (For notational convenience, we omit the subscript 0 from the parameter $\theta$.) Hence, we are considering $\partial_{\lambda_i}\gamma(\lambda(\epsilon), \mu)$, and

$$\partial_{\lambda_i} L(\lambda(\epsilon), \mu) = \sum_{k=0}^{\tau-1} \frac{\partial_{\lambda_i} P((\lambda(\epsilon), \mu), X_k, X_{k+1})}{P((\lambda(\epsilon), \mu), X_k, X_{k+1})} \tag{9}$$

is the partial derivative with respect to $\lambda_i$ of the likelihood ratio evaluated at $\theta(\epsilon) = (\lambda(\epsilon), \mu)$.

It is important to note that the partial derivatives obtained depend on the way the problem is parameterized. Therefore, a certain amount of care must be taken to ensure that the parameterization used will result in the desired partial derivatives. For example, we could specify a failure rate of a component to be $\lambda = 1/\nu$, where $\nu$ is the mean lifetime of the component. If we then take the derivative of the performance measure with respect to $\nu$, this will result in a different quantity than if we differentiated with respect to $\lambda$. However, the actual (non-derivative) performance measure itself is independent of how the problem is parameterized.

## 4.1 Estimating the Partial Derivatives w.r.t. Component Failure Rates Using Naive Simulation

In this section, we derive asymptotic expressions for the derivatives of the denominator with respect to the $\lambda_i$ and also for the variance of the estimators obtained using naive simulation. To establish our results, we need to make some additional technical assumptions. (The assumptions are not required to prove the results for the non-derivative performance measure estimates.)

**A4** *If* $p(y; x, i) > 0$ *and* $p(y; x, j) > 0$, *then* $b_i = b_j$.

**A5** *If there exists a component type $i$ such that $b_i = b_0$ and $p(y;0,i) > 0$, then there exists another component type $j \neq i$ such that $b_j = b_0$ and $p(y;0,j) \neq p(y;0,i)$.*

Assumption A4 stipulates that if the failure of either of two types of components can trigger a transition from state $x$ to $y$, then the failure rates of the two component types are of the same $\epsilon$-order. This implies that the $Q$-matrix has a certain form: for $x \neq y$,

$$q(\theta(\epsilon), x, y) = \begin{cases} c(x,y)\epsilon^{d(x,y)} & \text{if } y \succ x \\ \mu(x,y) & \text{if } y \prec x \\ 0 & \text{otherwise} \end{cases},$$

where $c(x,y) > 0$, $d(x,y) \geq b_0$ are integer-valued, and $\epsilon > 0$. Hence, the transition rates for failure transitions consist of a single term rather than a sum as in (6). Furthermore, we can determine the exact value of $d(x,y)$ since there must exist some component type $i$ such that $p(y;x,i) > 0$, and so Assumption A4 ensures that $d(x,y) = b_i$ in this case. We use this fact to determine the order of magnitude of derivatives with respect to $\lambda_i$; see the proof of Lemma 12.

Assumption A5 states that if there is some component type $i$ having failure rate of the order $\epsilon^{b_0}$ whose failure can cause a transition from state 0 to state $y$ with some positive probability, then there must be some other component type $j$ also having failure rate of the order $\epsilon^{b_0}$ which causes the same transition with a different probability. This condition is not unreasonable when we are considering large reliability systems. It should be noted that Assumption A5 holds if there exists a component type $j$ such that $b_j = b_0$ and $p(y;0,j) = 0$. Assumption A5 is a technical assumption needed to ensure that there is no cancellation when we compute certain expressions; see the proof of Lemma 12 for further details.

In the situation when there is no failure propagation, then Assumption A4 is automatically satisfied, and Assumption A5 reduces to requiring that there are two different component types which both have failure rates of the order $\epsilon^{b_0}$; i.e., there exists $i$ and $j$ such that $i \neq j$ and $b_i = b_j = b_0$. If we allow for failure propagation but with the restriction that any given failure transition can be triggered by the failure of only one type of component (i.e., for each $(x,y) \in \Gamma$ with $y \succ x$, there is at most one component type $i$ for which $p(y;x,i) > 0$), then again it is easy to verify that Assumption A4 holds and Assumption A5 reduces to requiring that there exist component types $i$ and $j$ such that $i \neq j$ and $b_i = b_j = b_0$.

Let $\hat{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu)$ denote the estimator of $\partial_{\lambda_i}\gamma(\lambda(\epsilon), \mu)$ obtained using naive simulation. The following theorem gives an expression in terms of $\epsilon$ for the partial derivative with respect to the failure rate of component type $i$ and the variance of its estimator when using naive simulation.

**Theorem 3** *Consider a model of any highly reliable Markovian system (as described in Section 2) which satisfies Assumptions A1–A5. For all $\epsilon$ sufficiently small, there exists $\tilde{a}_i \neq 0$, $r_i \geq r$, and $\bar{r}_i \geq r$ (which depend on the model) such that*

**(i)** $\partial_{\lambda_i}\gamma(\lambda(\epsilon), \mu) = E_{\theta(\epsilon)}[1\{\tau_F < \tau_0\}\partial_{\lambda_i}L(\lambda(\epsilon), \mu)] = \tilde{a}_i\epsilon^{\min\{r_i - b_i,\ \bar{r}_i - b_0\}} + o(\epsilon^{\min\{r_i - b_i,\ \bar{r}_i - b_0\}})$, *where $b_i$ is defined in (5).*

*Also, when using naive simulation, there exists $\tilde{a}_i' > 0$ such that*

**(ii)** $\sigma^2(\hat{\partial}_{\lambda_i}\gamma(\lambda(\epsilon),\mu)) \equiv Var_{\theta(\epsilon)}[1\{\tau_F < \tau_0\}\partial_{\lambda_i}L(\lambda(\epsilon),\mu)] = \tilde{a}_i'\epsilon^{\min\{r_i-2b_i,\bar{r}_i-2b_0\}}+o(\epsilon^{\min\{r_i-2b_i,\bar{r}_i-2b_0\}})$.

In general, we cannot say whether $r_i - b_i \le \bar{r}_i - b_0$ or $r_i - b_i > \bar{r}_i - b_0$, or whether $r_i - 2b_i \le \bar{r}_i - 2b_0$ or $r_i - 2b_i > \bar{r}_i - 2b_0$. Also, the expression for $\partial_{\lambda_i}\gamma(\lambda(\epsilon),\mu)$ is independent of whether we use naive simulation or importance sampling. However, the variance of the estimator depends on the simulation method being employed.

The basic idea of the proof of Theorem 3 is as follows. (The complete proof is given in Section 9.) Consider component type $i$, and define

$$\tau_i = \inf\{k > 0 : X_k \succ X_{k-1},\ n_i(X_{k-1})p(X_k; X_{k-1}, i) > 0\}. \tag{10}$$

From (2) we see that $\tau_i$ is the first failure transition along the path $X_0, X_1, \ldots$ which could have been triggered by a failure of a component of type $i$. We first show (see Lemma 12 in Section 9) that the order of magnitude of the summands in (9) depend on whether or not the transition could have been triggered by a failure of component type $i$; i.e., if $n_i(X_k)p(X_{k+1}; X_k, i) > 0$. (We need Assumptions A4 and A5 to show this.) We then decompose the event $\{\tau_F < \tau_0\}$ as

$$\{\tau_F < \tau_0\} = \{\tau_i \le \tau_F < \tau_0\} \cup \{\tau_F < \min\{\tau_i, \tau_0\}\}, \tag{11}$$

where the union is over disjoint events. Note that $\{\tau_i \le \tau_F < \tau_0\}$ is the event that the system fails before returning to the fully operational state and a component of type $i$ may have triggered one of the failures along the path. Also, $\{\tau_F < \min\{\tau_i, \tau_0\}\}$ is the event that the system fails before returning to the fully operational state and none of the failure transitions could have been triggered by a failure of a component of type $i$. Thus, the order of magnitude of $\partial_{\lambda_i}L(\lambda(\epsilon),\mu)$ depends on whether $\tau_i \le \tau_F < \tau_0$ or $\tau_F < \min\{\tau_i, \tau_0\}$ occurs. We analyze these cases separately by writing

$$\begin{aligned} & E_{\theta(\epsilon)}[1\{\tau_F < \tau_0\}\ \partial_{\lambda_i}L(\lambda(\epsilon),\mu)] \\ = \ & E_{\theta(\epsilon)}[\partial_{\lambda_i}L(\lambda(\epsilon),\mu) \mid \tau_i \le \tau_F < \tau_0]\ P_{\theta(\epsilon)}\{\tau_i \le \tau_F < \tau_0\} \\ & + E_{\theta(\epsilon)}[\partial_{\lambda_i}L(\lambda(\epsilon),\mu) \mid \tau_F < \min\{\tau_i, \tau_0\}]\ P_{\theta(\epsilon)}\{\tau_F < \min\{\tau_i, \tau_0\}\}. \end{aligned} \tag{12}$$

(We can decompose the expression for the second moment in a similar manner.) Employing arguments similar to the ones used in the proof of Theorem 2, we can establish that

$$P_{\theta(\epsilon)}\{\tau_i \le \tau_F < \tau_0\} = a_0^i\epsilon^{r_i} + o(\epsilon^{r_i}), \tag{13}$$

$$P_{\theta(\epsilon)}\{\tau_F < \min\{\tau_i, \tau_0\}\} = \bar{a}_0^i\epsilon^{\bar{r}_i} + o(\epsilon^{\bar{r}_i}), \tag{14}$$

where $a_0^i$ and $\bar{a}_0^i$ are are independent of $\epsilon$ and $r_i$ and $\bar{r}_i$ are as defined in Theorem 3. Furthermore, we can prove that both $P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) \in \cdot \mid \tau_i \le \tau_F < \tau_0\}$ and $P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) \in \cdot \mid \tau_F < \min\{\tau_0, \tau_i\}\}$ respectively converge to some limiting probability measures; see Theorem 13 in Section 9. We take advantage of this fact by then expressing each conditional expectation in (12) as a sum of two terms. The first term corresponds to the largest order terms when conditioning on either $\tau_i \le \tau_F < \tau_0$ or $\tau_F < \min\{\tau_i, \tau_0\}$. We use our results on the convergence of the conditional distributions to analyze these quantities (see Lemmas 14 and 15 in Section 9). Essentially, we accomplish this by examining the behavior of the derivative estimator on the most likely paths when $\tau_i \le \tau_F < \tau_0$ or $\tau_F < \min\{\tau_i, \tau_0\}$. The second term consists of the lower order terms, which we show vanishes as $\epsilon \to 0$. Putting all of this together then establishes the validity of Theorem 3. We work out Example 1 using this approach in the end of Section 4.2.

Before continuing, we first need to further analyze the constants $r_i$ and $\bar{r}_i$. Note that $\{\tau_F < \tau_0\} = \cup_{i=1}^{C} \{\tau_i \le \tau_F < \tau_0\}$, which implies

$$\min_{1 \le i \le C} r_i = r, \tag{15}$$

since the number of types of components $C < \infty$. Moreover,

$$\min\{r_i, \bar{r}_i\} = r \tag{16}$$

follows from (11).

## 4.2 Examining the Difficulty of Estimating the Partial Derivatives Using Naive Simulation

Theorem 3 provides expressions in terms of $\epsilon$ for the partial derivatives of $\gamma(\lambda(\epsilon), \mu)$ with respect to the different component failure rates. We also obtained similar expressions for the variances associated with estimating these quantities using naive simulation. Hence, we are now in a position to examine how difficult it is to evaluate the various partial derivatives.

The following corollary shows that when using naive simulation, the relative error of our estimates of the partial derivatives increases without bound. Thus, as in the case of estimating the performance measure, we see that it is also difficult to estimate derivatives by using naive simulation.

**Corollary 4** *Suppose the assumptions of Theorem 3 hold. Then when using naive simulation,* $RE(\hat{\partial}_{\lambda_i} \gamma(\lambda(\epsilon), \mu)) \to \infty$ *as* $\epsilon \to 0$.

**Proof.** First suppose that $r_i - b_i \le \bar{r}_i - b_0$, and so $\partial_{\lambda_i} \gamma(\lambda(\epsilon), \mu) \sim \epsilon^{r_i - b_i}$. Note that $\min\{r_i - 2b_i, \bar{r}_i - 2b_0\} \le r_i - 2b_i$, and so $\sigma^2(\hat{\partial}_{\lambda_i} \gamma(\lambda(\epsilon), \mu)) \sim \epsilon^d$, where $d \le r_i - 2b_i$. This implies that $d/2 - (r_i - b_i) \le (r_i - 2b_i)/2 - (r_i - b_i) = -r_i/2$, and so $RE(\hat{\partial}_{\lambda_i} \gamma(\lambda(\epsilon), \mu)) \sim \epsilon^{d/2 - (r_i - b_i)} \to \infty$ as $\epsilon \to 0$ since $r_i > 0$. Similarly, we can show the result holds when $r_i - b_i > \bar{r}_i - b_0$. ■

The next corollary shows that the partial derivatives of the performance measure with respect to the failure rates of certain types of components are no more difficult to estimate than the performance measure itself. More specifically, consider component type $i$. Suppose either a failure of a component of type $i$ can trigger some failure transition on one of the most likely paths to system failure (i.e., $r_i = r$) or components of type $i$ have one of the largest failure rates (i.e., $b_i = b_0$), where $r_i$ and $b_i$ are defined in (13) and (5), respectively. Then the following result shows that we can estimate the partial derivative of $\gamma(\lambda(\epsilon), \mu)$ with respect to $\lambda_i$ as accurately as we can estimate the performance measure itself when naive simulation is employed.

**Corollary 5** *Suppose the assumptions of Theorem 3 hold. Also, suppose either $r_i = r$ or $b_i = b_0$. Then, when using naive simulation, $RE(\hat{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu))/RE(\hat{\gamma}(\lambda(\epsilon), \mu))$ remains bounded as $\epsilon \to 0$.*

**Proof.** First suppose $r_i = r$. Since $\bar{r}_i \geq r$ by (16) and $b_i \geq b_0$, we have $r - pb_i \leq \bar{r}_i - pb_0$ for $p \geq 1$. Thus, $\partial_{\lambda_i}\gamma(\lambda(\epsilon), \mu) \sim \epsilon^{r-b_i}$ and $\sigma^2(\hat{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu)) \sim \epsilon^{r-2b_i}$, which implies $RE(\hat{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu)) \sim \epsilon^{-r/2}$ when $r_i = r$.

Now suppose that $b_i = b_0$. We may assume that $r_i > r$, and so $\bar{r}_i = r$ by (16). Hence, $r_i - b_i = r_i - b_0 > r - b_0 = \bar{r}_i - b_0$ and $r_i - 2b_i = r_i - 2b_0 > r - 2b_0 = \bar{r}_i - 2b_0$, which implies $\partial_{\lambda_i}\gamma(\lambda(\epsilon), \mu) \sim \epsilon^{r-b_0}$ and $\sigma^2(\hat{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu)) \sim \epsilon^{r-2b_0}$. Thus, we again have that $RE(\hat{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu)) \sim \epsilon^{-r/2}$ when $b_i = b_0$.

Finally, by Proposition 1, $RE(\hat{\gamma}(\lambda(\epsilon), \mu) \sim \epsilon^{-r/2}$, from which the result follows. ∎

Because of the importance of the Corollary 5, we now develop a condition which is equivalent to the hypothesis of Corollary 5. To do this, we first define the *sensitivity* of $\gamma(\lambda, \mu)$ with respect to $\lambda_i$, evaluated at $\theta(\epsilon) = (\lambda(\epsilon), \mu)$ with $\mu$ fixed, to be

$$s_i(\epsilon) = \lambda_i(\epsilon) \cdot \partial_{\lambda_i}\gamma(\lambda(\epsilon), \mu).$$

Sensitivities measure the effects on the overall system performance of relative changes in the value of a parameter. Based on this interpretation of sensitivities, the derivatives corresponding to the largest sensitivities are the most "important" ones.

To make this more rigorous, we say that the sensitivity with respect to $\lambda_i$ has the *largest asymptotic magnitude* if for all other component types $j \neq i$, $\liminf_{\epsilon \to 0} |s_i(\epsilon)/s_j(\epsilon)| > 0$. The next corollary shows that the sensitivities having the largest asymptotic magnitude correspond to components that either can trigger a failure transition by failing on one of the most likely paths to failure or have one of the largest failure rates. Consequently, we can estimate the partial derivatives corresponding to the largest sensitivities as accurately as the performance measure itself.

**Corollary 6** *Suppose the assumptions of Theorem 3 hold, and consider component type $i$. Then the sensitivity with respect to $\lambda_i$ has the largest asymptotic magnitude if and only if either $r_i = r$ or $b_i = b_0$. In this case, $\lambda_i \cdot \partial_{\lambda_i}\gamma(\lambda(\epsilon), \mu) = c_i\epsilon^r + o(\epsilon^r)$, where $c_i \neq 0$. Thus, it follows from Corollary 5 that when using naive simulation, $RE(\hat{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu))/RE(\hat{\gamma}(\lambda(\epsilon), \mu))$ remains bounded as $\epsilon \to 0$ for sensitivities with respect to $\lambda_i$ having a largest asymptotic magnitude.*

**Proof.** Using Theorem 3 and (5), $s_j(\epsilon) \sim \epsilon^{\min\{r_j,\ \bar{r}_j + b_j - b_0\}}$ for all component types $j$. Note that $r \leq r_j$ for all component types $j$ by (16). Since $b_j \geq b_0$ for all $j$ and $\bar{r}_i \geq r$ by (16), we have $r \leq \bar{r}_j + b_j - b_0$. Thus,

$$r \leq \min\{r_j,\ \bar{r}_j + b_j - b_0\} \tag{17}$$

for all component types $j$. Furthermore, if $r_j = r$, then $\min\{r_j,\ \bar{r}_j + b_j - b_0\} = r$. Hence, to prove our result, we need to show that $\min\{r_i,\ \bar{r}_i + b_i - b_0\} = r$ if and only if either $r_i = r$ or $b_i = b_0$.

If $r_i = r$, then $\min\{r_i,\ \bar{r}_i + b_i - b_0\} = r$ by (17). Now suppose $b_i = b_0$. We may assume that $r_i > r$, and so $\bar{r}_i = r$ by (16). Hence, $\bar{r}_i + b_i - b_0 = r$, giving us $\min\{r_i,\ \bar{r}_i + b_i - b_0\} = r$ by (17).

Now suppose $\min\{r_i,\ \bar{r}_i + b_i - b_0\} = r$. If $r_i$ is the minimum, then $r_i = r$. If $\bar{r}_i + b_i - b_0$ is the minimum, then $\bar{r}_i + b_i - b_0 = r$. We may assume $r_i > r$, and so $\bar{r}_i = r$ by (16). Thus, $b_i = b_0$. Hence, $\min\{r_i,\ \bar{r}_i + b_i - b_0\} = r$ implies either $r_i = r$ or $b_i = b_0$, which completes the proof. ■

We now want to compare the difficulties associated with estimating the different partial derivatives. Before doing so, we make another definition. We say that the sensitivity with respect to the failure rate of component type $i$ is *asymptotically strictly larger* than the sensitivity with respect to the failure rate of component type $j$ if $\liminf_{\epsilon \to 0} |s_i(\epsilon)/s_j(\epsilon)| = +\infty$. The following corollary shows that the partial derivatives with respect to the failure rates of the component types corresponding to the largest sensitivities are no more difficult (and possibly easier) to estimate than the other partial derivatives.

**Corollary 7** *Suppose the assumptions of Theorem 3 hold, and consider any component types $i$ and $j \neq i$. Assume the following:*

**(i)** *$s_i(\epsilon)$ has the largest asymptotic magnitude,*

**(ii)** *$s_i(\epsilon)$ is asymptotically strictly larger than $s_j(\epsilon)$.*

*Then, when using naive simulation, $RE(\hat{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu))/RE(\hat{\partial}_{\lambda_j}\gamma(\lambda(\epsilon), \mu))$ remains bounded (and possibly goes to 0) as $\epsilon \to 0$.*

**Proof.** The first assumption and Corollaries 5 and 6 imply that $RE(\hat{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu)) \sim \epsilon^{-r/2}$. Thus, to prove our result, we must show that $RE(\hat{\partial}_{\lambda_j}\gamma(\lambda(\epsilon), \mu))$ has $\epsilon$-order no greater than $-r/2$. By the two assumptions and Corollary 6, we must have $r_j > r$ and $b_j > b_0$. Hence, $\bar{r}_j = r$ by (16). There are two cases to consider: $\min\{r_j - b_j,\ \bar{r}_j - b_0\} = r_j - b_j$ and $\min\{r_j - b_j,\ \bar{r}_j - b_0\} = \bar{r}_j - b_0$.

First, suppose $\min\{r_j - b_j,\ \bar{r}_j - b_0\} = r_j - b_j$, and so $r_j - b_j \le \bar{r}_j - b_0 = r - b_0$. Then, $b_j > b_0$ implies that $r_j - 2b_j < \bar{r}_j - 2b_0$, giving us $\partial_{\lambda_j}\gamma(\lambda(\epsilon), \mu) \sim \epsilon^{r_j - b_j}$ and $\sigma^2(\hat{\partial}_{\lambda_j}\gamma(\lambda(\epsilon), \mu)) \sim \epsilon^{r_j - 2b_j}$. Hence, $RE(\hat{\partial}_{\lambda_j}\gamma(\lambda(\epsilon), \mu)) \sim \epsilon^{-r_j/2}$. Since $r_j > r$, $RE(\hat{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu))/RE(\hat{\partial}_{\lambda_j}\gamma(\lambda(\epsilon), \mu)) \to 0$ as $\epsilon \to 0$.

Now suppose $\min\{r_j - b_j,\ \bar{r}_j - b_0\} = \bar{r}_j - b_0 = r - b_0$. Observe that $\min\{r_j - 2b_j,\ \bar{r}_j - 2b_0\} \le \bar{r}_j - 2b_0 = r - 2b_0$. Therefore, $\partial_{\lambda_j}\gamma(\lambda(\epsilon), \mu) \sim \epsilon^{r - b_0}$ and $\sigma^2(\hat{\partial}_{\lambda_j}\gamma(\lambda(\epsilon), \mu)) \sim \epsilon^{d_1}$, where $d_1 \le r - 2b_0$. Hence, $RE(\hat{\partial}_{\lambda_j}\gamma(\lambda(\epsilon), \mu)) \sim \epsilon^{d_2}$, where $d_2 \le -r/2$, and so $RE(\hat{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu))/RE(\hat{\partial}_{\lambda_j}\gamma(\lambda(\epsilon), \mu))$ remains bounded (and possibly goes to 0) as $\epsilon \to 0$. ■

Now we examine how Corollaries 5–7 manifest themselves in our previous example.

**Example 1 (continued)** Consider component type 1. We first derive asymptotic expressions for the derivative with respect to $\lambda_1$ and the variance of its naive estimator. Note that $\tau_1 \le \tau_F < \tau_0$

for the path $(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle)$, and

$$P_{\theta(\epsilon)}\{(X_0,\ldots,X_{\tau_F}) = (\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle)\} = (1/2)\epsilon + o(\epsilon).$$

We can show that the sum of the probabilities of all other paths that satisfy $\tau_1 \le \tau_F < \tau_0$ is $o(\epsilon)$. Thus, $P_{\theta(\epsilon)}\{\tau_1 \le \tau_F < \tau_0\} = \epsilon/2 + o(\epsilon)$, and $r_1 = 1$. Since the set of most likely paths for which $\tau_1 \le \tau_F < \tau_0$ consists of only one path, we only have to evaluate $\partial_{\lambda_1} L(\lambda(\epsilon), \mu)$ on the one path to determine the asymptotic expression for $E_{\theta(\epsilon)}[\partial_{\lambda_1} L(\lambda(\epsilon), \mu) \mid \tau_i \le \tau_F < \tau_0]$. Observe that

$$\begin{aligned}
&\partial_{\lambda_1} L(\lambda(\epsilon), \mu)(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle) \\
&= \frac{2\lambda_2(\epsilon) + \lambda_3(\epsilon)}{\lambda_1(\epsilon)(2\lambda_1(\epsilon) + 2\lambda_2(\epsilon) + \lambda_3(\epsilon))} + \frac{1 + 2\lambda_2(\epsilon) + \lambda_3(\epsilon)}{\lambda_1(\epsilon)(1 + \lambda_1(\epsilon) + 2\lambda_2(\epsilon) + \lambda_3(\epsilon))} \\
&= (3/2)\epsilon^{-1} + o(\epsilon^{-1}),
\end{aligned}$$

and so $E_{\theta(\epsilon)}[\partial_{\lambda_1} L(\lambda(\epsilon), \mu) \mid \tau_1 \le \tau_F < \tau_0] = (3/2)\epsilon^{-1} + o(\epsilon^{-1})$, and $E_{\theta(\epsilon)}[(\partial_{\lambda_1} L(\lambda(\epsilon), \mu))^2 \mid \tau_1 \le \tau_F < \tau_0] = (9/4)\epsilon^{-2} + o(\epsilon^{-2})$. Similarly, by considering the path $(\langle 0,0,0\rangle, \langle 0,1,0\rangle, \langle 0,2,0\rangle)$, we can show that $P_{\theta(\epsilon)}\{\tau_F < \min\{\tau_1, \tau_0\}\} = (1/2)\epsilon + o(\epsilon)$, and $\bar{r}_1 = 1$. Also, we have $E_{\theta(\epsilon)}[\partial_{\lambda_1} L(\lambda(\epsilon), \mu) \mid \tau_F < \min\{\tau_1, \tau_0\}] = -(1/2)\epsilon^{-1} + o(\epsilon^{-1})$ and $E_{\theta(\epsilon)}[(\partial_{\lambda_1} L(\lambda(\epsilon), \mu))^2 \mid \tau_F < \min\{\tau_1, \tau_0\}] = (1/4)\epsilon^{-2} + o(\epsilon^{-2})$. Using (12), we get

$$\begin{aligned}
\partial_{\lambda_1} \gamma(\lambda(\epsilon), \mu) &= \left((3/2)\epsilon^{-1} + o(\epsilon^{-1})\right)((1/2)\epsilon + o(\epsilon)) + \left(-(1/2)\epsilon^{-1} + o(\epsilon^{-1})\right)((1/2)\epsilon + o(\epsilon)) \\
&= 1/2 + o(1)
\end{aligned}$$

and

$$\begin{aligned}
&E_{\theta(\epsilon)}[1\{\tau_F < \tau_0\}(\partial_{\lambda_1} L(\lambda(\epsilon), \mu))^2] \\
&= ((9/4)\epsilon^{-2} + o(\epsilon^{-2}))((1/2)\epsilon + o(\epsilon)) + ((1/4)\epsilon^{-2} + o(\epsilon^{-2}))((1/2)\epsilon + o(\epsilon)) \\
&= (5/4)\epsilon^{-1} + o(\epsilon^{-1})
\end{aligned}$$

Hence, $\sigma^2(\hat{\partial}_{\lambda_1} \gamma(\lambda(\epsilon), \mu)) = (5/4)\epsilon^{-1} + o(\epsilon^{-1})$ and $RE(\hat{\partial}_{\lambda_1} \gamma(\lambda(\epsilon), \mu)) = \sqrt{5}\epsilon^{-1/2} + o(\epsilon^{-1/2})$. Also, we can show that $r_2 = \bar{r}_2 = \bar{r}_3 = 1$, $r_3 = 2$, and

$$\begin{aligned}
\partial_{\lambda_2} \gamma(\lambda(\epsilon), \mu) &= 1/2 + o(1) \\
\sigma^2(\hat{\partial}_{\lambda_2} \gamma(\lambda(\epsilon), \mu)) &= (5/4)\epsilon^{-1} + o(\epsilon^{-1}) \\
\partial_{\lambda_3} \gamma(\lambda(\epsilon), \mu) &= (1/4)\epsilon^{-1} + o(\epsilon^{-1}) \\
\sigma^2(\hat{\partial}_{\lambda_3} \gamma(\lambda(\epsilon), \mu)) &= (1/4)\epsilon^{-4} + o(\epsilon^{-4}).
\end{aligned}$$

It then follows that $RE(\hat{\partial}_{\lambda_2} \gamma(\lambda(\epsilon), \mu)) = \sqrt{5}\epsilon^{-1/2} + o(\epsilon^{-1/2})$ and $RE(\hat{\partial}_{\lambda_3} \gamma(\lambda(\epsilon), \mu)) = 2\epsilon^{-1} + o(\epsilon^{-1})$. Note that $s_1(\epsilon) = s_2(\epsilon) = \epsilon/2 + o(\epsilon)$ and $s_3(\epsilon) = \epsilon^2/4 + o(\epsilon^2)$. Therefore, $s_1(\epsilon)$ and $s_2(\epsilon)$ have the largest asymptotic magnitudes, and they are asymptotically strictly larger than $s_3(\epsilon)$. We previously established that $RE(\hat{\gamma}(\lambda(\epsilon), \mu)) = \epsilon^{-1/2} + o(\epsilon^{-1/2})$, and so we can estimate the derivative with respect to $\lambda_1$ or $\lambda_2$ with the same relative accuracy as the performance measure, which agrees with Corollaries 5 and 6. Also, the partial derivatives with respect to $\lambda_1$ or $\lambda_2$ are strictly easier to estimate than the partial with respect to $\lambda_3$, as $\epsilon \to 0$, in accordance with Corollary 7. ■

# 5 Estimating the Performance Measure Using Importance Sampling

In the previous sections, we showed that we can estimate certain derivatives of the denominator of the ratio expression for the MTTF as accurately as we can estimate the denominator itself. However, the relative errors of all of these estimates diverge to infinity in the limit when using naive simulation. Thus, variance reduction techniques must be utilized to obtain efficient estimators. We now describe an importance sampling scheme known as the "balanced failure biasing" method, which was proposed by Shahabuddin (1991). If balanced failure biasing is used, Shahabuddin showed that we can obtain stable estimates of $\gamma(\theta(\epsilon))$, independently of the rareness of system failures (under the original measure); i.e., its relative error remains bounded as the failure rates go to zero and the repair rates remain fixed. In the following section, we show that balanced failure biasing can also be applied to the estimation of the derivatives of $\gamma(\theta(\epsilon))$ to obtain similar results. As in Section 3, we do not require Assumptions A4 and A5 to hold in this section.

The basic idea behind importance sampling is as follows. Our goal is to estimate $\gamma(\theta(\epsilon)) = E_{\theta(\epsilon)}[1\{\tau_F < \tau_0\}]$. (The numerator of the ratio expression for the MTTF in (4) can be efficiently estimated using naive simulation, so we do not use importance sampling to estimate this quantity.) Let $\Omega$ be the set of all paths of the embedded DTMC $X$ starting in state 0, and let $\mathcal{F}$ be the corresponding $\sigma$-field. Recall that our original probability measure on $\mathcal{F}$ was $P_{\theta(\epsilon)}$, and let $P_*$ be some other probability measure defined on $\mathcal{F}$ such that $P_{\theta(\epsilon)}(d\omega) = 0$ implies $P_*(d\omega) = 0$ for all $\omega \in \Omega$ for which $1\{\tau_F < \tau_0\}(\omega) = 1$. Finally, let $E_*$ denote the expectation operator induced by $P_*$. Then,

$$\begin{aligned} E_{\theta(\epsilon)}[1\{\tau_F < \tau_0\}] &= \int 1\{\tau_F < \tau_0\}(\omega)\ dP_{\theta(\epsilon)}(\omega) = \int 1\{\tau_F < \tau_0\}(\omega)\ \frac{dP_{\theta(\epsilon)}}{dP_*}(\omega)\ dP_*(\omega) \\ &= E_*[1\{\tau_F < \tau_0\}\ L_*(\theta(\epsilon))], \end{aligned}$$

where $L_*(\theta(\epsilon), \omega) \equiv \frac{dP_{\theta(\epsilon)}}{dP_*}(\omega)$ is the Radon-Nykodym derivative of $P_{\theta(\epsilon)}$ with respect to $P_*$, or simply the likelihood ratio. Hence, to evaluate $\gamma(\theta(\epsilon))$, we now compute the expectation of $1\{\tau_F < \tau_0\}\ L_*(\theta(\epsilon))$ with respect to the probability measure $P_*$. This transformation is known as a "change of measure." (The reader is referred to Glynn and Iglehart 1989 for a more detailed description of importance sampling.) By properly selecting $P_*$, we can obtain large reductions in the variance of our estimators. Therefore, the main thrust of the research in importance sampling is in determining how $P_*$ should be chosen.

Shahabuddin (1991) proposed the balanced failure biasing algorithm as a method of implementing importance sampling in simulations of highly reliable Markovian systems. The general intuition behind this method is to alter the transition probabilities of the embedded DTMC so as to increase the probability of the event $\{\tau_F < \tau_0\}$. Thus, $\{\tau_F < \tau_0\}$ will no longer be a "rare event" under the importance sampling distribution, making its probability easier to estimate.

A description of the balanced failure biasing method is as follows. First, consider state 0, from which there are only failure transitions possible. Under the change of measure, we make all transitions from this state occur with equal probability. For example, if from state 0 there are only $m$ failure transitions possible, then each of the failure transitions would be assigned probability $1/m$ under balanced failure biasing. Now, consider any state from which there are only repair transitions possible. Balanced failure biasing does not change the transition probabilities of the (repair) transitions from this state. For all of the other states not yet considered, there are both failure transitions and repair transitions possible. For these states, under the original measure, the total probability of taking a repair transition typically is much greater than the total probability of taking a failure transition. To increase the probability of a system failure under the new measure, we inflate the total probability of a failure transition to $p_1$, where $0 \ll p_1 < 1$; i.e. $p_1$ is independent of $\epsilon$. The individual failure transitions are made to occur with equal probability. The total probability of a repair transition from the state is reduced to $1-p_1$, and the ratio of the individual repair transition probabilities remains the same as when $P(\theta(\epsilon))$ is used. A more detailed description of balanced failure biasing is given in Shahabuddin (1991).

In general, we have $p_1 = \bar{p}_1 + o(1)$, although typically in practice, $p_1$ is selected independently of $\epsilon$. Extensive empirical work suggests that we should select $0.5 \leq p_1 \leq 0.9$; see Goyal et al. (1992). (Shahabuddin 1991 also developed some heuristics for selecting $p_1$.) Hence, under the importance sampling measure, all transitions $(x, y) \in \Gamma$ have probability of the order 1; i.e.,

$$P_*(\theta(\epsilon), x, y) = p_*(x, y) + o(1), \tag{18}$$

where $p_*(x, y) > 0$ is independent of $\epsilon$. This implies that all sample paths $(x_0, \ldots, x_n) \in \Delta$ also have probability of order 1 under the importance sampling distribution, and so the event of a system failure occurring before a regeneration is no longer rare under the probability measure corresponding to balanced failure biasing.

Let $L_*(\theta(\epsilon))$ denote the likelihood ratio corresponding to balanced failure biasing; i.e.,

$$L_*(\theta(\epsilon)) = \prod_{k=0}^{\tau-1} \frac{P(\theta(\epsilon), X_k, X_{k+1})}{P_*(\theta(\epsilon), X_k, X_{k+1})}. \tag{19}$$

Also, let $\tilde{\gamma}(\theta(\epsilon))$ denote the estimator of $\gamma(\theta(\epsilon))$ obtained by using balanced failure biasing, and let $\sigma_*^2(\tilde{\gamma}(\theta(\epsilon)))$ denote its variance. Then, Shahabuddin (1991) showed the following results:

**Proposition 8 (Shahabuddin)** *Consider a model of any highly reliable Markovian system (as described in Section 2) which satisfies Assumptions A1–A3. When using balanced failure biasing, there exists $a_1 > 0$ (which depends on the system) such that*

**(i)** $\sigma_*^2(\tilde{\gamma}(\theta(\epsilon))) = Var_*[1\{\tau_F < \tau_0\} L_*(\theta(\epsilon))] = a_1 \epsilon^{2r} + o(\epsilon^{2r})$,

**(ii)** $RE(\tilde{\gamma}(\theta(\epsilon))) = \frac{\sigma_*(\tilde{\gamma}(\theta(\epsilon)))}{\gamma(\theta(\epsilon))} = \frac{\sqrt{a_1 \epsilon^{2r} + o(\epsilon^{2r})}}{(a_0 \epsilon^r + o(\epsilon^r))} = \frac{\sqrt{a_1 + o(1)}}{a_0 + o(1)}$

*for all $\epsilon$ sufficiently small, where $r$ is as defined in (22).*

Proposition 11 established the asymptotic expression for $\gamma(\theta(\epsilon))$, which is independent of the simulation method used to estimate it. However, the variance of the estimate of $\gamma(\theta(\epsilon))$ depends on the simulation method employed, and an expression for the variance associated with estimating $\gamma(\theta(\epsilon))$ using balanced failure biasing is given in Proposition 81. Proposition 82 shows that by using balanced failure biasing, the relative error of the performance measure estimator remains bounded as the failure rates go to zero and the repair rates remain fixed. Hence, we can obtain good estimates of $\gamma(\theta(\epsilon))$ by using importance sampling, no matter how rarely system failures occur.

Now let us reconsider our previous example.

**Example 1 (continued)** Figure 2 is the transition probability diagram of the system under balanced failure biasing. Note that

$$\begin{aligned} E_*[1\{\tau_F < \tau_0\}(L_*(\theta(\epsilon)))^2] &= E_{\theta(\epsilon)}[1\{\tau_F < \tau_0\}L_*(\theta(\epsilon))] \\ = \sum_{\substack{(x_0,\ldots,x_n)\in\Delta \\ n\geq 1}} & L_*(\theta(\epsilon))(x_0,\ldots,x_n)P_{\theta(\epsilon)}\{(X_0,\ldots,X_{\tau_F}) = (x_0,\ldots,x_n)\}. \end{aligned}$$

For the paths $(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle)$ and $(\langle 0,0,0\rangle, \langle 0,1,0\rangle, \langle 0,2,0\rangle)$, we have that

$$L_*(\theta(\epsilon))(x_0,\ldots,x_n)P_{\theta(\epsilon)}\{(X_0,\ldots,X_{\tau_F}) = (x_0,\ldots,x_n)\} = \frac{9}{p_1}\epsilon^2 + o(\epsilon^2).$$

We can show that all other paths have $L_*(\theta(\epsilon))(x_0,\ldots,x_n)P_{\theta(\epsilon)}\{(X_0,\ldots,X_{\tau_F}) = (x_0,\ldots,x_n)\} = o(\epsilon^2)$ and that their sum over all of these paths is $o(\epsilon)$. Thus,

$$\sigma_*^2(\tilde{\gamma}(\theta(\epsilon))) = \left(\frac{9}{p_1} - 1\right)\epsilon^2 + o(\epsilon^2)$$

and $RE(\tilde{\gamma}(\theta(\epsilon))) = \sqrt{\frac{9}{p_1} - 1} + o(1)$. ■

# 6 Estimating Derivatives Using Importance Sampling

We now show that balanced failure biasing can also be applied to the estimation of the partial derivatives of $\gamma(\lambda(\epsilon), \mu)$ to obtain large reductions in the variances. As in Section 4, we require Assumptions A4 and A5 to hold as well as A1–A3.

## 6.1 Estimating the Partial Derivatives w.r.t. Component Failure Rates Using Importance Sampling

In Section 4.1 we denoted the partial derivative of $\gamma(\lambda(\epsilon), \mu)$ with respect to $\lambda_i$, evaluated at the parameter $\theta(\epsilon) = (\lambda(\epsilon), \mu)$, by $\partial_{\lambda_i}\gamma(\lambda(\epsilon), \mu) \equiv \frac{\partial}{\partial\lambda_i}\gamma(\lambda(\epsilon), \mu) = E_{(\lambda(\epsilon),\mu)}[1\{\tau_F < \tau_0\}\ \partial_{\lambda_i}L(\lambda(\epsilon), \mu)]$, where $\partial_{\lambda_i}L(\lambda(\epsilon), \mu)$ is defined in (9). Using the change of measure corresponding to balanced failure biasing, we obtain

$$\partial_{\lambda_i}\gamma(\lambda(\epsilon), \mu) = E_*[1\{\tau_F < \tau_0\}\ (\partial_{\lambda_i}L(\lambda(\epsilon), \mu))\ L_*(\lambda(\epsilon), \mu)],$$

where $L_*(\lambda(\epsilon), \mu)$ is defined in (19).

Let $\tilde{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu)$ denote the estimator of $\partial_{\lambda_i}\gamma(\lambda(\epsilon), \mu)$ obtained by using balanced failure biasing. The following theorem gives an expression in terms of $\epsilon$ for the variance of this estimator; see Section 9 for the proof.

**Theorem 9** *Consider a model of any highly reliable Markovian system (as described in Section 2) which satisfies Assumptions A1–A5. Then when using balanced failure biasing, there exists $\tilde{a}_i'(*) > 0$ such that*

$$\begin{aligned}\sigma_*^2(\tilde{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu) &\equiv Var_*[1\{\tau_F < \tau_0\}\ (\partial_{\lambda_i} L(\lambda(\epsilon), \mu))\ L_*(\lambda(\epsilon), \mu)] \\ &= \tilde{a}_i'(*)\epsilon^{\min\{2r_i - 2b_i,\ 2\bar{r}_i - 2b_0\}} + o(\epsilon^{\min\{2r_i - 2b_i,\ 2\bar{r}_i - 2b_0\}})\end{aligned}$$

*for all $\epsilon$ sufficiently small.*

Our basic approach for proving Theorem 9 is as follows. Note that

$$E_*[1\{\tau_F < \tau_0\}\ (\partial_{\lambda_i} L(\lambda(\epsilon), \mu))^2\ (L_*(\lambda(\epsilon), \mu))^2] = E_{\theta(\epsilon)}[1\{\tau_F < \tau_0\}\ (\partial_{\lambda_i} L(\lambda(\epsilon), \mu))^2\ L_*(\lambda(\epsilon), \mu)],$$

and so we can analyze the second moment of the balanced failure biasing estimator using the same method we employ to examine the second moment of the naive simulation derivative estimator described in Section 4.1. More specifically, we express

$$\begin{aligned}&E_*[1\{\tau_F < \tau_0\}\ (\partial_{\lambda_i} L(\lambda(\epsilon), \mu))^2\ (L_*(\lambda(\epsilon), \mu))^2] \\ &= E_{\theta(\epsilon)}[(\partial_{\lambda_i} L(\lambda(\epsilon), \mu))^2\ L_*(\lambda(\epsilon), \mu) \mid \tau_i \le \tau_F < \tau_0]\ P_{\theta(\epsilon)}\{\tau_i \le \tau_F < \tau_0\} \\ &\quad E_{\theta(\epsilon)}[(\partial_{\lambda_i} L(\lambda(\epsilon), \mu))^2\ L_*(\lambda(\epsilon), \mu) \mid \tau_F < \min\{\tau_i, \tau_0\}]\ P_{\theta(\epsilon)}\{\tau_F < \min\{\tau_i, \tau_0\}\}. \qquad (20)\end{aligned}$$

After obtaining bounds for the likelihood ratio, we can establish asymptotic expressions for the conditional expectations using the technique employed to prove Theorem 3. We can also prove Proposition 8 in a similar manner.

## 6.2 Examining the Difficulty of Estimating the Partial Derivatives Using Importance Sampling

Theorem 9 provides an expression in terms of $\epsilon$ for the variance obtained using balanced failure biasing of the estimator of the partial derivative of $\gamma(\lambda(\epsilon), \mu)$ with respect to various component failure rates. Hence, we are now in a position to examine the difficulties of estimating the different partial derivatives.

The following corollary shows that when using balanced failure biasing, we can obtain stable estimates of the partial derivative of $\gamma(\lambda(\epsilon), \mu)$ with respect to the failure rate of *any* component type.

**Corollary 10** *Suppose the assumptions of Theorem 9 hold, and consider any component type $i$. Then, when using balanced failure biasing, $RE(\tilde{\partial}_{\lambda_i}\gamma(\lambda(\epsilon), \mu))$ remains bounded as $\epsilon \to 0$.*

**Proof.** If $r_i - b_i \leq \bar{r}_i - b_0$, then $2r_i - 2b_i \leq 2\bar{r}_i - 2b_0$, implying $\partial_{\lambda_i}\gamma(\lambda(\epsilon),\mu) \sim \epsilon^{r_i - b_i}$ and $\sigma_*^2(\tilde{\partial}_{\lambda_i}\gamma(\lambda(\epsilon),\mu)) \sim \epsilon^{2r_i - 2b_i}$. Hence, $RE(\tilde{\partial}_{\lambda_i}\gamma(\lambda(\epsilon),\mu))$ is of order 1 in this case. Similarly, we can show the result holds when $r_i - b_i > \bar{r}_i - b_0$, which completes the proof. ∎

In the next corollary, which is a direct consequence of the previous result, it is shown that when balanced failure biasing is used, we can estimate *all* partial derivatives of $\gamma(\lambda(\epsilon),\mu)$ with equal accuracy. This is in sharp contrast to the situation that arose when using naive simulation, in which certain derivatives were easier to estimate than others (see Corollary 7).

**Corollary 11** *Suppose the assumptions of Theorem 9 hold, and consider any component types i and j. Then, when using balanced failure biasing,* $RE(\tilde{\partial}_{\lambda_i}\gamma(\lambda(\epsilon),\mu))/RE(\tilde{\partial}_{\lambda_j}\gamma(\lambda(\epsilon),\mu))$ *remains bounded as* $\epsilon \to 0$.

Now let us reconsider our previous example.

**Example 1 (continued)** By considering the most likely paths when $\tau_i \leq \tau_F < \tau_0$ and $\tau_F < \min\{\tau_i, \tau_0\}$ for $i = 1, 2, 3$, we can show that

$$\begin{aligned}
\sigma_*^2(\tilde{\partial}_{\lambda_1}\gamma(\lambda(\epsilon),\mu)) &= \left(\frac{45}{8p_1} - \frac{1}{4}\right) + o(1) \\
\sigma_*^2(\tilde{\partial}_{\lambda_2}\gamma(\lambda(\epsilon),\mu)) &= \left(\frac{45}{8p_1} - \frac{1}{4}\right) + o(1) \\
\sigma_*^2(\tilde{\partial}_{\lambda_3}\gamma(\lambda(\epsilon),\mu)) &= (1/8)\epsilon^{-2} + o(\epsilon^{-2}),
\end{aligned}$$

and so $RE(\tilde{\partial}_{\lambda_1}\gamma(\lambda(\epsilon),\mu)) = RE(\tilde{\partial}_{\lambda_2}\gamma(\lambda(\epsilon),\mu)) = 2\sqrt{\frac{45}{8p_1} - \frac{1}{4}} + o(1)$ and $RE(\tilde{\partial}_{\lambda_3}\gamma(\lambda(\epsilon),\mu)) = \sqrt{2} + o(1)$. Recalling that $RE(\tilde{\gamma}(\theta(\epsilon))) = \sqrt{\frac{9}{p_1} - 1} + o(1)$, we see that each of the derivatives can be estimated with the same asymptotic relative accuracy as the performance measure, which is what we proved in Corollary 10. Also, we can estimate all derivatives with equal asymptotic relative accuracy, which agrees with Corollary 11. ∎

## 7 Empirical Results

In this section we present some results from simulating a large computing system. Goyal et al. (1992) and Nakayama, Goyal, and Glynn (1991) previously studied the same system with different failure rates for the components. The system consists of two types of processors, $A$ and $B$, each having a redundancy of two; two sets of disk controllers, each having a redundancy of two; and six disk clusters consisting of four disks each. The data is replicated in each disk cluster in such a way that one of the disks can fail and all of the data in the cluster is still accessible. The processors of one type access the data through one set of disk controllers, and the other type of processors access data through the other set. When a processor of one type fails, it causes a processor of the other type to fail simultaneously with probability 0.01. The system fails when either both processors of

| Performance Measure | Numerical Result | Naive Simulation | Balanced Failure Biasing |
|---|---|---|---|
| MTTF | $0.1609 \times 10^6$ | $0.1588 \times 10^6$ $\pm$ 25.8% | $0.1595 \times 10^6$ $\pm$ 6.2% |
| $pAfr \cdot \frac{\partial}{\partial pAfr}$MTTF | $-.7174 \times 10^5$ | $-.6957 \times 10^5$ $\pm$ 54.9% | $-.7006 \times 10^5$ $\pm$ 13.4% |
| $d1fr \cdot \frac{\partial}{\partial d1fr}$MTTF | $-.2772 \times 10^3$ | $-.7117 \times 10^3$ $\pm$ 569.7% | $-.2735 \times 10^3$ $\pm$ 34.6% |

Table 1: Estimates of MTTF and sensitivities with relative 99% confidence intervals

one type are failed or both disk controllers in one set are failed or two or more disks in any cluster are failed. Figure 3 is a block diagram of the system.

The failure rate of the processors of types $A$ and $B$ are 1/1500 and 1/2000 per hour, respectively. All of the disk controllers have failure rate 1/2000 per hour. The disks in the first cluster have failure rate 1/60000 per hour, and the disks in all of the other clusters have a failure rate of 1/6000 per hour. All of the components can fail in two different modes, where the repair rate in the first mode is 1 per hour and the repair rate in the second mode is 1/2 per hour. There is a single repairperson who repairs failed components in random order service.

Using the SAVE package (see Goyal and Lavenberg 1987), we estimated the mean time to failure of the system and its sensitivities with respect to $pAfr$, which is the failure rate of processors of type $A$, and $d1fr$, the failure rate of the disks in the first cluster. (Note that the theory we developed in the previous sections only considered the denominator term in the ratio expression for the MTTF and its derivatives, whereas now we estimate the actual MTTF and its derivatives.) We computed the values using a (non-simulation) numerical method, naive simulation, and simulation using balanced failure biasing. All of the simulation results were obtained from simulating 1,000,000 events, where an event is either a component failure or repair. The results are given in Table 1.

There are several interesting points to note in Table 1. First, the absolute value of the numerical value of the sensitivity with respect to $pAfr$ is much larger than that with respect to $d1fr$, and in fact, it turns out that the sensitivity with respect to $pAfr$ was one of the largest sensitivities in absolute value. When naive simulation was used, we were able to estimate this sensitivity with about the same relative error (as measured by the relative width of the confidence interval) as the MTTF. This agrees with the theory established in Corollaries 5 and 6. Furthermore, when using naive simulation, the estimate of the sensitivity with respect to $d1fr$ is very poor since it has such a large confidence interval. Thus, we were able to obtain a better estimate of the sensitivity with respect to $pAfr$ than with respect to $d1fr$, which is in accord with Corollary 7. Proposition 8 and Corollary 10 suggest that we should be able to obtain stable estimates for all of the performance measures by using balanced failure biasing, and indeed this is the case. Finally, we were able to estimate both sensitivities with comparable relative errors; see Corollary 11. Nakayama, Goyal, and Glynn (1990) also present similar results and a more comprehensive empirical study of derivative estimators.

# 8 Conclusions and Directions for Future Research

In this paper we considered the estimation of various partial derivatives of the performance measure mean time to failure in highly reliable Markovian systems. Analyzing the most likely paths to failure enabled us to derive asymptotic expressions for the derivative estimators and their variances. Hence, we could compare the difficulties of estimating the different derivatives. In particular we showed that the performance measure and some of its derivatives can be estimated with the same limiting accuracy, thus showing that the likelihood ratio method can be a quite efficient technique for estimating derivatives when used in an appropriate problem setting.

A number of directions for future research are possible. Shahabuddin (1991) demonstated the strong connection between estimating the mean time to failure and the steady-state unavailability in highly reliable Markovian systems. Consequently, we expect that results similar to the ones shown here can be proven for the derivatives of the steady-state unavailability. Transient performance measures such as the system reliability present additional complications due to the fact that conditional Monte Carlo cannot be used, and so the random holding times must be taken into account. However, the empirical findings of Nakayama, Goyal, and Glynn (1990) for derivatives of transient measures suggest that similar results can be obtained in this context. Furthermore, examining the characteristics of likelihood ratio derivative estimators in highly reliable non-Markovian systems may prove to be fruitful. Finally, there has been recent theoretical work on determining the asymptotic distribution of the time to system failure as system failures become rare; e.g., see Arisimov and Sztrik (1989), Keilson (1979), Ushakov (1985), and Sztrik (1989). It would be interesting to investigate if these types of results could be developed for derivatives. [1]

# 9 Proofs

Here we provide the proofs for all of our results.

**Proof of Theorem 2.** Consider the sample path $(x_0, \ldots, x_n) \in \Delta$ of the embedded DTMC.

[1]This paper is based directly on material from the author's Ph.D. dissertation, which was completed while the author was in the Operations Research Department at Stanford University. The research at Stanford University was supported by IBM under SUR-SST Contract 12480042. The author is very grateful to his advisor Peter W. Glynn, with whom he had many enlightening discussions. Also, the author would like to thank Ambuj Goyal, Steve Lavenberg, and Perwez Shahabuddin, the anonymous referees, and the area and associate editors for their helpful comments.

Using (7), the probability of the sample path for all sufficiently small $\epsilon$ is given by

$$P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau)=(x_0,\ldots,x_n)\}=\prod_{k=0}^{n-1}P(\theta(\epsilon),x_k,x_{k+1})=a(x_0,\ldots,x_n)\epsilon^m+o(\epsilon^m),$$

for some integer integer $m \geq 0$, where $a(x_0,\ldots,x_n)>0$ is independent of $\epsilon$. Consequently, we can decompose $\Delta$ as $\Delta=\cup_{m=0}^{\infty}\Delta_m$, where

$$\Delta_m=\{(x_0,\ldots,x_n)\in\Delta : n\geq 1,\ P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau)=(x_0,\ldots,x_n)\}\sim\epsilon^m\}. \tag{21}$$

$\Delta_m$ is the set of sample paths in $\Delta$ which have probability of the order $\epsilon^m$ and is independent of $\epsilon$ for all $\epsilon$ sufficiently small. Using (1), (7), and Assumption A3, we can easily show that for all $(x_0,\ldots,x_n)\in\Delta$, there exists at least one $k$, $0\leq k<n$, such that $x_{k+1}\succ x_k$ and $P(\theta(\epsilon),x_k,x_{k+1})\sim\epsilon^{d(x_k,x_{k+1})}$, where $d(x_k,x_{k+1})\geq 1$. Hence, $P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau)=(x_0,\ldots,x_n)\}$ is at least of the order $\epsilon$ for $(x_0,\ldots,x_n)\in\Delta$, implying $\Delta_0=\emptyset$. Therefore,

$$\Delta=\cup_{m=r}^{\infty}\Delta_m, \tag{22}$$

where $r\geq 1$ is an integer. We define $r$ such that $\Delta_r\neq\emptyset$ and $\Delta_m=\emptyset$ for all $m<r$. (It turns out that this $r$ is exactly the same as the $r$ given in Proposition 1.)

Now we show that $\Delta_m$, for each $m\geq r$, consists of a finite number of sample paths. Let $(x_0,\ldots,x_n)\in\Delta_m$, and consider $x_k$, $0<k<n$. Since $\tau_F<\tau_0$, we must have that $x_k\neq 0$. If $x_{k+1}\succ x_k$, then $P(\theta(\epsilon),x_k,x_{k+1})\sim\epsilon^{d(x_k,x_{k+1})}$, where $d(x_k,x_{k+1})\geq 1$, by (7). Moreover, $x_{k+1}\prec x_k$ implies that $P(\theta(\epsilon),x_k,x_{k+1})=c(x_k,x_{k+1})+o(1)$, where $0<c(x_k,x_{k+1})\leq 1$. Hence, since $(x_0,\ldots,x_n)\in\Delta_m$ has probability of the order $\epsilon^m$, the path can have at most $m+1$ failure transitions, including $(x_0,x_1)$. As we have allowed for failure propagation, each failure transition can result in the failure of at most $K$ failed components, where $K\equiv\sum_{i=1}^{C}n_i<\infty$ is the total number of components in the system. Since $\tau_F<\tau_0$, at most $K-1$ repair transitions can occur after every failure transition, and so there must be no more than $(m+1)(K-1)$ repair transitions on the path. Hence, each path in $\Delta_m$ has at most $(m+1)K$ total transitions; i.e., for $(x_0,\ldots,x_n)\in\Delta_m$,

$$n\leq(m+1)K. \tag{23}$$

It then follows that

$$|\Delta_m|\leq|E|^{(m+1)K}, \tag{24}$$

and since $|E|<\infty$ and $K<\infty$, we have $|\Delta_m|<\infty$ for all $m$.

Note that

$$\begin{aligned}P_{\theta(\epsilon)}\{\tau_F<\tau_0\} &= \sum_{\substack{(x_0,\ldots,x_n)\in\Delta_r\\ n\geq 1}}P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau)=(x_0,\ldots,x_n)\}\\ &\quad+\sum_{m=r+1}^{\infty}\sum_{\substack{(x_0,\ldots,x_n)\in\Delta_m\\ n\geq 1}}P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau)=(x_0,\ldots,x_n)\}.\end{aligned}$$

Since $|\Delta_r| < \infty$,

$$\sum_{\substack{(x_0,\ldots,x_n)\in\Delta_r \\ n\geq 1}} P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (x_0,\ldots,x_n)\} \;=\; a_0\epsilon^r + o(\epsilon^r) \tag{25}$$

for $\epsilon$ sufficiently small, where $a_0 > 0$.

Now we want to show that

$$\sum_{m=r+1}^{\infty} \sum_{\substack{(x_0,\ldots,x_n)\in\Delta_m \\ n\geq 1}} P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (x_0,\ldots,x_n)\} \;=\; o(\epsilon^r). \tag{26}$$

Note that by (7),

$$P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (x_0,\ldots,x_n)\} = \prod_{k=0}^{n-1} P(\theta(\epsilon), x_k, x_{k+1}) = \prod_{k=0}^{n-1} c(x_k, x_{k+1})\epsilon^{d(x_k,x_{k+1})} + o(\epsilon^{d(x_k,x_{k+1})}), \tag{27}$$

where $c(x_k, x_{k+1}) > 0$ for $0 \leq k < n$. Now define $t(x_k, x_{k+1}) = 2c(x_k, x_{k+1})$, where $c(x_k, x_{k+1})$ is in (27). Thus,

$$P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (x_0,\ldots,x_n)\} \leq \prod_{k=0}^{n-1} t(x_k, x_{k+1})\epsilon^{d(x_k,x_{k+1})} \tag{28}$$

for all sufficiently small $\epsilon > 0$. Define $\bar{t} = \max\{t(x,y) : (x,y) \in \Gamma\}$, and note that $\bar{t} < \infty$ since $|E| < \infty$. Now let

$$w = \max\{\bar{t}, 1\}. \tag{29}$$

Finally, note that for $(x_0,\ldots,x_n) \in \Delta_m$, $\sum_{k=0}^{n-1} d(x_k, x_{k+1}) = m$, where $d(x_k, x_{k+1})$ are as defined in (27).

Recall that

$$P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (x_0,\ldots,x_n)\} = \prod_{k=0}^{n-1} P(\theta(\epsilon), x_k, x_{k+1}) = \prod_{k=0}^{n-1} \frac{q(\theta(\epsilon), x_k, x_{k+1})}{q(\theta(\epsilon), x_k)}$$

and $q(\theta(\epsilon), x) = \sum_{y:(x,y)\in\Gamma} q(\theta(\epsilon), x, y)$. Since $|E| < \infty$, we can express $q(\theta(\epsilon), x) = c(x)\epsilon^{d(x)} + o(\epsilon^{d(x)})$, where $c(x) > 0$ and $d(x) \geq 0$. Now define $q_*(\theta(\epsilon), x) = c(x)\epsilon^{d(x)}$; i.e., $q_*(\theta(\epsilon), x)$ is the same as $q(\theta(\epsilon), x)$ without the $o(\epsilon^{d(x)})$ term. The $o(\epsilon^{d(x)})$ term in the expression for $q(\theta(\epsilon), x)$ is non-negative since $q(\theta(\epsilon), x, y) = \sum_{k=d_1(x,y)}^{d_2(x,y)} c_k(x,y)\epsilon^k$ with $c_k(x,y) \geq 0$ and $d_1(x,y) \geq 0$ for all $(x,y) \in \Gamma$ by (6). (If $(x,y)$ is a repair transition, then $q(\theta(\epsilon), x, y) = c(x,y)\mu = \bar{c}(x,y)\epsilon^0$ since $\mu$ is fixed.) Thus, $q_*(\theta(\epsilon), x_k) \leq q(\theta(\epsilon), x_k)$. Also, for $(x,y) \in \Gamma$, define $q^*(\theta(\epsilon), x, y)$ as follows. For $y \prec x$, let $q^*(\theta(\epsilon), x, y) = q(\theta(\epsilon), x, y)$. For $y \succ x$, let $q^*(\theta(\epsilon), x, y) = 2c_{d_1(x,y)}(x,y)\epsilon^{d_1(x,y)}$, where $c_{d_1(x,y)}(x,y)$ and $d_1(x,y)$ are defined such that $q(\theta(\epsilon), x, y) = \sum_{k=d_1(x,y)}^{d_2(x,y)} c_k(x,y)\epsilon^k$ with $c_{d_1(x,y)}(x,y) > 0$. For all sufficiently small $\epsilon$, $q^*(\theta(\epsilon), x, y) \geq q(\theta(\epsilon), x, y)$, implying that

$$P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (x_0,\ldots,x_n)\} \leq \prod_{k=0}^{n-1} \frac{q^*(\theta(\epsilon), x_k, x_{k+1})}{q_*(\theta(\epsilon), x_k)} \equiv \prod_{k=0}^{n-1} t(x_k, x_{k+1})\epsilon^{d(x_k,x_{k+1})}, \tag{30}$$

where $t(x_k, x_{k+1}) > 0$ is independent of $\epsilon$ and $d(x_k, x_{k+1}) \geq 0$. Note that there is no $o(\epsilon^{d(x_k,x_{k+1})})$ term in (30) and for $(x_0, \ldots, x_n) \in \Delta_m$, $\sum_{k=0}^{n-1} d(x_k, x_{k+1}) = m$. Define $\bar{t} = \max\{t(x, y) : (x, y) \in \Gamma\} < \infty$, since $|E| < \infty$. Finally, define

$$w = \max\{\bar{t}, 1\}. \tag{31}$$

Since each path in $\Delta_m$ has at most $(m + 1)K$ total transitions as shown in (23),

$$\prod_{k=0}^{n-1} P(\theta(\epsilon), x_k, x_{k+1}) \;\leq\; \prod_{k=0}^{n-1} \frac{q^*(\theta(\epsilon), x_k, x_{k+1})}{q_*(\theta(\epsilon), x_k)} \;\leq\; w^{(m+1)K}\epsilon^m \tag{32}$$

for $(x_0, \ldots, x_n) \in \Delta_m$. Consequently, using (24), we have

$$\sum_{\substack{(x_0,\ldots,x_n)\in\Delta_m \\ n\geq 1}} P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_n)\} \;\leq\; \psi(\epsilon) \;\equiv\; (w|E|)^{(m+1)K}\epsilon^m, \tag{33}$$

which gives us

$$\epsilon^{-r} \sum_{m=r+1}^{\infty} \sum_{\substack{(x_0,\ldots,x_n)\in\Delta_m \\ n\geq 1}} P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_n)\} \;\leq\; \phi(\epsilon) \;\equiv\; \sum_{m=r+1}^{\infty} (w|E|)^{(m+1)K}\epsilon^{m-r}.$$

Note that $\limsup_{m\to\infty} \sqrt[m]{(w|E|)^{(m+1)K}} = (w|E|)^K < \infty$, so $\phi(\epsilon)$ is finite for all sufficiently small $\epsilon$ (specifically, the sum is finite if $\epsilon < \epsilon_0 \equiv (w|E|)^{-K}$); see Theorem 3.39 of Rudin (1976). Now for $\epsilon < \epsilon_1 < \epsilon_0$, we have $\psi(\epsilon) < \psi(\epsilon_1)$ and $\phi(\epsilon_1) < \infty$, and so

$$\begin{aligned}
&\lim_{\epsilon\to 0} \epsilon^{-r} \sum_{m=r+1}^{\infty} \sum_{\substack{(x_0,\ldots,x_n)\in\Delta_m \\ n\geq 1}} P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_n)\} \\
&\quad = \sum_{m=r+1}^{\infty} \sum_{\substack{(x_0,\ldots,x_n)\in\Delta_m \\ n\geq 1}} \lim_{\epsilon\to 0} \frac{a(x_0, \ldots, x_n)\epsilon^m + o(\epsilon^m)}{\epsilon^r} \;=\; 0
\end{aligned}$$

by the dominated convergence theorem (Theorem 16.4 of Billingsley 1986). Thus, (26) holds and

$$P_{\theta(\epsilon)}\{\tau_F < \tau_0\} \;=\; \sum_{\substack{(x_0,\ldots,x_n)\in\Delta_r \\ n\geq 1}} a(x_0, \ldots, x_n)\epsilon^r + o(\epsilon^r) \;=\; a_0\epsilon^r + o(\epsilon^r), \tag{34}$$

where $a_0 > 0$.

Finally, let $(x_0, \ldots, x_n) \in \Delta_m$ for some $m \geq r$. Then

$$\begin{aligned}
&P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_n) \mid \tau_F < \tau_0\} \\
&\quad = \frac{P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_n)\}}{P_{\theta(\epsilon)}\{\tau_F < \tau_0\}} \\
&\quad = \frac{a(x_0, \ldots, x_n)\epsilon^m + o(\epsilon^m)}{\sum_{\substack{(y_0,\ldots,y_k)\in\Delta_r \\ k>0}} a(y_0, \ldots, y_k)\epsilon^r + o(\epsilon^r)} \\
&\quad \to \begin{cases} \frac{a(x_0,\ldots,x_n)}{\sum_{\substack{(y_0,\ldots,y_k)\in\Delta_r \\ k>0}} a(y_0,\ldots,y_k)} & \text{if } (x_0, \ldots, x_n) \in \Delta_r \\ 0 & \text{otherwise} \end{cases},
\end{aligned} \tag{35}$$

as $\epsilon \to 0$. Hence, $P_0\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_n)\}$ is given by (35). ∎

The constant $a_0$ in Proposition 11 can be calculated in terms of the limiting probability measure $P_0$ from Theorem 2 as follows. We had defined $a(x_0, \ldots, x_n)$ such that $P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_n)\} = a(x_0, \ldots, x_n)\epsilon^m + o(\epsilon^m)$ for $(x_0, \ldots, x_n) \in \Delta_m$. It can be easily shown that $a_0 = \sum_{(x_0,\ldots,x_k)\in\Delta_r} a(x_0, \ldots, x_k)$. From Theorem 2, we have $(x_0, \ldots, x_k) \in \Delta_r$ if and only if $P_0\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_k)\} > 0$.

We now establish Theorem 3. We prove this by first showing a number of preliminary results. (An outline of the proof is given after the statement of Theorem 3.) Our first lemma describes the forms of the summands in the expression for $\partial_{\lambda_i} L(\lambda(\epsilon), \mu)$ given in (9).

**Lemma 12** *Suppose the assumptions of Theorem 3 hold, and consider any $(x, y) \in \Gamma$ with $x \in U$. Then for all $\epsilon$ sufficiently small, $\partial_{\lambda_i} P((\lambda(\epsilon), \mu), x, y)/P((\lambda(\epsilon), \mu), x, y)$ is of the form*

**(i)** $c_0\epsilon^{-b_i} + o(\epsilon^{-b_i})$, *where $c_0 \neq 0$, if $y \succ x$ with $p(y; x, i) > 0$,*

**(ii)** $c_0 + o(1)$, *where $c_0 < 0$, if either $x \neq 0$ and $y \prec x$, or $x \neq 0$ and $y \succ x$ with $p(y; x, i) = 0$,*

**(iii)** $c_0\epsilon^{-b_0} + o(\epsilon^{-b_0})$, *where $c_0 < 0$, if $x = 0$ and $y \succ 0$ with $p(y; 0, i) = 0$.*

**Proof.** Suppose $(0, y) \in \Gamma$. Then,

$$P((\lambda, \mu), 0, y) = \frac{\sum_{k=1}^C n_k(0)\lambda_k p(y; 0, k)}{\sum_{l=1}^C n_l(0)\lambda_l},$$

and so

$$\frac{\partial_{\lambda_i} P((\lambda, \mu), 0, y)}{P((\lambda, \mu), 0, y)} = \frac{n_i(0)\sum_{k=1}^C n_k(0)\lambda_k(p(y; 0, i) - p(y; 0, k))}{\sum_{k=1}^C n_k(0)\lambda_k p(y; 0, k) \sum_{l=1}^C n_l(0)\lambda_l}.$$

Now substitute $\lambda_k = \tilde{\lambda}_k\epsilon^{b_k}$ for all component types $k$. Recall that $b_k \geq b_0$ for all $k$ with at least one $b_j = b_0$ by the definition of $b_0$. Thus, $\sum_{l=1}^C n_l(0)\lambda_l(\epsilon) \sim \epsilon^{b_0}$. If $p(y; 0, i) = 0$, then

$$\frac{\partial_{\lambda_i} P((\lambda(\epsilon), \mu), 0, y)}{P((\lambda(\epsilon), \mu), 0, y)} = \frac{-n_i(0)}{\sum_{l=1}^C n_l(0)\lambda_l(\epsilon)} = c(0, y)\epsilon^{-b_0} + o(\epsilon^{-b_0}),$$

where $c(0, y) < 0$. Now assume that $p(y; 0, i) > 0$. First suppose $b_i > b_0$. Then there exists some component type $j$ such that $b_j = b_0$ by the definition of $b_0$, and so $p(y; 0, i) \neq p(y; 0, j) = 0$ by Assumption A4. Consequently, $n_i(0)\sum_{k=1}^C n_k(0)\lambda_k(\epsilon)(p(y; 0, i) - p(y; 0, k)) \sim \epsilon^{b_0}$. By Assumption A4, $\sum_{k=1}^C n_k(0)\lambda_k(\epsilon)p(y; 0, k) \sim \epsilon^{b_i}$, which implies that when $b_i > b_0$,

$$\frac{\partial_{\lambda_i} P((\lambda(\epsilon), \mu), 0, y)}{P((\lambda(\epsilon), \mu), 0, y)} = c(0, y)\epsilon^{-b_i} + o(\epsilon^{-b_i}),$$

where $c(0, y) > 0$. Now suppose $b_i = b_0$. Then $\sum_{k=1}^C n_k(0)\lambda_k(\epsilon)p(y; 0, k) \sim \epsilon^{b_0}$, and by Assumption A5, $n_i(0)\sum_{k=1}^C n_k(0)\lambda_k(\epsilon)(p(y; 0, i) - p(y; 0, k)) \sim \epsilon^{b_0}$. Hence,

$$\frac{\partial_{\lambda_i} P((\lambda(\epsilon), \mu), 0, y)}{P((\lambda(\epsilon), \mu), 0, y)} = c(0, y)\epsilon^{-b_0} + o(\epsilon^{-b_0}),$$

when $b_i = b_0$, where $c(0,y) \neq 0$.

Now suppose $(x,y) \in \Gamma$ with $x \neq 0$ and $y \succ x$. Then,

$$P((\lambda,\mu),x,y) = \frac{\sum_{k=1}^{C} n_k(x)\lambda_k p(y;x,k)}{\sum_{z \prec x} \mu(x,z) + \sum_{l=1}^{C} n_l(x)\lambda_l},$$

and so

$$\frac{\partial_{\lambda_i} P((\lambda,\mu),x,y)}{P((\lambda,\mu),x,y)} = \frac{n_i(x)(\sum_{k=1}^{C} n_k(x)\lambda_k(p(y;x,i) - p(y;x,k)) + \sum_{z \prec x} \mu(x,z)p(y;x,i))}{\sum_{k=1}^{C} n_k(x)\lambda_k p(y;x,k)(\sum_{z \prec x} \mu(x,z) + \sum_{l=1}^{C} n_l(x)\lambda_l)}.$$

Substituting $\lambda_k = \tilde{\lambda}_k \epsilon^{b_k}$, since $\mu$ is fixed and by Assumption A4, we obtain

$$\frac{\partial_{\lambda_i} P((\lambda(\epsilon),\mu),x,y)}{P((\lambda(\epsilon),\mu),x,y)} = c(x,y)\epsilon^{-b_i} + o(\epsilon^{-b_i}),$$

when $p(y;x,i) > 0$, where $c(x,y) > 0$. If $p(y;x,i) = 0$, then

$$\frac{\partial_{\lambda_i} P((\lambda(\epsilon),\mu),x,y)}{P((\lambda(\epsilon),\mu),x,y)} = c(x,y) + o(1),$$

where $c(x,y) < 0$.

Finally, suppose $(x,y) \in \Gamma$ with $x \neq 0$ and $y \prec x$. Then,

$$P((\lambda,\mu),x,y) = \frac{\mu(x,y)}{\sum_{z \prec x} \mu(x,z) + \sum_{k=1}^{C} n_k(x)\lambda_k},$$

where $c_0 > 0$ and $c_1 > 0$, and so

$$\frac{\partial_{\lambda_i} P((\lambda,\mu),x,y)}{P((\lambda,\mu),x,y)} = \frac{-n_i(x)}{\sum_{z \prec x} \mu(x,z) + \sum_k n_k(x)\lambda_k}$$

Now substituting $\lambda_k = \tilde{\lambda}_k \epsilon^{b_k}$, since $\mu$ is fixed, we obtain

$$\frac{\partial_{\lambda_i} P((\lambda(\epsilon),\mu),x,y)}{P((\lambda(\epsilon),\mu),x,y)} = c(x,y) + o(1),$$

where $c(x,y) < 0$. ∎

Before stating the next result, we first develop some notation. Consider component type $i$, and define

$$\Delta^i = \{(x_0,\ldots,x_n) \in \Delta : n \geq 1,\ n_i(x_{k-1})p(x_k;x_{k-1},i) > 0 \text{ for some } 0 < k \leq n \text{ such that } x_k \succ x_{k-1}\}, \quad (36)$$

which is the set of sample paths for which $\tau_i \leq \tau_F < \tau_0$, where $\tau_i$ is defined in (10). Similarly, we define

$$\bar{\Delta}^i = \{(x_0,\ldots,x_n) \in \Delta : n \geq 1,\ p(x_{k+1};x_k,i) = 0 \text{ for all } 0 \leq k < n \text{ such that } x_{k+1} \succ x_k\},$$

which is the set of sample paths for which $\tau_F < \min\{\tau_0, \tau_i\}$.

We have the following theorem. The proof is omitted since these results can be proven using an argument similar to the one employed to show Theorem 2.

**Theorem 13** *Suppose the assumptions of Theorem 3 hold. Then*

**(i)** $P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau)\in\cdot \mid \tau_i\le\tau_F<\tau_0\}\Rightarrow P_0^i\{(X_0,\ldots,X_\tau)\in\cdot\ \}$,

**(ii)** $P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau)\in\cdot \mid \tau_F<\min\{\tau_0,\tau_i\}\}\Rightarrow \bar{P}_0^i\{(X_0,\ldots,X_\tau)\in\cdot\ \}$,

*as* $\epsilon\to 0$*, where* $P_0^i$ *and* $\bar{P}_0^i$ *are some limiting probability measures on* $\Delta^i$ *and* $\bar{\Delta}^i$*, respectively.*

We can decompose $\Delta^i$ as $\Delta^i=\cup_{m=r}^{\infty}\Delta_m^i$, where $\Delta_m^i=\Delta^i\cap\Delta_m$ and $\Delta_m$ is defined in (21). Note that $\Delta_m^i$ is the set of sample paths in $\Delta^i$ that have probability of the order $\epsilon^m$. Since $|\Delta_m|<\infty$, $\Delta_m^i\subset\Delta_m$ implies $|\Delta_m^i|<\infty$. Also, $\Delta^i=\cup_{m=r_i}^{\infty}\Delta_m^i$, where $r_i\ge r$ since $\Delta^i\subset\Delta$. Similarly, we can decompose $\bar{\Delta}^i$ as $\bar{\Delta}^i=\cup_{m=\bar{r}_i}^{\infty}\bar{\Delta}_m^i$, where $\bar{\Delta}_m^i=\bar{\Delta}^i\cap\Delta_m$ and $\bar{r}_i\ge r$ since $\bar{\Delta}^i\subset\Delta$. We can show that $P_0^i\{(X_0,\ldots,X_\tau)=(x_0,\ldots,x_n)\}>0$ if and only if $(x_0,\ldots,x_n)\in\Delta_{r_i}^i$, and that $\bar{P}_0^i\{(X_0,\ldots,X_\tau)=(x_0,\ldots,x_n)\}>0$ if and only if $(x_0,\ldots,x_n)\in\bar{\Delta}_{\bar{r}_i}^i$.

Employing arguments similar to the ones used in the proof of Theorem 13, we can show that the constants in (13) and (14) are given by $a_0^i=\sum_{(x_0,\ldots,x_n)\in\Delta_{r_i}^i}a(x_0,\ldots,x_n)>0$ and $\bar{a}_0^i=\sum_{(x_0,\ldots,x_n)\in\bar{\Delta}_{\bar{r}_i}^i}a(x_0,\ldots,x_n)>0$.

The next lemma shows that given $\tau_i\le\tau_F<\tau_0$, $\partial_{\lambda_i}L(\lambda(\epsilon),\mu)$ is of the order $\epsilon^{-b_i}$, and given $\tau_F<\min\{\tau_i,\tau_0\}$, $\partial_{\lambda_i}L(\lambda(\epsilon),\mu)$ is of the order $\epsilon^{-b_0}$.

**Lemma 14** *Suppose the assumptions of Theorem 3 hold. Then there exist limiting random variables* $L'_{\lambda_i}$ *and* $\bar{L}'_{\lambda_i}$ *such that*

**(i)** $P_{\theta(\epsilon)}\{\epsilon^{b_i}\ \partial_{\lambda_i}L(\lambda(\epsilon),\mu)\in\cdot \mid \tau_i\le\tau_F<\tau_0\}\Rightarrow P_0^i\{L'_{\lambda_i}\in\cdot\ \}$

**(ii)** $P_{\theta(\epsilon)}\{\epsilon^{b_0}\ \partial_{\lambda_i}L(\lambda(\epsilon),\mu)\in\cdot \mid \tau_F<\min\{\tau_i,\tau_0\}\}\Rightarrow \bar{P}_0^i\{\bar{L}'_{\lambda_i}\in\cdot\ \}$

*as* $\epsilon\to 0$*, where* $P_0^i\{|L'_{\lambda_i}|<M_i\}=1$ *for some constant* $M_i$*,* $\bar{P}_0^i\{|\bar{L}'_{\lambda_i}|<\bar{M}_i\}=1$ *for some constant* $\bar{M}_i$*, and* $P_0^i$ *and* $\bar{P}_0^i$ *are the limiting probability distributions given in Theorem 13.*

**Proof.** We only show 1, as 2 can be proven in a similar manner. By Lemma 12, we can express

$$\partial_{\lambda_i}L(\lambda(\epsilon),\mu) = \sum_{k=0}^{\tau-1}\frac{\partial_{\lambda_i}P((\lambda(\epsilon),\mu),X_k,X_{k+1})}{P((\lambda(\epsilon),\mu),X_k,X_{k+1})} = \sum_{k=0}^{\tau-1}r(X_k,X_{k+1})\epsilon^{-b_i}+\sum_{k=0}^{\tau-1}\rho(\epsilon,X_k,X_{k+1}), \quad (37)$$

where

$$r(x,y)=\begin{cases} c(x,y) & \text{if } \frac{\partial_{\lambda_i}P((\lambda(\epsilon),\mu),x,y)}{P((\lambda(\epsilon),\mu),x,y)}=c(x,y)\epsilon^{-b_i}+o(\epsilon^{-b_i}),\\ & \text{where } c(x,y)\neq 0\\ \\ 0 & \text{if } \frac{\partial_{\lambda_i}P((\lambda(\epsilon),\mu),x,y)}{P((\lambda(\epsilon),\mu),x,y)}=o(\epsilon^{-b_i}) \end{cases}, \quad (38)$$

and

$$\rho(\epsilon,x,y)=\frac{\partial_{\lambda_i}P((\lambda(\epsilon),\mu),x,y)}{P((\lambda(\epsilon),\mu),x,y)}-r(x,y)\epsilon^{-b_i}. \quad (39)$$

Note that $r(x,y)$ is independent of $\epsilon$ and $\rho(\epsilon,x,y) = o(\epsilon^{-b_i})$ for all $(x,y) \in \Gamma$. Since we are working with discrete point distributions,

$$P_{\theta(\epsilon)}\left\{\sum_{k=0}^{\tau-1} r(X_k, X_{k+1}) \in \cdot \;\middle|\; \tau_i \le \tau_F < \tau_0\right\} \Rightarrow P_0^i\left\{\sum_{k=0}^{\tau-1} r(X_k, X_{k+1}) \in \cdot\;\right\}$$

as $\epsilon \to 0$, where $P_0^i$ is the limiting distribution in Theorem 13. All sample paths with $\tau_i \le \tau_F < \tau_0$ must include a transition $(x,y)$ such that $n_i(x)p(y;x,i) > 0$. By Lemma 12, each of these paths has at least one failure transition $(x,y)$ with $r(x,y) \ne 0$. Furthermore, $|\Delta^i_{r_i}| < \infty$ and all paths in $\Delta^i_{r_i}$ have finitely many transitions by (23). Thus, $P_0^i\left\{|\sum_{k=0}^{\tau-1} r(X_k, X_{k+1})| < M_i\right\} = 1$ for some $M_i$.

Now we show that the remainder term in (37) goes to 0 in probability. Let $\delta > 0$, and note that

$$\begin{aligned}
&P_{\theta(\epsilon)}\left\{\left|\sum_{k=0}^{\tau-1} \epsilon^{b_i}\rho(\epsilon, X_k, X_{k+1})\right| < \delta \;\middle|\; \tau_i \le \tau_F < \tau_0\right\} \\
&= \sum_{\substack{(x_0,\dots,x_n)\in\Delta^i \\ n\ge 1}} P_{\theta(\epsilon)}\left\{\left|\sum_{k=0}^{\tau-1} \epsilon^{b_i}\rho(\epsilon, X_k, X_{k+1})\right| < \delta,\ (X_0,\dots,X_\tau) = (x_0,\dots,x_n) \;\middle|\; \tau_i \le \tau_F < \tau_0\right\}.
\end{aligned}$$

For all paths $(x_0,\dots,x_n) \in \Delta^i_{r_i}$, observe that $\left|\sum_{k=0}^{n-1} \epsilon^{b_i}\rho(\epsilon, x_k, x_{k+1})\right| < \delta$ for all sufficiently small $\epsilon > 0$ since $\rho(\epsilon,x,y) = o(\epsilon^{-b_i})$ for all $(x,y) \in \Gamma$ and by (23). Furthermore, $|\Delta^i_{r_i}| < \infty$, and so by Theorem 13,

$$\begin{aligned}
&P_{\theta(\epsilon)}\left\{\left|\sum_{k=0}^{\tau-1} \epsilon^{b_i}\rho(\epsilon, X_k, X_{k+1})\right| < \delta,\ (X_0,\dots,X_\tau) = (x_0,\dots,x_n) \;\middle|\; \tau_i \le \tau_F < \tau_0\right\} \\
&\to\ P_0^i\{(X_0,\dots,X_\tau) = (x_0,\dots,x_n)\}
\end{aligned}$$

as $\epsilon \to 0$ for all paths $(x_0,\dots,x_n) \in \Delta^i_{r_i}$. All other paths $(x_0,\dots,x_n) \in \Delta^i$ satisfy

$$\begin{aligned}
&P_{\theta(\epsilon)}\left\{\left|\sum_{k=0}^{\tau-1} \epsilon^{b_i}\rho(\epsilon, X_k, X_{k+1})\right| < \delta,\ (X_0,\dots,X_\tau) = (x_0,\dots,x_n) \;\middle|\; \tau_i \le \tau_F < \tau_0\right\} \\
&\le\ P_{\theta(\epsilon)}\left\{(X_0,\dots,X_\tau) = (x_0,\dots,x_n) \mid \tau_i \le \tau_F < \tau_0\right\} \\
&\to\ P_0^i\{(X_0,\dots,X_\tau) = (x_0,\dots,x_n)\} \ = \ 0
\end{aligned}$$

as $\epsilon \to 0$ by Theorem 13. The bounded convergence theorem then implies that

$$\begin{aligned}
&\lim_{\epsilon\to 0} P_{\theta(\epsilon)}\left\{\left|\sum_{k=0}^{\tau-1} \epsilon^{b_i}\rho(\epsilon, X_k, X_{k+1})\right| < \delta \;\middle|\; \tau_i \le \tau_F < \tau_0\right\} \\
&= \sum_{\substack{(x_0,\dots,x_n)\in\Delta^i \\ n\ge 1}} P_0^i\{(X_0,\dots,X_\tau) = (x_0,\dots,x_n)\} \ = \ 1,
\end{aligned}$$

proving that the remainder term goes to 0 in probability. Hence, using the converging together lemma, we have $L'_{\lambda_i} = \sum_{k=0}^{\tau-1} r(X_k, X_{k+1})$, and the proof is complete. ■

Our final lemma shows that for $k \ge 1$, $E_{\theta(\epsilon)}[(\partial_{\lambda_i} L(\lambda(\epsilon),\mu))^k \mid \tau_i \le \tau_F < \tau_0] \sim \epsilon^{-kb_i}$ and $E_{\theta(\epsilon)}[(\partial_{\lambda_i} L(\lambda(\epsilon),\mu))^k \mid \tau_F < \min\{\tau_i, \tau_0\}] \sim \epsilon^{-kb_0}$.

**Lemma 15** *Suppose the assumptions of Theorem 3 hold. Then, for $k \geq 1$,*

**(i)** $E_{\theta(\epsilon)}[\epsilon^{kb_i}\ (\partial_{\lambda_i} L(\lambda(\epsilon),\mu))^k \mid \tau_i \leq \tau_F < \tau_0] \to E_0^i[{L'_{\lambda_i}}^k]$,

**(ii)** $E_{\theta(\epsilon)}[\epsilon^{kb_0}\ (\partial_{\lambda_i} L(\lambda(\epsilon),\mu))^k \mid \tau_F < \min\{\tau_i,\tau_0\}] \to \bar{E}_0^i[\bar{L}'^k_{\lambda_i}]$,

*as $\epsilon \to 0$, where $E_0^i$ and $\bar{E}_0^i$ are the expectation operators under the limiting probability measures $P_0^i$ and $\bar{P}_0^i$ given in Theorem 13 and $L'_{\lambda_i}$ and $\bar{L}'_{\lambda_i}$ are the limiting random variables in Lemma 14.*

**Proof.** We will only prove that 1 holds, as 2 can be shown in a similar manner. First, we will show that $E_{\theta(\epsilon)}[|\epsilon^{b_i}\partial_{\lambda_i} L(\lambda(\epsilon),\mu)|^p \mid \tau_i \leq \tau_F < \tau_0]$ is bounded for all sufficiently small $\epsilon$, where $p \equiv k+\delta$ and $\delta > 0$ is some constant. By the triangle inequality,

$$\begin{aligned} &E_{\theta(\epsilon)}\left[|\epsilon^{b_i}\ \partial_{\lambda_i} L(\lambda(\epsilon),\mu)|^p \mid \tau_i \leq \tau_F < \tau_0\right] \\ &\leq \sum_{m=r_i}^{\infty} \sum_{\substack{(x_0,\ldots,x_n)\in\Delta_m^i \\ n\geq 1}} \left(\sum_{k=0}^{n-1} \epsilon^{b_i} \left|\frac{\partial_{\lambda_i} P((\lambda(\epsilon),\mu),x_k,x_{k+1})}{P((\lambda(\epsilon),\mu),x_k,x_{k+1})}\right|\right)^p \\ &\qquad \cdot P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (x_0,\ldots,x_n) \mid \tau_i \leq \tau_F < \tau_0\}. \end{aligned}$$

Define $\bar{r} = \max\{|r(x,y)| : (x,y) \in \Gamma\}$, where $r(x,y)$ is defined in (38). Since $|E| < \infty$, $|\bar{r}| < \infty$. Also, define $\bar{\rho}(\epsilon) = \max\{|\rho(\epsilon,x,y)| : (x,y) \in \Gamma\}$, where $\rho(\epsilon,x,y)$ is defined in (39). Note that $\bar{\rho}(\epsilon) = o(\epsilon^{-b_i})$ for all sufficiently small $\epsilon$. Thus, for $(x,y) \in \Gamma$,

$$\frac{\partial_{\lambda_i} P((\lambda(\epsilon),\mu),x,y)}{P((\lambda(\epsilon),\mu),x,y)} \ = \ r(x,y)\epsilon^{-b_i} + \rho(\epsilon,x,y) \ \leq \ \bar{r}\epsilon^{-b_i} + \bar{\rho}(\epsilon) \ \leq \ 2\bar{r}\epsilon^{-b_i}$$

for all $\epsilon$ sufficiently small. Hence, using (23), for $(x_0,\ldots,x_n) \in \Delta_m^i$,

$$\sum_{k=0}^{n-1} \epsilon^{b_i} \left|\frac{\partial_{\lambda_i} P((\lambda(\epsilon),\mu),x_k,x_{k+1})}{P((\lambda(\epsilon),\mu),x_k,x_{k+1})}\right| \leq 2(m+1)K\bar{r}, \tag{40}$$

giving us

$$\begin{aligned} &E_{\theta(\epsilon)}\left[|\epsilon^{b_i}\partial_{\lambda_i} L(\lambda(\epsilon),\mu)|^p \mid \tau_i < \tau_F < \tau_0\right] \\ &\leq \sum_{m=r_i}^{\infty} \sum_{\substack{(x_0,\ldots,x_n)\in\Delta_m^i \\ n\geq 1}} (2(m+1)K\bar{r})^p \frac{P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (x_0,\ldots,x_n)\}}{P_{\theta(\epsilon)}\{\tau_i \leq \tau_F < \tau_0\}} \\ &= \frac{(2K\bar{r})^p}{a_0^i\epsilon^{r_i} + o(\epsilon^{r_i})} \sum_{m=r_i}^{\infty} (m+1)^p \sum_{\substack{(x_0,\ldots,x_n)\in\Delta_m^i \\ n\geq 1}} P_{\theta(\epsilon)}\{(X_0,\ldots,X_\tau) = (x_0,\ldots,x_n)\} \end{aligned}$$

by (13). Using (33) and the fact that $\Delta^i \subset \Delta$,

$$\begin{aligned} E_{\theta(\epsilon)}\left[|\epsilon^{b_i}\partial_{\lambda_i} L(\lambda(\epsilon),\mu)|^p \mid \tau_i \leq \tau_F < \tau_0\right] &\leq \frac{(2K\bar{r})^p}{a_0^i\epsilon^{r_i} + o(\epsilon^{r_i})} \sum_{m=r_i}^{\infty} (m+1)^p (w|E|)^{(m+1)K}\epsilon^m \\ &= \frac{(2K\bar{r})^p}{a_0^i + o(1)} (w|E|)^{(r+1)K}\,\phi(\epsilon) \end{aligned}$$

where $w < \infty$ is defined in (31) and $\phi(\epsilon) = \sum_{m=0}^{\infty}(m + r + 1)^p \left(w^K|E|^K\epsilon\right)^m$. Thus, to prove our result, we first need to show $\phi(\epsilon)$ is bounded as $\epsilon \to 0$.

Since $w < \infty$, $|E| < \infty$, and $K < \infty$, there exists $\epsilon_0 > 0$ such that $w^K|E|^K\epsilon_0 < 1$, implying $\phi(\epsilon_0) < \infty$. Also, $(m + r + 1)^p \left(w^K|E|^K\epsilon\right)^m \leq (m + r + 1)^p \left(w^K|E|^K\epsilon_0\right)^m$ for all $0 < \epsilon < \epsilon_0$. Consequently, by the dominated convergence theorem, $\phi(\epsilon) \to c_0$ as $\epsilon \to 0$, where $c_0 < \infty$ is some constant, proving the boundedness. Hence, $\phi(\epsilon) = c_0 + o(1)$ for all sufficiently small $\epsilon$, and so

$$E_{\theta(\epsilon)}\left[|\epsilon^{b_i}\partial_{\lambda_i}L(\lambda(\epsilon), \mu)|^p \mid \tau_i \leq \tau_F < \tau_0\right] \;\leq\; \frac{(2K\bar{r})^p}{a_0^i + o(1)}\,(w|E|)^{(r+1)K}\,(c_0 + o(1)),$$

which is bounded for all sufficiently small $\epsilon$ since $K < \infty$ and $\bar{r} < \infty$. Finally, using Lemma 14 and the corollary to Theorem 25.12 of Billingsley (1986), we have our result. ∎

Now we can finally prove Theorem 3.

**Proof of Theorem 3.** By Lemma 15, for $p \geq 1$,

$$\begin{aligned} E_{\theta(\epsilon)}[(\partial_{\lambda_i}L(\lambda(\epsilon), \mu))^p \mid \tau_i \leq \tau_F < \tau_0] &= c_0(p)\epsilon^{-pb_i} + o(\epsilon^{-pb_i}) \\ E_{\theta(\epsilon)}[(\partial_{\lambda_i}L(\lambda(\epsilon), \mu))^p \mid \tau_F < \min\{\tau_i, \tau_0\}] &= c_1(p)\epsilon^{-pb_0} + o(\epsilon^{-pb_0}) \end{aligned}$$

for all sufficiently small $\epsilon$, where $c_0, c_1 \neq 0$. Thus, using (12), (13), and (14),

$$\begin{aligned} E_{\theta(\epsilon)}[1\{\tau_F < \tau_0\}\;(\partial_{\lambda_i}L(\lambda(\epsilon), \mu))^p] &= c_0(p)a_0^i\epsilon^{r_i - pb_i} + o(\epsilon^{r_i - pb_i}) + c_1(p)\bar{a}_0^i\epsilon^{\bar{r}_i - pb_0} + o(\epsilon^{\bar{r}_i - pb_0}) \\ &= c_2(p)\epsilon^{\min\{r_i - pb_i,\; \bar{r}_i - pb_0\}} + o(\epsilon^{\min\{r_i - pb_i,\; \bar{r}_i - pb_0\}}) \end{aligned}$$

for all sufficiently small $\epsilon$, where $c_2(p) \neq 0$. From this the result easily follows. ∎

**Proof of Theorem 9.** Consider the first conditional expectation in (20). (We can analyze the other conditional expectation in the same manner.) We first establish a bound on the likelihood ratio. For any $(x_0, \ldots, x_n) \in \Delta_m$, $n \geq 1$,

$$\begin{aligned} L_*(\theta(\epsilon))(x_0, \ldots, x_n) &= P_{\theta(\epsilon)}\{(X_0, \ldots, X_\tau) = (x_0, \ldots, x_n)\} \prod_{k=0}^{n-1} \frac{1}{P_*(\theta(\epsilon), x_k, x_{k+1})} \\ &\leq w^{(m+1)K}\epsilon^m \prod_{k=0}^{n-1} \frac{1}{P_*(\theta(\epsilon), x_k, x_{k+1})} \end{aligned}$$

by (32). By (18), we can find a constant $0 < h < 1$ such that $P_*(\theta(\epsilon), x, y) \geq h$ for all sufficiently small $\epsilon > 0$ since $|E| < \infty$. This implies that

$$L_*(\theta(\epsilon))(x_0, \ldots, x_n) \leq w^{(m+1)K}\epsilon^m \left(\frac{1}{h}\right)^n \leq \left(\frac{w}{h}\right)^{(m+1)K}\epsilon^m \tag{41}$$

by (23). Also, we can show that

$$L_*(\theta(\epsilon))(x_0, \ldots, x_n) \sim \epsilon^m \tag{42}$$

when $(x_0, \ldots, x_n) \in \Delta_m$, $m \geq r$.

Consider any path $(x_0, \ldots, x_n) \in \Delta^i$, $n \geq 1$. By (42), we can express

$$L_*(\lambda(\epsilon), \mu)(x_0, \ldots, x_n) = u(x_0, \ldots, x_n)\epsilon^{r_i} + \upsilon(x_0, \ldots, x_n),$$

where

$$u(x_0, \ldots, x_n) = \begin{cases} c(x_0, \ldots, x_n) & \text{if } L_*(\lambda(\epsilon), \mu)(x_0, \ldots, x_n) = c(x_0, \ldots, x_n)\epsilon^{r_i} + o(\epsilon^{r_i}), \\ & \text{where } c(x_0, \ldots, x_n) > 0 \\ & \\ 0 & \text{if } L_*(\lambda(\epsilon), \mu)(x_0, \ldots, x_n) = o(\epsilon^{r_i}) \end{cases},$$

and

$$\upsilon(\epsilon, x_0, \ldots, x_n) = L_*(\lambda(\epsilon), \mu)(x_0, \ldots, x_n) - u(x_0, \ldots, x_n)\epsilon^{r_i}.$$

Note that $u(x_0, \ldots, x_n)$ is independent of $\epsilon$ and $\upsilon(\epsilon, x_0, \ldots, x_n) = o(\epsilon^{r_i})$ for all $(x_0, \ldots, x_n) \in \Delta^i$. Since we are working with discrete point distributions,

$$P_{\theta(\epsilon)} \left\{ \left( \sum_{k=0}^{\tau-1} r(X_k, X_{k+1}) \right)^2 u(X_0, \ldots, X_\tau) \in \cdot \;\middle|\; \tau_i \leq \tau_F < \tau_0 \right\}$$
$$\Rightarrow \quad P_0^i \left\{ \left( \sum_{k=0}^{\tau-1} r(X_k, X_{k+1}) \right)^2 u(X_0, \ldots, X_\tau) \in \cdot \right\}$$

as $\epsilon \to 0$, where $P_0^i$ is the limiting distribution in Theorem 13. As we argued in the proof of Lemma 14, all sample paths satisfying $\tau_i \leq \tau_F < \tau_0$ must include some transition $(x, y)$ such that $r(x, y) \neq 0$. Furthermore, all paths $(x_0, \ldots, x_n) \in \Delta_{r_i}^i$ have $u(x_0, \ldots, x_n) > 0$ by (42). Therefore, there exists some constant $\tilde{M}_i < \infty$ such that $P_0^i \left\{ \left( \sum_{k=0}^{\tau-1} r(X_k, X_{k+1}) \right)^2 u(X_0, \ldots, X_\tau) < \tilde{M}_i \right\} = 1$ since $|\Delta_{r_i}^i| < \infty$ and by (23).

Using an argument similar to that employed in the proof of Lemma 14, we can show that

$$P_{\theta(\epsilon)}\{\epsilon^{2b_i - 2r_i} \; (\partial_{\lambda_i} L(\lambda(\epsilon), \mu))^2 \; L_*(\lambda(\epsilon), \mu) \in \cdot \mid \tau_i \leq \tau_F < \tau_0\} \Rightarrow P_0^i\{{L'_{\lambda_i}}^2 \; L_i \in \cdot \}$$

as $\epsilon \to 0$, where $L'_{\lambda_i}$ is defined in Lemma 14 and $L_i = u(X_0, \ldots, X_\tau)$. Likewise, using (41), we can prove that

$$E_{\theta(\epsilon)}[\epsilon^{2b_i - 2r_i} \; (\partial_{\lambda_i} L(\lambda(\epsilon), \mu))^2 \; L_*(\lambda(\epsilon), \mu) \mid \tau_i \leq \tau_F < \tau_0] \to E_0^i[{L'_{\lambda_i}}^2 L_i]$$

as $\epsilon \to 0$ in the same manner we established Lemma 15. The result then easily follows. ∎

# References


Anantharam, V., "How Large Delays Build Up in a GI/G/1 Queue," *Queueing Systems*, 5 (1990), 345–368.

Anantharam, V., P. Heidelberger, and P. Tsoucas, "Analysis of Rare Events in Continuous Time Markov Chains Via Time Reversal and Fluid Approximation," Research Report RC 16280, IBM Research Division, T. J. Watson Research Center, Yorktown Heights, NY, (1990).

Arisimov, V. V. and J. Sztrik, "Asymptotic Analysis of Some Controlled Finite Source Queuing Systems," *Acta Cybernetica*, 9 (1989), 27–38.

Asmussen, S., "Conditioned Limit Theorems Relating a Random Walk to its Associate," *Adv. in Appl. Probab.*, 14 (1982), 143–170.

Billingsley, P., *Probability and Measure*, Second Edition. John Wiley, New York, 1986.

Crane, M. A. and D. L. Iglehart, "Simulating Stable Stochastic Systems, II: Markov Chains," *J. Assoc. Comput. Mach.*, 21 (1974), 114–123.

Csiszár, I., T. M. Cover, and B.-S. Choi, "Conditional Limit Theorems under Markov Conditioning," *IEEE Transactions on Information Theory*, 33 (1987), 788–801.

Fox, B. L. and P. W. Glynn, "Discrete Time Conversion For Simulating Semi-Markov Processes," *Oper. Res. Lett.*, 5 (1986), 191–196.

Gertsbakh, I. B., "Asymptotic Methods in Reliability Theory: A Review," *Adv. in Appl. Probab.*, 16 (1984), 147–175.

Glasserman, P., *Gradient Estimation Via Perturbation Analysis*, Kluwer Academic, Norwell, Massachusetts, 1990a.

Glasserman, P., "Stochastic Monotonicity and Conditional Monte Carlo for Likelihood Ratios," *Adv. in Appl. Probab.*, 25 (1993), 103–115.

Glasserman, P., "Derivative Estimates From Simulation of Continuous Time Markov Chains," *Oper. Res.*, 40 (1992), 292–308.

Glynn, P. W., "Stochastic Approximation For Monte Carlo Optimization," *Proceedings of the 1986 Winter Simulation Conference*, Wilson, J., Henriksen, J. and Roberts, S. (eds.), IEEE Press, 1986, 356–364.

Glynn, P. W., "Likelihood Ratio Gradient Estimation: An Overview," *Proceedings of the 1987 Winter Simulation Conference*, Thesen, A., Grant, H. and Kelton, W. D. (eds.), IEEE Press, 1987, 366–375.

Glynn, P. W., "Likelihood Ratio Derivative Estimators for Stochastic Systems," *Comm. ACM*, 33 (1990), 75–84.

Glynn, P. W. and D. L. Iglehart, "Importance Sampling for Stochastic Simulations," *Management Sci.*, 35 (1989), 1367–1393.

Glynn, P. W. and W. Whitt, "Asymptotic Efficiency of Simulation Estimators," *Oper. Res.*, 40 (1992), 505–520.

Gnedenko, D. B. and A. D. Solovyev, "Estimation of the Reliability of Complex Renewable Systems," *Engineering Cybernetics*, 13-3 (1975), 89–96.

Goyal, A., P. Shahabuddin, P. Heidelberger, V. F. Nicola, and P. W. Glynn, "A Unified Framework for Simulating Markovian Models of Highly Dependable Systems," *IEEE*

*Trans. Comput.*, C-41 (1992), 36–51.

GOYAL, A. AND S. S. LAVENBERG, “Modelling and Analysis of Computer System Availability,” *IBM Journal of Res. and Dev.*, 31 (1987), 651–664.

HEIDELBERGER, P., X. R. CAO, M. A. ZAZANIS, AND R. SURI, “Convergence Properties of Infinitesimal Perturbation Analysis Estimates,” *Management Sci.*, 34 (1988), 1281–1302.

HO, Y. C., “Performance Evaluation and Perturbation Analysis of Discrete Event Dynamic Systems,” *IEEE Trans. Automat. Control*, AC-17 (1987), 563–572.

HO, Y. C. AND X. R. CAO, “Optimization and Perturbation Analysis of Queueing Networks,” *J. Optim. Theory Appl.*, 17 (1983), 559–582.

HORDIJK, A., D. L. IGLEHART, AND R. SCHASSBERGER, “Discrete-Time Methods For Simulating Continuous-Time Markov Chains,” *Adv. in Appl. Probab.*, 8 (1976), 772–788.

KATEHAKIS, M. N. AND C. DERMAN, “On the Maintenance of Systems Composed of Highly Reliable Components,” *Management Sci.*, 35 (1989), 551–560.

KEILSON, J., *Markov Chain Models — Rarity and Exponentiality*, Springer-Verlag, New York, 1979.

L’ECUYER, P., “A Unified View of the IPA, SF, and LR Gradient Estimation Techniques,” *Management Sci.*, 36 (1990), 1364–1383.

L’ECUYER, P., N. GIROUX, AND P. W. GLYNN, “Stochastic Optimization by Simulation: Some Experiments with a Simple Steady-State Queue,” *Management Sci.*, to appear, (1989).

LEWIS, E. E. AND F. BÖHM, “Monte Carlo Simulation of Markov Unreliability Models,” *Nuclear Engineering and Design*, 77 (1984), 49–62.

NAKAYAMA, M. K., *Simulation of Highly Reliable Markovian and Non-Markovian Systems.* Ph.D. Thesis, Dept. of Operations Research, Stanford Univ., Stanford, California, 1991.

NAKAYAMA, M. K., A. GOYAL, AND P. W. GLYNN, “Likelihood Ratio Sensitivity Analysis For Markovian Models of Highly Dependable Systems,” *Oper. Res.*, to appear, (1990).

REIMAN, M. I. AND A. WEISS, “Sensitivity Analysis for Simulations Via Likelihood Ratios,” *Oper. Res.*, 37 (1989), 830–844.

RUBINSTEIN, R. Y., “The Score Function Approach for Sensitivity Analysis of Computer Simulation Models,” *Math. Comput. Simulation*, 28 (1986), 351–379.

RUBINSTEIN, R. Y., “Sensitivity Analysis and Performance Extrapolation for Computer Simulation Models,” *Oper. Res.*, 37 (1989), 72–81.

RUDIN, W., *Principles of Mathematical Analysis*, Third Edition. McGraw-Hill, Inc., New York, 1976.

SHAHABUDDIN, P., *Simulation and Analysis of Highly Reliable Systems.* Ph.D. Thesis, Dept. of Operations Research, Stanford Univ., Stanford, California, 1990.

SHAHABUDDIN, P., “Importance Sampling for the Simulation of Highly Reliable Markovian Systems,” *Management Sci.*, to appear, (1991).

SHAHABUDDIN, P., V. F. NICOLA, P. HEIDELBERGER, A. GOYAL, AND P. W. GLYNN, “Variance Reduction in Mean Time to Failure Simulations,” *Proceedings of the 1988 Winter Simulation*

*Conference*, Abrams, M. A., Haigh, P. L. and Comfort, J. C. (eds.), IEEE Press, 1988, 491–499.

Suri, R., "Perturbation Analysis: The State of the Art and Research Issues Explained via the GI/G/1 Queue," *Proc. IEEE*, 77 (1989), 114–137.

Suri, R. and M. A. Zazanis, "Perturbation Analysis Gives Strongly Consistent Sensitivity Estimates for the M/G/1 Queue," *Management Sci.*, 34 (1988), 39–64.

Sztrik, J., "Asymptotic Reliability Analysis of Some Complex Systems with Repair Operating in Random Environments," *J. Inf. Process*, EIK 25 (1989), 37–43.

Ushakov, I. A. (editor), *Reliability of Complex Systems Handbook*, Radio i Svjaz, Moscow (in Russian), 1985.