Monte Carlo Estimation of Economic Capital

Yajuan Li[1], Zachary T. Kaplan[1], and Marvin K. Nakayama[1]

[1]Computer Science Department
New Jersey Institute of Technology

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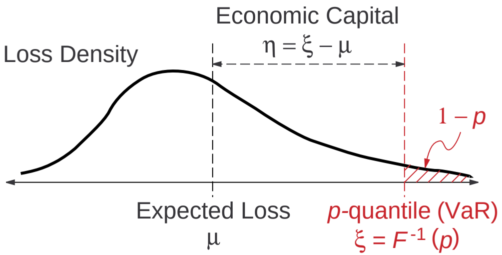

- Credit portfolio
  - $m = 10^3$ or 10^4 obligors: loans, bonds, etc., subject to default
  - Obligors dependent
  - Determine capital to protect against large losses with high probability.
- **Goal:** use Monte Carlo to estimate **economic capital** $\eta = \xi - \mu$
  - $\xi = F^{-1}(p)$ is p-quantile or value-at-risk (VaR) of loss CDF F.
  - Deutsche Bank (2018): $p = 0.999$ or 0.9998

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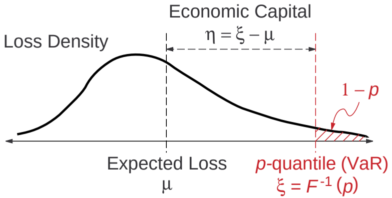
Notation



- Loss $Y = c(\boldsymbol{X}) \sim F$ over some time horizon (e.g., 1 year)
  - $c : \Re^d \to \Re$, with $\Re^d$-valued $\boldsymbol{X} \sim G$.
  - Factor model: Glasserman & Li (2005), Bassamboo et al. (2008)
- Unknown
  - **CDF:** F with derivative f (when it exists)
  - **Mean:** $\mu = \mathbb{E}[Y]$
  - ***p*-quantile (value-at-risk):** $\xi = F^{-1}(p) = \inf\{\, x : F(x) \geq p \,\}$
  - **Economic capital (EC):** $\eta = \xi - \mu$
    - Klaassen & van Eeghen (2009), Lütkebohmert (2009), Scandizzo (2016)
    - AKA **credit**, **relative** or **mean-adjusted VaR**: Jorion (2003,2007), McNeil et al. (2015)

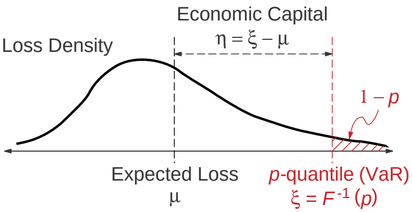
Simple Random Sampling (SRS)

- Generate inputs $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ i.i.d. from G, compute loss $Y_i = c(\boldsymbol{X}_i) \sim F$.

Estimand	**Expression**	**SRS Estimator**
Mean	$\mu = \mathbb{E}[\,Y\,]$	$\widehat{\mu}_{\mathrm{SRS},n} = \frac{1}{n}\sum_{i=1}^{n} Y_i$
CDF	$F(y) = \mathbb{P}(Y \le y) = \mathbb{E}[\,I(Y \le y)\,]$	$\widehat{F}_{\mathrm{SRS},n}(y) = \frac{1}{n}\sum_{i=1}^{n} I(Y_i \le y)$
p-quantile	$\xi = F^{-1}(p)$	$\widehat{\xi}_{\mathrm{SRS},n} = \widehat{F}_{\mathrm{SRS},n}^{-1}(p)$
EC	$\eta = \xi - \mu$	$\widehat{\eta}_{\mathrm{SRS},n} = \widehat{\xi}_{\mathrm{SRS},n} - \widehat{\mu}_{\mathrm{SRS},n}$

- $\widehat{\eta}_{\mathrm{SRS},n}$ satisfies CLT as $n \to \infty$.

- $\widehat{\eta}_{\mathrm{SRS},n} = \widehat{\xi}_{\mathrm{SRS},n} - \widehat{\mu}_{\mathrm{SRS},n}$ has large variance, $p \approx 1$.
- Recall: $Y = c(\boldsymbol{X}) \sim F, \quad \boldsymbol{X} \sim G.$
- **Importance Sampling** (IS) [Glynn 1996]
  - Sample $\boldsymbol{X} \sim H$ so event of interest more likely.
  - Unbias results by multiplying by correction factor.



- Rewrite **tail CDF** $1 - F(y) = \mathbb{E}[\, I(Y > y)\,]$ using change of measure

$$1 - F(y) = \mathbb{E}_G\left[I\Big(c(\boldsymbol{X}) > y\Big) \right] = \int I\Big(c(\boldsymbol{x}) > y\Big)\, \mathrm{d}G(\boldsymbol{x})$$

$$= \int I\Big(c(\boldsymbol{x}) > y\Big) \frac{\mathrm{d}G(\boldsymbol{x})}{\mathrm{d}H(\boldsymbol{x})}\, \mathrm{d}H(\boldsymbol{x}) = \mathbb{E}_H\left[I\Big(c(\boldsymbol{X}) > y\Big)\, L(\boldsymbol{X}) \right]$$

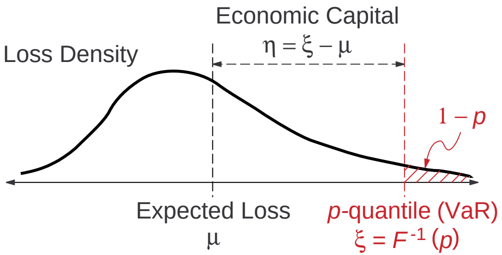
where $L(\boldsymbol{x}) = \frac{\mathrm{d}G(\boldsymbol{x})}{\mathrm{d}H(\boldsymbol{x})}$ is likelihood ratio (LR).

Importance Sampling (IS)

- IS algorithm: generate $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ i.i.d. H

Estimand	Expression	IS Estimator
CDF	$F(y) = 1 - \mathbb{E}_H\left[\, I\left(c(\boldsymbol{X}) > y\right) L(\boldsymbol{X})\,\right]$	$\widehat{F}_{\mathrm{IS},n}(y) = 1 - \frac{1}{n}\sum_{i=1}^{n} I\Big(c(\boldsymbol{X}_i) > y\Big)\, L(\boldsymbol{X}_i)$
p-quantile	$\xi = F^{-1}(p)$	$\widehat{\xi}_{\mathrm{IS},n} = \widehat{F}^{-1}_{\mathrm{IS},n}(p)$
Mean	$\mu = \mathbb{E}_G\left[\, c(\boldsymbol{X})\,\right] = \mathbb{E}_H\left[\, c(\boldsymbol{X})\, L(\boldsymbol{X})\,\right]$	$\widehat{\mu}_{\mathrm{IS},n} = \frac{1}{n}\sum_{i=1}^{n} c(\boldsymbol{X}_i)\, L(\boldsymbol{X}_i)$
EC	$\eta = \xi - \mu$	$\widehat{\eta}_{\mathrm{IS},n} = \widehat{\xi}_{\mathrm{IS},n} - \widehat{\mu}_{\mathrm{IS},n}$

- $\widehat{\eta}_{\mathrm{IS},n}$ obeys CLT as $n \to \infty$.



- SRS: Estimates μ well, but ξ poorly
- IS: Estimates ξ well, but μ poorly
- Combine IS and SRS
  - Measure-specific IS (MSIS) [Shahabuddin et al. 1988]
  - IS with defensive mixture (ISDM) [Hesterberg 1995, Owen & Zhou 2000]
  - Double estimator (DE)

Measure-Specific Importance Sampling (MSIS)

- **Measure-specific IS** (MSIS) [Shahabuddin et al. 1988]
  - Estimate ξ using IS.
  - Independently estimate μ using SRS.
- Fix overall sample size n and allocation $\delta \in (0, 1)$.

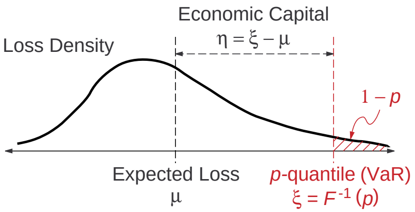
Method	Sample Size	Estimators
IS	δn	$\widehat{\xi}_{\mathrm{IS},\delta n}$
SRS	$(1-\delta)n$	$\widehat{\mu}_{\mathrm{SRS},(1-\delta)n}$

- MSIS EC estimator $\widehat{\eta}_{\mathrm{MSIS},n} = \widehat{\xi}_{\mathrm{IS},\delta n} - \widehat{\mu}_{\mathrm{SRS},(1-\delta)n}$
  - CLT as $n \to \infty$.

- Problem with IS: LR $L(\boldsymbol{x}) = \frac{dG(\boldsymbol{x})}{dH(\boldsymbol{x})}$ can be huge.
- Instead sample $\boldsymbol{X}$ from **mixture distribution**:

$$\boldsymbol{X} \sim H_{\text{ISDM}} = \delta H + (1-\delta) G$$

[Hesterberg 1995, Owen and Zhou 2000]



- **IS with defensive mixture** (ISDM)

$$L_{\text{ISDM}}(\boldsymbol{x}) = \frac{dG(\boldsymbol{x})}{dH_{\text{ISDM}}(\boldsymbol{x})} = \frac{dG(\boldsymbol{x})}{\delta\, dH(\boldsymbol{x}) + (1-\delta)\, dG(\boldsymbol{x})} \le \frac{1}{1-\delta}$$

- ISDM algorithm: generate $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ i.i.d. H_{ISDM}
  - Estimate both ξ and μ from ISDM data.
- ISDM EC estimator $\widehat{\eta}_{\text{ISDM},n} = \widehat{\xi}_{\text{ISDM},n} - \widehat{\mu}_{\text{ISDM},n}$
  - CLT: special case of IS

- Use **both** IS and SRS to estimate **both** ξ and μ.

Method	Sample Size	Estimators		
IS	δn	$\widehat{\xi}_{\mathrm{IS},\delta n}$	and	$\widehat{\mu}_{\mathrm{IS},\delta n}$
SRS	$(1-\delta)n$	$\widehat{\xi}_{\mathrm{SRS},(1-\delta)n}$	and	$\widehat{\mu}_{\mathrm{SRS},(1-\delta)n}$

- **DE: linear combination** of the 4 estimators using weights $\upsilon_1, \upsilon_2 \in [0, 1]$,

$$\widehat{\eta}_{\mathrm{DE},n} = \underbrace{\left[\upsilon_1\ \widehat{\xi}_{\mathrm{IS},\delta n} + (1-\upsilon_1)\ \widehat{\xi}_{\mathrm{SRS},(1-\delta)n}\right]}_{\widehat{\xi}_{\mathrm{DE},n}} - \underbrace{\left[\upsilon_2\ \widehat{\mu}_{\mathrm{IS},\delta n} + (1-\upsilon_2)\ \widehat{\mu}_{\mathrm{SRS},(1-\delta)n}\right]}_{\widehat{\mu}_{\mathrm{DE},n}}$$

  - CLT as $n \to \infty$.
  - Derived optimal weights υ_1, υ_2 to minimize $\mathrm{Var}[\widehat{\eta}_{\mathrm{DE},n}]$.

Asymptotic Analysis of i.i.d. Sum Model

- Compare 5 methods
  - **SRS** Simple random sampling
  - **IS** Importance sampling
  - **MSIS** Measure-specific importance sampling
  - **ISDM** IS with defensive mixture
  - **DE** Double estimator
- Loss: $Y \equiv Y_m = \sum_{k=1}^{m} X_k \sim F_m$ with density f_m
  - $X_k \sim G_0$ light tailed
  - $Q_0(\theta) = \ln \mathbb{E}[e^{\theta X_k}]$ is CGF of G_0
  - $Q_0'(\theta) = \frac{d}{d\theta} Q_0(\theta)$
- EC $\eta_m = \xi_m - \mu_m$
- Analyze as $m \to \infty$
  - Quantile level $p \equiv p_m = 1 - e^{-\beta m}$, fixed $\beta > 0$ [Glynn 1996]
- IS via exponential twist
  - i.i.d. $X_k \sim \widetilde{G}_{0,\theta}, \quad \mathrm{d}\widetilde{G}_{0,\theta}(x) = e^{\theta x - Q_0(\theta)} \, \mathrm{d}G_0(x)$
  - Glynn (1996): Estimate ξ_m with $\theta = \theta_\star$ as root of

$$-\theta_\star Q_0'(\theta_\star) + Q_0(\theta_\star) = -\beta$$

Asymptotic Analysis of i.i.d. Sum Model

- MSIS, ISDM, DE: fixed $\delta, \upsilon_1, \upsilon_2 \in (0,1)$ as $m \to \infty$.
- For generic estimand φ_m, compare estimators $\widehat{\varphi}_m$ in terms of *relative error* (RE)

$$\mathrm{RE}[\widehat{\varphi}_m] = \frac{\sqrt{\mathrm{Var}[\widehat{\varphi}_m]}}{|\varphi_m|}$$

- *Approximate RE* (R$\check{\mathrm{E}}$) for EC and ξ
  - **Quantile approximation** [Glynn 1996]

$$\check{\xi}_m = mQ_0'(\theta_\star), \quad \text{which satisfies} \quad \frac{\check{\xi}_m - \xi_m}{m} \to 0 \text{ as } m \to \infty$$

  - **Saddlepoint approximation** [Jensen 1995] to density f_m

$$\check{f}_m(x) = \frac{1}{\sqrt{2\pi mQ_0''(\theta_x)}} \exp\left[mQ_0(\theta_x) - x\theta_x\right], \quad \text{for } mQ_0'(\theta_x) = x$$

    - $f_m(\xi_m)$ appears in $\mathrm{Var}[\widehat{\xi}_m]$ and $\mathrm{Var}[\widehat{\eta}_m]$

Asymptotic Analysis of i.i.d. Sum Model

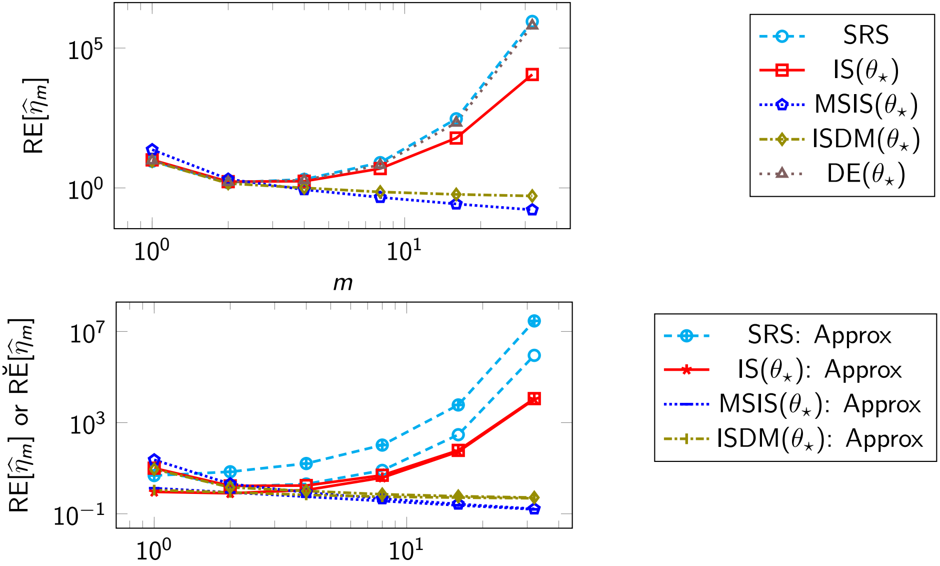
Theorem

- *Suppose loss* $Y_m = \sum_{k=1}^{m} X_k$, *and quantile level* $p_m = 1 - e^{-\beta m}$, $\beta > 0$.
- *Under regularity conditions, the estimators satisfy the following as* $m \to \infty$*:*

Method	Approx Relative Error ($\check{RE}$)		
	Mean μ_m $(\theta \neq 0)$	**Quantile** ξ_m $(\theta = \theta_\star)$	**EC** η_m $(\theta = \theta_\star)$
SRS	$O(m^{-1/2})$	*Expo* $\uparrow$	*Expo* $\uparrow$
IS	*Expo* $\uparrow$	$O(m^{-1/2})$	*Expo* $\uparrow$
MSIS	$O(m^{-1/2})$	$O(m^{-1/2})$	$O(m^{-1/2})$
ISDM	$O(1)$	$O(m^{-1/2})$	$O(1)$
DE	*Expo* $\uparrow$	*Expo* $\uparrow$	*Expo* $\uparrow$

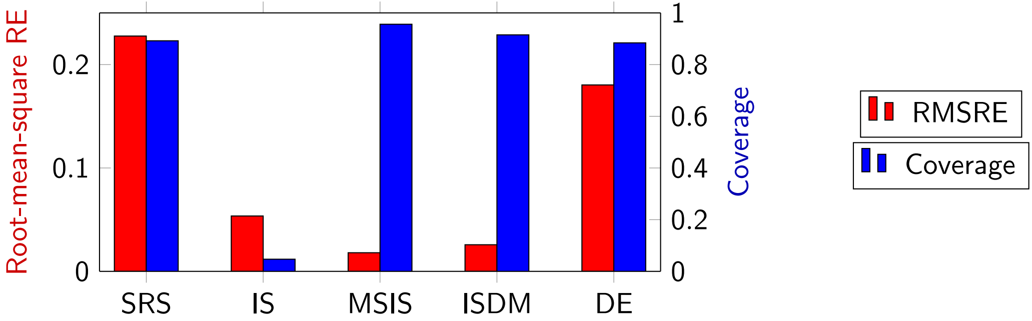
- "*Expo* $\uparrow$" = *exponentially increasing in m*

Numerical (Non-Simulation) Results: i.i.d. sum model with $X_k \sim$ exponential(1)



Credit portfolio with $m = 1000$ **dependent** obligors, 10 factors, Gaussian copula

- EC with quantile level $p = 0.999$
- IS: modification of Glasserman and Li (2005) for estimating $\mathbb{P}(Y > x)$
- Root-mean-square relative error (RMSRE) $\sqrt{\mathbb{E}[(\widehat{\eta}_n - \eta)^2]}/\eta$ for $n = 2000$
- Coverage of nominal 95% confidence intervals (10^3 indep reps)
  - IS results unreliable: coverage ≈ 0.05



"Green Simulation"

- Feng & Staum (2017), Dong, Feng & Nelson (2018), ...
- Goal: for $c : \Re^d \to \Re$ "expensive" to compute, estimate

$$\mu(\theta) \equiv \mathbb{E}_{G_\theta}\left[\, c(\boldsymbol{X}) \,\right] = \int c(\boldsymbol{x}) \, \mathrm{d}\, G_\theta(\boldsymbol{x}), \quad \forall\, \theta \in \Theta.$$

- Idea: **reuse** existing data $(c(\boldsymbol{X}_i), \boldsymbol{X}_i)$ for $\boldsymbol{X}_i \sim G_{\theta_0}$ by change of measure

$$\mu(\theta) = \int c(\boldsymbol{x}) \, \frac{\mathrm{d}\, G_\theta(\boldsymbol{x})}{\mathrm{d}\, G_{\theta_0}(\boldsymbol{x})} \, \mathrm{d}\, G_{\theta_0}(\boldsymbol{x}) = \int c(\boldsymbol{x}) \, \frac{\mathrm{d}\, G_\theta(\boldsymbol{x})}{\mathrm{d}\, G_{\mathrm{ISDM},\theta,\theta_0}(\boldsymbol{x})} \, \mathrm{d}\, G_{\mathrm{ISDM},\theta,\theta_0}(\boldsymbol{x})$$

  to get unbiased estimator of $\mu(\theta)$, where

$$G_{\mathrm{ISDM},\theta,\theta_0} = \delta\, G_{\theta_0} + (1 - \delta)\, G_\theta$$

- i.i.d. sum $c(\boldsymbol{X}) = \sum_{k=1}^{m} X_k$ for **any** $\theta \neq \theta_0$ as $m \to \infty$:

Method	RE of $\widehat{\mu}_m(\theta)$
IS	Expo ↑
ISDM	$O(1)$

Concluding Remarks

- EC: $\eta = F^{-1}(p) - \mu$ for $p \approx 1$
- i.i.d. sum model: theoretical analysis
  - MSIS has **vanishing** relative error (RE), and ISDM has **bounded** RE
  - SRS, IS, and DE have **unbounded** RE
- Portfolio credit risk model with dependent obligors
  - Similar empirical behavior
- "Green simulation"

Questions?