A Characterization of the Simple Failure-Biasing Method for Simulations of Highly Reliable Markovian Systems

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Simple failure biasing is an importance-sampling technique used to reduce the variance of estimates of performance measures and their gradients in simulations of highly reliable Markovian systems. Although simple failure biasing yields bounded relative error for the performance measure estimate when the system is balanced, it may not provide bounded relative error when the system is unbalanced.

In this article, we provide a characterization of when the simple failure-biasing method produces estimators of a performance measure and its derivatives with bounded relative error. We derive a necessary and sufficient condition on the structure of the system for when the performance measure can be estimated with bounded relative error when using simple failure biasing. Furthermore, a similar condition for the derivative estimators is established. One interesting aspect of the conditions is that it shows that to obtain bounded relative error, not only the most likely paths to system failure must be examined but also some secondary paths leading to failure as well. We also show by example that the necessary and sufficient conditions for a derivative estimator do not imply those for the performance measure estimator; i.e., it is possible to estimate a derivative more efficiently than the performance measure when using simple failure biasing.

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1. INTRODUCTION

There is an increasing demand for systems, such as transaction-processing systems and communications systems, that have high levels of reliability. It is thus of utmost importance to be able to predict during the design stage the

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reliability of a system, to ensure that the actual system will perform at an acceptable level. Typically, a designer constructs a stochastic model of the system, and with methods to analyze it, he or she can identify critical components and explore different design tradeoffs. Due to the typically large state spaces of models of these types of systems, numerical methods are not practical, and so the designer must resort to using simulation. However, standard simulation (i.e., simulation without the use of any variance reduction techniques) is inefficient because of the rareness of system failures. Hence, the need arises for effective variance reduction techniques to be used in simulations of any proposed reliable systems.

One such method is importance sampling; see Glynn and Iglehart [1989] for an overview. The basic idea behind this approach is to change the dynamics (i.e., the underlying probability distributions) governing the system so as to make the event of interest (in our case, system failures) occur more frequently. This is known as a "change of measure" since the system is now simulated under a different probability measure. To obtain unbiased estimates, we must multiply our estimator by a correction factor called the likelihood ratio. The proper choice of importance-sampling distribution can lead to a large variance reduction. However, selecting an appropriate change of measure is not straightforward since it depends on the system being simulated, and if an unsuitable importance-sampling distribution is chosen, we may increase the variance. Thus, the thrust of the research in this area is to determine appropriate importance-sampling distributions.

For highly reliable Markovian systems, there is a class of importance-sampling techniques available known as failure-biasing schemes. These include simple failure biasing (Lewis and Böhm [1984], Goyal et al. [1992], Shahabuddin et al. [1988]), bias2 failure biasing (Goyal et al. [1992]), balanced failure biasing (Shahabuddin [1994] and Goyal et al. [1992]), and failure distance biasing (Carrasco [1991; 1992]). The basic idea behind each of these methods is as follows. From any state having both repair and failure transitions, the probability of choosing a failure transition over a repair transition is very small under the original dynamics of the system. Thus, in failure biasing, we increase the total probability of a failure transition to δ, where $0 \ll \delta < 1$ and decrease the total probability of a repair transition to $1 - \delta$. The various failure-biasing schemes differ in the way they allocate δ to the individual failure transitions. In this article we provide a mathematical analysis of the simple failure-biasing method, from which we will gain insight into other techniques.

Shahabuddin [1994] was the first to study mathematically the asymptotic properties of performance measure estimates obtained using failure-biasing methods. In particular, Shahabuddin proved that when using balanced failure biasing, certain performance measure estimates always have bounded relative error (i.e., for a fixed number of samples, the expected width of the relative confidence interval for the performance measure estimate remains bounded as the failure rates of the components tend to zero). Shahabuddin also showed that if the system being simulated is balanced (i.e., all of the failure rates of the components are of the same order of magnitude), then using simple failure biasing will give rise to bounded relative error for the performance measure estimate. Furthermore, Shahabuddin devised an example showing that if simple failure biasing is applied to a system which is unbalanced, then the relative error may increase without bound as the failure rates vanish.

Since balanced failure biasing always yields performance measure estimators that have bounded relative error and simple failure biasing does not, balanced failure biasing is a more robust method. However, we show both by example and empirically that when simple failure biasing does give rise to bounded relative error, it sometimes yields a smaller variance than balanced failure biasing. Thus, in certain contexts, simple failure biasing may be more appropriate than balanced failure biasing, and this is an important part of our motivation for now studying simple failure biasing.

It was not previously established exactly when simple failure biasing leads to bounded relative error for the performance measure estimate. In this article we develop a necessary and sufficient condition on the structure of the system that characterizes when simple failure biasing yields bounded relative error for the performance measure estimate. One interesting aspect of this condition is that in order to determine if simple failure biasing will give bounded relative error, we must examine not only the most likely paths to system failure but also some secondary paths to failure as well. (The total number of paths needed to be considered is finite.) This is in contrast to what others have encountered when analyzing the balanced failure-biasing methodology. For example, by considering the most likely paths to failure, Shahabuddin [1990, Section 2.5.1] derived bounds on the variance of the performance measure estimate when using balanced failure biasing. Also, Nakayama [1991] formalized the idea of "most likely paths to failure" into a theorem and used it to establish results about derivative estimators obtained using balanced failure biasing.

In some sense the need to examine secondary paths to failure in our current setting illustrates the difference between the importance-sampling distributions we consider here and those used to simulate large-deviations-type rare events, such as buffer overflows and excessive backlogs in queuing networks. Specifically, in the large-deviations context, the optimal (i.e., minimum variance subject to certain constraints) importance-sampling distribution is selected solely with regard to the most likely path to failure; e.g., see Cottrell et al. [1983] and Parekh and Walrand [1989]. On the other hand, our necessary and sufficient condition for simple failure biasing shows that we must take into account the secondary paths to failure when designing an importance-sampling scheme for the types of reliability systems studied in this article.

We also consider estimating derivatives with respect to the failure rates of the components using the likelihood ratio method. (For details on the likelihood ratio method of derivative estimation, see Glynn [1986], Reiman and Weiss [1989], and Rubinstein [1989].) We establish a necessary and sufficient condition that characterizes when simple failure biasing will give bounded relative error for the estimate of a derivative. The condition for the performance measure differs from that for the derivatives. In fact, we show by an example that the condition for one derivative may hold whereas the condition for the performance measure does not. Thus, by using simple failure biasing, we may actually obtain better estimates for a derivative than for the performance measure itself. However, we prove that if we can estimate all of the derivatives with bounded relative error, then we can also do so for the performance measure. This situation contrasts that which arises when using balanced failure biasing to estimate derivatives since Nakayama [1991] proved that with balanced failure biasing, we can always estimate derivatives with respect to any component failure rate with bounded relative error and with the same relative error as the performance measure estimate.

We also establish a number of other conditions which are used to further analyze the derivatives. Finally, we apply all of our conditions to examine several examples to give a characterization of the simple failure-biasing method.

The rest of the article is organized as follows. In Section 2 we describe the mathematical model of a highly reliable Markovian system. Section 3 reviews the basic concepts of importance sampling and explicitly describes the simple failure-biasing and balanced failure-biasing methods. We then examine some simple examples to motivate why we study simple failure biasing. In Section 4 we present some tables which summarize all of the results shown in this article. Section 5 contains the analysis of estimating the performance measure using the simple failure-biasing method. In Section 6 we examine estimators of the derivatives using both standard simulation and simple failure biasing. We also consider several examples in order to complete our characterization of simple failure biasing. Section 7 presents some empirical results from simulating a large computing system. We state our conclusions and some directions for future research in Section 8. All of the proofs are placed in the Appendix.

2. MATHEMATICAL MODEL AND NOTATION

The following is a modification of a model originally developed by Shahabuddin [1994] to study performance measures of highly reliable systems and later adapted by Nakayama [1991] to analyze derivatives. We assume that the system is composed of C different types of components, labeled $1, \ldots, C$, with n_i components of type i. We let $N \equiv \sum_{i=1}^{C} n_i$, which is the total number of components of all types in the system. As time evolves, components randomly fail and are sent to a repair facility to be repaired, where the type of service discipline for the failed components is arbitrary.

A state x of the system keeps track of the number of failed components of each type and any information about the queuing at the repair facility, and we let S be the state space of all such states, where we assume that $|S| < \infty$. Our analysis in the following sections is independent of the actual form of the state space. We define $n_i(x)$ to be the number of components of type i to be operational when the system is in state x. We decompose the state space as $S = U \cup F$, where U is the set of states for which the system is considered operational, and F is the set of failed states. The system starts in state 0, the state with all components operational, and we assume that $0 \in U$. We also assume that if $x \in U$ and $y \in E$ with $n_i(y) \geq n_i(x)$ for all i, then $y \in U$.

We allow for the possibility of *component failure propagation*; i.e., the failure of one component causes some other components to fail simultaneously. More precisely, consider some component type i and state $x \in S$, and let $S_i(x) \equiv \{y \in S : n_j(y) \leq n_j(x) \text{ for all } j,\ n_i(y) < n_i(x)\}$, which is the set of states y which have at least as many components of each type failed as in state x, with at least one more component of type i failed in state y than in x. Suppose a component of type i fails when the system is in state x, and let $p(\cdot;\ x,\ i)$ be a probability mass function on $S_i(x)$. Then the system immediately enters state $y \in S_i(x)$ with probability $p(y;\ x,\ i)$. Thus, $n_j(x) - n_j(y)$ components of type j, $1 \leq j \leq C$, fail on the transition caused by the failure of the component of type i. In this case, we say that component i *triggered* the transition $(x,\ y)$. This situation may occur, for example, in a computing system, where the failure of a processor contaminates the data and causes the disks to fail simultaneously.

Similarly, we allow for the possibility that the repair facility completes repair on more than one component at a time. This may occur if some component is made up of a number of subcomponents, and the entire component is replaced when enough of the subcomponents fail. However, we do not allow for some components to complete repair and other components to fail at the same instant. (For example, this may happen when the repairperson fixes some component but breaks something else when replacing the repaired component in the system.)

We define a transition $(x,\ y)$ to be a *failure transition* if $n_j(y) \leq n_j(x)$ for all $j = 1, \ldots, C$, with $n_k(y) < n_k(x)$ for some k. We use the notation "$y \succ x$" if $(x,\ y)$ is a failure transition. Furthermore, we define $(x,\ y)$ to be a *repair transition* if $n_j(y) \geq n_j(x)$ for all $j = 1, \ldots, C$, with $n_k(y) > n_k(x)$ for some k. We use the notation "$y \prec x$" if $(x,\ y)$ is a repair transition.

When the system is in state x, the components of type i have exponentially distributed lifetimes with rate $\lambda_i(x) \geq 0$. Also, the system makes a repair transition $(x,\ y)$ with rate $\mu(x,\ y) \geq 0$. We model the evolution of the system as a continuous-time Markov chain $Y = \{Y(s) : s \geq 0\}$ on the state space S. The infinitesimal generator matrix $Q = \{q(x,\ y) : x,\ y \in S\}$ of Y is given by

$$q(x, y) = \begin{cases} \sum_{k=1}^{C} n_k(x)\lambda_k(x)p(y;\ x,\ k) & \text{if } y \succ x \\ \mu(x,\ y) & \text{if } y \prec x \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for $x \neq y$, and $q(x,\ x) = \sum_{y \neq x} q(x,\ y)$. We let

$$q(x) \equiv -q(x,\ x).$$

We denote the embedded discrete-time Markov chain (DTMC) of Y by $X = \{X_n : n \geq 0\}$. The transition matrix of X is then given by $P = \{P(x,\ y) : x,\ y \in S\}$, where $P(x,\ y) = q(x,\ y)/q(x)$ for $x \neq y$, and $P(x,$

$x) = 0$. We define $\Gamma = \{(x, y): x, y \in S, P(x, y) > 0\}$, which is the set of possible transitions the system can make.

We assume that our highly reliable system is composed of highly reliable components (i.e., the component failure rates are much smaller than the repair rates). (High reliability for the system can also be achieved by having high redundancies.) We model this by introducing a rarity parameter ϵ and assume that the failure rates of the components of type i are

$$\lambda_i(x, \epsilon) = \tilde{\lambda}_i(x)\epsilon^{b_i(x)},$$

where $\tilde{\lambda}_i(x) \geq 0$ and $b_i(x) \geq 1$ are independent of ϵ, and $b_i(x)$ is integer valued. Also, we allow $p(\cdot; x, i)$ to depend on ϵ; i.e., for all $(x, y) \in \Gamma$ such that $y \succ x$,

$$p(y; x, i) = p_\epsilon(y; x, i) = c_i(x, y)\epsilon^{d_i(x,y)},$$

where $d_i(x, y) \geq 0$ is integer valued, $c_i(x, y) \geq 0$, and $\sum_{y \in S_i(x)} p_\epsilon(y; x, i) = 1$. We assume that repair rates, $\mu(x, y)$, are independent of ϵ. We will examine the behavior of the system as $\epsilon \to 0$.

We say that the system is *balanced* if all values $\lambda_i(x, \epsilon)$ are of the same order of magnitude (i.e., if for all $1 \leq i \leq C$ and all $x \in S$, $b_i(x) = b$ for some $b \geq 1$) and the $p_\epsilon(y; x, i)$ are independent of ϵ. In this situation, we can assume without loss of generality that $b = 1$. If the system is not balanced, we call it *unbalanced*.

We now make some definitions. For some constant d, a function f is $o(\epsilon^d)$ if $f(\epsilon)/\epsilon^d \to 0$ as $\epsilon \to 0$. Also, we use the notation $f(\epsilon) = \Theta(\epsilon^d)$ if we can express $f(\epsilon) = c\epsilon^d + o(\epsilon^d)$ as $\epsilon \to 0$, where $c \neq 0$ is independent of ϵ. Similarly, we say $f(\epsilon) = \underline{O}(\epsilon^d)$ if we can express $f(\epsilon) = c\epsilon^{\bar{d}} + o(\epsilon^{\bar{d}})$ as $\epsilon \to 0$, where $c \neq 0$ and $\bar{d} \geq d$ is independent of ϵ. Finally, we say $f(\epsilon) = \overline{O}(\epsilon^d)$ if we can express $f(\epsilon) = c\epsilon^{\bar{d}} + o(\epsilon^{\bar{d}})$ as $\epsilon \to 0$, where $c \neq 0$ and $\bar{d} \leq d$ is independent of ϵ.

We define

$$b_0 \equiv \min_{1 \leq i \leq C} b_i(0),$$

and so $q(0) = \Theta(\epsilon^{b_0})$. For any $x \in S$, we define

$$s(x) \equiv \min_{1 \leq i \leq C} \left\{b_i(x): n_i(x)\tilde{\lambda}_i(x) > 0\right\}.$$

Note that $s(x)$ is the exponent of the order of magnitude of the total failure rate out of state x, and so $s(x) = d$ if $\sum_{y \succ x} q(x, y) = \Theta(\epsilon^d)$. Furthermore, $s(0) = b_0$. For any $(x, y) \in \Gamma$, we define

$$b(x, y) \equiv \begin{cases} \min_{1 \leq i \leq C}\left\{b_i(x) + d_i(x, y): n_i(x)\tilde{\lambda}_i(x)p_\epsilon(y; x, i) > 0\right\} & \text{if } y \succ x \\ 0 & \text{if } y \prec x \end{cases},$$

which is the exponent of the order of magnitude of the rate of a transition (x, y). Thus, for any $(x, y) \in \Gamma$, $b(x, y) = d$ if $q(x, y) = \Theta(\epsilon^d)$ and $b(x, y) \geq 1$ if $y \succ x$.

To prove our results, we need to assume some structure on the model being considered.

(A1) *The DTMC X is irreducible over the state space S.*

(A2) *For each state $x \in S$ with $x \neq 0$, there exists $y \in S$ such that $(x, y) \in \Gamma$ and $y \prec x$.*

(A3) *For each state $z \in F$ such that $(0, z) \in \Gamma$, $q(0, z) = o(\epsilon^{b_0})$.*

Assumption A2 states that there is at least one repair transition possible from any state $x \neq 0$. Hence, for $x \neq 0$, $q(x) = \Theta(1)$. This implies that all failure transitions (x, y) with $x \neq 0$ have transition probability $P(x, y) = \Theta(\epsilon^{b(x,y)})$. Assumption A2 is a critical assumption. If it does not hold, then none of the failure-biasing methods discussed in the introduction will work well; see Juneja and Shahabuddin [1992].

Assumption A3 guarantees that transitions which take the system from state 0 directly to a failed state have transition rates that are much smaller than ϵ^{b_0}, which is the magnitude of the largest transition rate from state 0. This ensures that system failures are rare events for the embedded DTMC when ϵ is small.

From these assumptions the elements of the transition matrix have a certain form. For any $(x, y) \in \Gamma$,

$$P(x, y) = \begin{cases} \Theta(1) & \text{if } x \neq 0 \text{ and } y \prec x \\ \Theta(\epsilon^{b(x,y)}) & \text{if } x \neq 0 \text{ and } y \succ x, \\ \Theta(\epsilon^{b(x,y)-b_0}) & \text{if } x = 0 \text{ and } y \succ x \end{cases} \tag{2}$$

as $\epsilon \to 0$.

For any set of states $A \subset S$, define $\tau_A = \inf\{n > 0 : X_n \in A\}$, which is the hitting time to the set A for the DTMC. We concentrate on estimating

$$\gamma \equiv P\{\tau_F < \tau_0\},$$

which is of interest for several reasons. First, we can express the mean time to failure as $MTTF = \xi/\gamma$, where $\xi = E[\sum_{k=0}^{\tau_{\min}-1} 1/q(X_k)]$ and $\tau_{\min} = \min\{n > 0 : X_n \in \{0, F\}\}$; see Goyal et al. [1992]. Also, let $U(t)$ be the unreliability of the system for time horizon t; i.e., $U(t)$ is the probability that the system fails before time t. Then Shahabuddin and Nakayama [1992] prove that $(1 - e^{-q(0)\gamma t})/U(t) \to 1$ as $\epsilon \to 0$, where $t = \Theta(\epsilon^{-r_t})$ with $r_t > 0$.

Shahabuddin [1994] showed that if Assumptions A1–A3 hold, then there exists some constant $r \geq 1$ (which depends on the model being considered) such that

$$\gamma = \Theta(\epsilon^r). \tag{3}$$

We define

$$\Delta = \{(x_0, \ldots, x_n) : n \geq 1,\ x_0 = 0,\ x_n \in F,\ x_i \notin \{0, F\} \text{ for } 1 \leq i < n,\ (x_i, x_{i+1}) \in \Gamma \text{ for } 0 \leq i < n\},$$

which is the set of sample paths of the embedded DTMC for which $\tau_F < \tau_0$.

Furthermore, we define

$$\Delta_m = \Big\{(x_0, \ldots, x_n) \in \Delta : n \geq 1,\ P\big\{(X_0, \ldots, X_{\tau_F}) = (x_0, \ldots, x_n)\big\} = \Theta(\epsilon^m)\Big\},$$

which is the set of sample paths for which $\tau_F < \tau_0$ and have probability (under the original measure) of the order ϵ^m. Note that

$$\Delta = \textstyle\bigcup_{m=r}^{\infty} \Delta_m, \tag{4}$$

where the r is as defined in (3). We call Δ_r the set of "most likely paths to failure" and any path $(x_0, \ldots, x_n) \in \Delta_m$ with $m > r$ a "secondary path to failure."

For any sample path $(x_0, \ldots, x_n) \in \Delta$, we define

$$f(x_0, \ldots, x_n) \equiv \sum_{k=0}^{n-1} 1\{x_{k+1} \succ x_k\} s(x_k),$$

which is the sum of the exponents of the orders of magnitude of the total failure rate out of each state x_k along the path such that $(x_k,\ x_{k+1})$ is a failure transition. The reason for introducing $f(x_0, \ldots, x_n)$ will become more apparent in the later sections. Note that since $s(0) = b_0$ and $x_1 \succ x_0 = 0$ is always true,

$$f(x_0, \ldots, x_n) \geq b_0 \tag{5}$$

as $s(x) \geq 0$ for all $x \in U$. In a balanced system, $f(x_0, \ldots, x_n)$ is the number of failure transitions along the path $(x_0, \ldots, x_n)$.

For our results on derivative estimation, we use the following additional notation. Let

$$\tau_i = \inf\{k > 0 : X_k \succ X_{k-1},\ n_i(X_{k-1}) \lambda_i(X_k) p(X_k; X_{k-1}, i) > 0\},$$

which is the first failure transition which may have been triggered by a failure of a component of type i. Nakayama [1991] showed that there exists constants $r_i \geq r$ and $\bar{r}_i \geq r$ such that

$$P\{\tau_i \leq \tau_F < \tau_0\} = \Theta(\epsilon^{r_i}) \tag{6}$$

and

$$P\{\tau_F < \min\{\tau_i, \tau_0\}\} = \Theta(\epsilon^{\bar{r}_i}). \tag{7}$$

If $P\{\tau_F < \min\{\tau_i, \tau_0\}\} = 0$, then we set $\bar{r}_i = \infty$. Additionally it was established in Nakayama [1991] that

$$\min\{r_i, \bar{r}_i\} = r. \tag{8}$$

Also we define the set

$$\Delta^i = \{(x_0, \ldots, x_n) \in \Delta : n \geq 1,\ n_i(x_k) \lambda_i(x_k) p(x_{k+1}; x_k, i) > 0 \text{ for some } 0 \leq k < n \text{ such that } x_{k+1} \succ x_k\},$$

which is the set of sample paths of the embedded DTMC for which $\tau_i \leq \tau_F < \tau_0$.

Similarly, we define

$$\bar{\Delta}^{\iota} = \{(x_0, \ldots, x_n) \in \Delta : n \geq 1,\ n_{\iota}(x_k)\lambda_{\iota}(x_k)p(x_{k+1}; x_k, i) = 0$$
$$\text{for all } 0 \leq k < n \text{ such that } x_{k+1} \succ x_k\},$$

which is the set of sample paths of the embedded DTMC for which $\tau_F < \min(\tau_0, \tau_{\iota})$. Furthermore, let $\Delta^{\iota}_m = \Delta^{\iota} \cap \Delta_m$ and $\bar{\Delta}^{\iota}_m = \bar{\Delta}^{\iota} \cap \Delta_m$, and note that $\Delta^{\iota} = \bigcup_{m=r_{\iota}}^{\infty} \Delta^{\iota}_m$ and $\bar{\Delta}^{\iota} = \bigcup_{m=\bar{r}_{\iota}}^{\infty} \bar{\Delta}^{\iota}_m$, where r_{ι} and $\bar{r}_{\iota}$ are as defined in (6) and (7), respectively.

3. BACKGROUND AND MOTIVATION

Let $(\Omega, \mathscr{F})$ be the probability space on which X is defined, and let P denote the probability measure on $(\Omega, \mathscr{F})$ induced by the Q-matrix given in (1). Letting $1\{\cdot\}$ denote the indicator function, we can write $\gamma = E[1\{\tau_F < \tau_0\}]$, which leads to the following approach to estimating γ by standard simulation. We first collect n i.i.d. samples $\hat{I}_1, \ldots, \hat{I}_n$ of $1\{\tau_F < \tau_0\}$, where the $\hat{I}_k$ are generated using the original probability measure P. Then we form the point estimate

$$\hat{\gamma}(n) = \frac{1}{n} \sum_{k=1}^{n} \hat{I}_k,$$

and the variance of $1\{\tau_F < \tau_0\}$ under the original probability measure P is

$$\sigma^2 = \gamma - \gamma^2 = \Theta(\epsilon^r) - \Theta(\epsilon^{2r}) = \Theta(\epsilon^r)$$

as $\epsilon \to 0$. We define the relative error RE of our estimator to be the expected relative width of its confidence interval for a fixed number of samples n and a given confidence level $1 - \psi$. Letting z_{ψ} be the $1 - \psi/2$ quantile of a standard normal distribution, we see that for n fixed,

$$RE = z_{\psi} \frac{\sqrt{\sigma^2/n}}{\gamma} = \frac{z_{\psi}}{\sqrt{n}} \frac{\Theta(\epsilon^{r/2})}{\Theta(\epsilon^r)} = \frac{z_{\psi}}{\sqrt{n}} \Theta(\epsilon^{-r/2}) \to \infty$$

as $\epsilon \to 0$, and so it becomes more difficult to estimate γ using standard simulation as system failures become rare.

Hence, we must utilize some variance reduction technique to obtain a better estimator. One such method is importance sampling, which we now describe. Recall that Δ is the (measurable) set of events ω for which $1\{\tau_F < \tau_0\}(\omega) = 1$. Consider another probability measure P' defined on $(\Omega, \mathscr{F})$ for which $P'(B) > 0$ for all $B \subset \Delta$, $B \in \mathscr{F}$, such that $P(B) > 0$. Then, note that

$$\begin{aligned} \gamma = E[1\{\tau_F < \tau_0\}] &= \sum_{\omega \in \Delta} P(\omega) = \sum_{\omega \in \Delta} \frac{P(\omega)}{P'(\omega)} P'(\omega) \\ &= \sum_{\omega \in \Delta} L(\omega) P'(\omega) = E'[1\{\tau_F < \tau_0\} L], \end{aligned} \tag{9}$$

where $L(\omega) = P(\omega)/P'(\omega)$, which is known as the Radon-Nikodym derivative, or simply the likelihood ratio, and E' is the expectation operator under the probability measure P'. See Hammersley and Handscomb [1964] and Glynn and Iglehart [1989] for more details.

Equation (9) forms the basis of importance sampling, which we apply as follows. Generate n i.i.d. samples $(\tilde{I}_1, \tilde{L}_1), \dots, (\tilde{I}_n, \tilde{L}_n)$ of $(1\{\tau_f < \tau_0\}, L)$ using the probability measure P'. Then our new point estimate is

$$\tilde{\gamma}(n) = \frac{1}{n} \sum_{k=1}^{n} \tilde{I}_k \tilde{L}_k,$$

and the variable of $1\{\tau_F < \tau_0\}L$ under the probability measure P' is

$$\sigma'^2 = E'\left[1\{\tau_F < \tau_0\}L^2\right] - \gamma^2.$$

The goal is to choose P' so that $E'[1\{\tau_F < \tau_0\}L^2] \ll E[1\{\tau_F < \tau_0\}]$, thus ensuring that the variance when using importance sampling is much smaller than that for standard simulation.

Now we describe the simple failure-biasing method of importance sampling. Under the original probability measure P, the probability of any failure transition from some state $x \in U$, $x \neq 0$, is $O(\epsilon)$, and the probability of a repair transition is $\Theta(1)$, as was shown in (2). The basic idea behind simple failure biasing is to increase the total probability to δ, where $\delta = \Theta(1)$ with $\delta < 1$, of a failure transition from x, and then allocate δ to the individual failure transitions in proportion to their original probabilities. Also, we decrease the total probability of a repair transition to $1 - \delta$ and allocate $1 - \delta$ to the individual repair transitions in proportion to their original probabilities. We do not alter the transition probabilities from state 0 or from any state $x \in F$. In some sense the simple failure-biasing method is a natural way of implementing importance sampling since it preserves the underlying structure of the system. A more precise description of simple failure biasing follows.

For any state $x \in S$, define $F(x) = \sum_{z \succ x} P(x, z)$ and $R(x) = \sum_{z \prec x} P(x, z)$. Note that $F(x)$ is the total probability of taking a failure transition from x, and $R(x)$ is the total probability of taking a repair transition from x. We construct the transition matrix P' for simple failure biasing from the original transition matrix P using the ensuing algorithm.

(i) For $(x, y) \notin \Gamma$,

$$P'(x, y) = 0;$$

(ii) For $(x, y) \in \Gamma$ and $x \in U$,

(a) With $x = 0$,

$$P'(x, y) = P(x, y);$$

(b) With $x \neq 0$,

$$P'(x, y) = \begin{cases} \delta P(x, y)/F(x) & \text{if } y \succ x \\ (1 - \delta)P(x, y)/R(x) & \text{if } y \prec x \\ 0 & \text{otherwise} \end{cases}; \tag{10}$$

(iii) For $(x, y) \in \Gamma$ and $x \in F$,

$$P'(x, y) = P(x, y).$$

Extensive empirical work suggests that δ should be chosen such that $0.5 \leq \delta \leq 0.9$; see Lewis and Böhm [1984], Goyal et al. [1992], and Shahabuddin et al. [1988] for further details. (For balanced failure-biasing items, (i) and (iii) are the same as above, but (ii) is changed to

(ii′) For $(x, y) \in \Gamma$ and $x \in U$,
(a) With $x = 0$,

$$P'(x, y) = 1/n_F(x);$$

(b) With $x \neq 0$,

$$P'(x, y) = \begin{cases} \delta/n_F(x) & \text{if } y \succ x \\ (1-\delta)P(x, y)/R(x) & \text{if } y \prec x \\ 0 & \text{otherwise} \end{cases};$$

where $n_F(x) = \sum_{y \succ x} 1\{(x, y) \in \Gamma\}$ is the number of failure transitions possible from state x. See Shahabuddin [1994] for further details on balanced failure biasing.)

Note that for $(x, y) \in \Gamma$ with $x \in U$,

$$P'(x, y) = \begin{cases} \Theta(\epsilon^{b(x,y)-s(x)}) & \text{if } y \succ x \\ \Theta(1) & \text{if } y \prec x \end{cases} \tag{11}$$
$$= \Theta(\epsilon^{1\{y \succ x\}(b(x,y)-s(x))})$$

under simple failure biasing.

Let σ'^2 be the variance of $1\{\tau_F < \tau_0\}L$ under simple failure biasing. Using (4), we obtain the following representation of the second moment term which will aid in our analysis throughout the article:

$$\begin{aligned} E'&\left[1\{\tau_F < \tau_0\}L^2\right] \\ &= E[1\{\tau_F < \tau_0\}L] \\ &= \sum_{m=r}^{\infty} \sum_{\substack{(x_0,\ldots,x_n)\in\Delta_m \\ n>0}} L(x_0,\ldots,x_n)P\{(X_0,\ldots,X_{\tau_F}) = (x_0,\ldots,x_n)\}. \end{aligned} \tag{12}$$

We define

$$RE' = z_{\psi}\frac{\sqrt{\sigma'^2/n}}{\gamma}, \tag{13}$$

which is the relative error of our simple failure-biasing estimator of γ. Then Shahabuddin [1994] proved the following result; see Shahabuddin [1994] for the proof.

PROPOSITION 1 (Shahabuddin). *If the system is balanced, then when using simple failure biasing, RE' remains bounded as $\epsilon \to 0$.*

Shahabuddin showed by example that if the system is not balanced, then simple failure biasing may not lead to bounded relative error for the performance measure estimate. Example 1 below also demonstrates this. (We will return to Example 1 and Example 2 below later to gain insight into why the simple failure biasing leads to bounded relative error for certain models but not for others.)

Example 1. Consider a system which has three types of components (i.e., $C = 3$), where the first two component types have a redundancy of two (i.e., $n_1 = n_2 = 2$), and the third type of component has a redundancy of one (i.e., $n_3 = 1$). Also, the components of type 1 and 2 have failure rate ϵ (i.e., $b_1 = b_2 = 1$), and the component of type 3 has failure rate ϵ^2 (i.e., $b_3 = 2$). Thus, $b_0 = 1$, and the system is unbalanced. There is a single repairperson who repairs components at rate 1 using a processor-sharing discipline. In our situation, it is sufficient to define the state of the system to be $x = \langle x_1, x_2, x_3 \rangle$, where x_i is the number of failed components of type i. Initially, all components are operational, and the system is considered to be operational if and only if there is at least one component of each type operational. We assume there is no failure propagation. Figure 1 is a state diagram of this model with the arcs having the original transition probabilities, and Figure 2 is the same when using simple failure biasing. We have lumped all of the states with $x_3 = 1$ into the single state $\langle x_1, x_2, 1 \rangle \in F$ for all x_1 and x_2 since the system fails as soon as the component of type 3 fails. Also, we have omitted the arcs from state $\langle x_1, x_2, 1 \rangle$ since these repair transitions do not play a role in our analysis.

It is easy to see that $\gamma = \Theta(\epsilon)$, and so $r = 1$. Now we examine the variance of the estimator of γ obtained using simple failure biasing. Consider the path $(\langle 0, 0, 0 \rangle, \langle 0, 0, 1 \rangle) \in \Delta$. From (12), the second moment satisfies

$$
\begin{aligned}
E'&\left[1\{\tau_F < \tau_0\} L^2\right] \\
&= \sum_{\substack{(x_0, \ldots, x_n) \in \Delta \\ n > 0}} L(x_0, \ldots, x_n) P\left\{(X_0, \ldots, X_{\tau_F}) = (x_0, \ldots, x_n)\right\} \\
&\geq L(\langle 0, 0, 0 \rangle, \langle 0, 0, 1 \rangle) P(\langle 0, 0, 0 \rangle, \langle 0, 0, 1 \rangle) \\
&= \frac{\epsilon}{4} + o(\epsilon) = \Theta(\epsilon).
\end{aligned}
$$

Thus, $\sigma'^2 = E'[1\{\tau_F < \tau_0\} L^2] - \gamma^2 \geq L(\langle 0, 0, 0 \rangle, \langle 0, 0, 1 \rangle) P(\langle 0, 0, 0 \rangle, \langle 0, 0, 1 \rangle) - \Theta(\epsilon^2) = \Theta(\epsilon)$. It then follows that for a fixed number of samples n,

$$
RE' = z_\psi \frac{\sigma'}{\gamma \sqrt{n}} \geq \frac{z_\psi}{\sqrt{n}} \frac{\Theta(\epsilon^{1/2})}{\Theta(\epsilon)} = \frac{z_\psi}{\sqrt{n}} \Theta(\epsilon^{-1/2}) \to \infty
$$

as $\epsilon \to 0$. Hence, simple failure biasing may not give bounded relative error when the system is unbalanced.

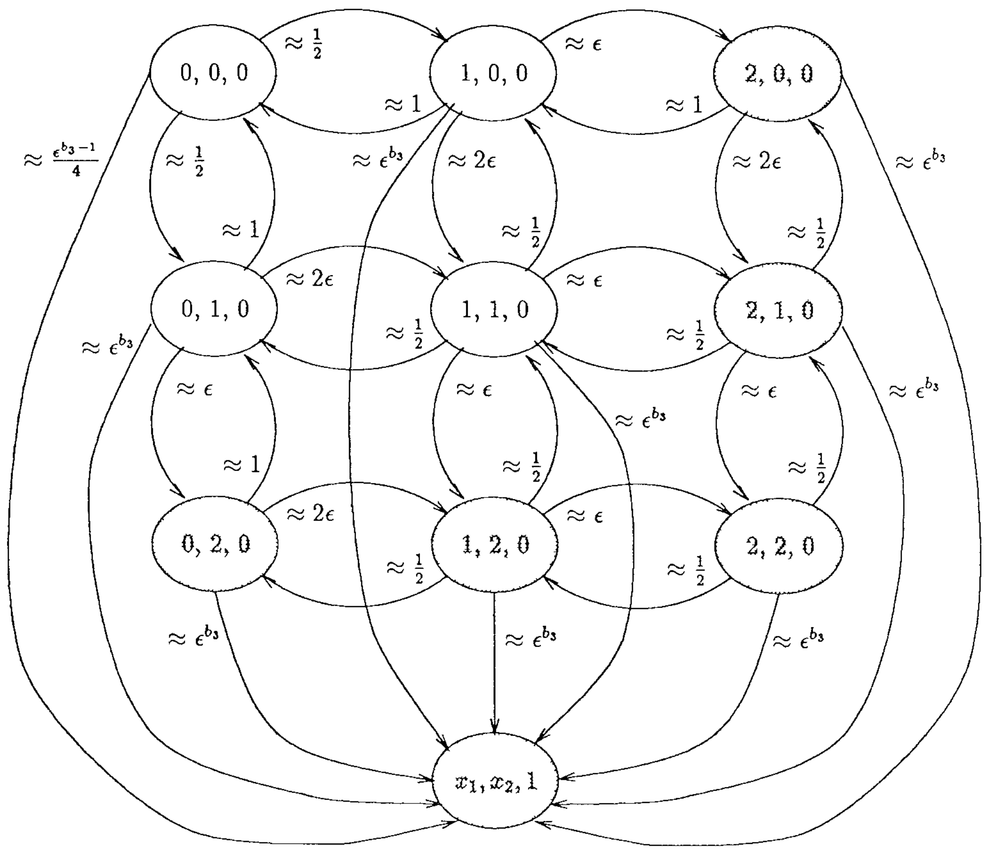

Fig. 1. Transition diagram for Examples 1 and 2 with original transition probabilities.

Now we examine another example, which was previously analyzed in Nakayama [1991].

Example 2. Consider the same system as in Example 1 except that we now change the failure rate of the third type of component to ϵ^3 (i.e., $b_3 = 3$). Thus, the system is still unbalanced. Figures 1 and 2 are state diagrams of this model with the arcs having the original transition probabilities and the simple failure-biasing transition probabilities, respectively.

It is easy to see that $\gamma = \epsilon + o(\epsilon)$, and so $r = 1$. Now consider the path $(\langle 0, 0, 0\rangle, \langle 0, 0, 1\rangle) \in \Delta$, which contributed $\Theta(\epsilon)$ to the second moment in Example 1, thereby giving the unbounded relative error. Note that now

$$L(\langle 0,0,0\rangle, \langle 0,0,1\rangle)P(\langle 0,0,0\rangle, \langle 0,0,1\rangle) = \frac{\epsilon^2}{4} + o(\epsilon^2) = \Theta(\epsilon^2),$$

and in fact, we can show that $E'[1\{\tau_F < \tau_0\}L^2] = ((12 + \delta)/4\delta)\epsilon^2 + o(\epsilon^2) = \Theta(\epsilon^2)$. (Recall that δ is the parameter used in failure biasing.) It then follows

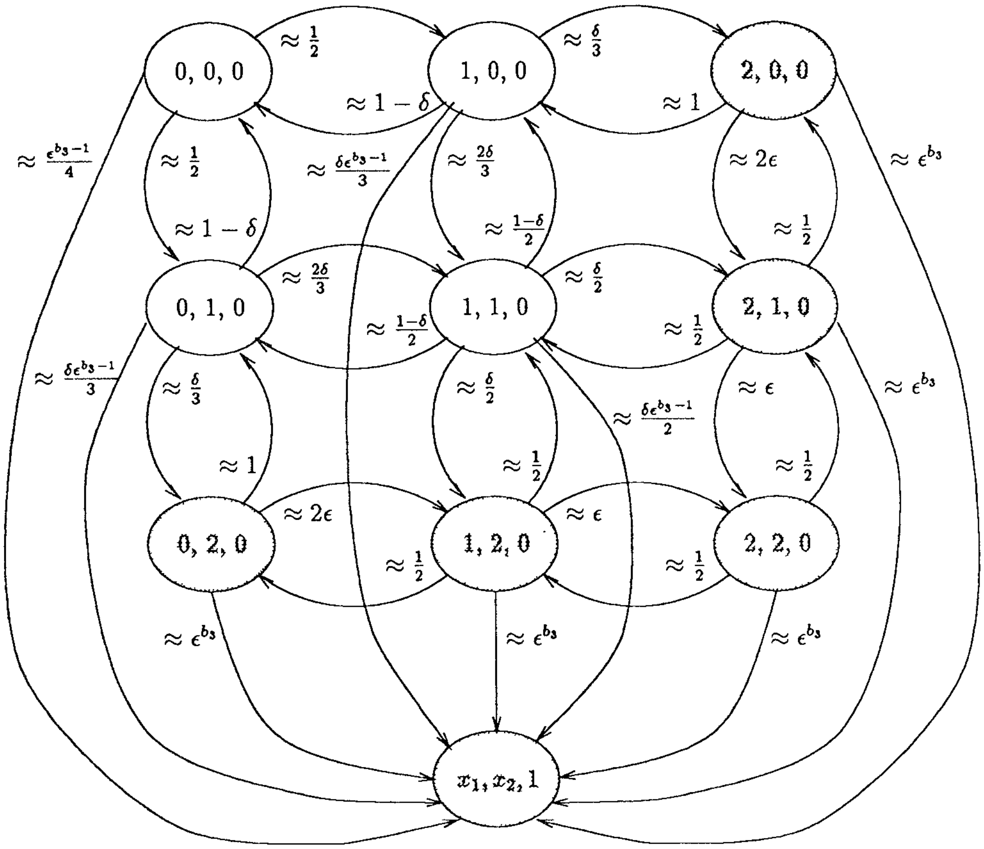

Fig. 2. Transition diagram for Examples 1 and 2 under simple failure biasing.

that $\sigma'^2 = ((12 + \delta)/(4\delta) - 1)\epsilon^2 + o(\epsilon^2)$ and

$$RE' = z_\psi \frac{\sigma'}{\gamma\sqrt{n}} = \frac{z_\psi}{\sqrt{n}} \frac{\sqrt{\dfrac{12+\delta}{4\delta} - 1}\,\epsilon + o(\epsilon)}{\epsilon + o(\epsilon)} = \frac{z_\psi}{\sqrt{n}} \sqrt{\frac{12+\delta}{4\delta} - 1} + o(1)$$

which remains bounded as $\epsilon \to 0$.

Nakayama [1991] showed that the variance of the estimator of γ when using balanced failure biasing is

$$\sigma_*^2 = \left(\frac{9}{\delta} - 1\right)\epsilon^2 + o(\epsilon^2).$$

Thus, although the variances obtained using simple and balanced failure biasing are of the same asymptotic order of magnitude, the variance when using simple failure biasing has a smaller constant for the lowest-order term in its asymptotic expansion.

The importance of Example 2 is twofold. First, it demonstrates that for certain models, simple failure biasing may yield smaller variances than balanced failure biasing, thereby motivating why we want to study simple failure biasing. Also, it shows that simple failure biasing may give bounded relative error even if the model being considered is unbalanced. This raises several questions: why does simple failure biasing work well for Example 2 but not for Example 1? For a given model can we determine if simple failure biasing will give bounded relative error?

In the following sections we will develop conditions on the structure of the model to determine when simple failure biasing will and will not give bounded RE'.

4. SUMMARY OF THE RESULTS

In this article we define conditions which, for a given model, characterize when the performance measure or its derivatives can be estimated with bounded relative error by using simple failure biasing. (For the results on derivative estimation, we impose some additional structure on the system; see Section 6.1. One restriction is that we do not allow state-dependent failure rates for the components, and so $b_i(x) = b_i$ for all $x \in S$.) The conditions, given below, are in terms of the asymptotic structure of the system being considered.

(**C**0′) $f(x_0, \ldots, x_n) \geq 2r - m + b_0$ for all $(x_0, \ldots, x_n) \in \Delta_m$, $r \leq m \leq 2r - 1$.

(**C***i*) $r_i = r$ or $b_i = b_0$ or $\bar{r}_i - 2b_0 \leq r_i - 2b_i$.

(**C***i*′) If $r_i - b_i \leq \bar{r}_i - b_0$, then

$$f(x_0, \ldots, x_n) \geq \begin{cases} 2r_i - m + b_0 \\ \quad \text{for all } (x_0, \ldots, x_n) \in \Delta^i_m,\ r_i \leq m \leq 2r_i - 1 \\ 2r_i - m + 3b_0 - 2b_i \\ \quad \text{for all } (x_0, \ldots, x_n) \in \bar{\Delta}^i_m,\ \bar{r}_i \leq m \leq 2r_i - 2b_i + 2b_0 + 1 \end{cases}.$$

If $r_i - b_i > \bar{r}_i - b_0$, then

$$f(x_0, \ldots, x_n) \geq \begin{cases} 2\bar{r}_i - m - b_0 + 2b_i \\ \quad \text{for all } (x_0, \ldots, x_n) \in \Delta^i_m,\ r_i \leq m \leq 2\bar{r}_i + 2b_i - 2b_0 - 1 \\ 2\bar{r}_i - m + b_0 \\ \quad \text{for all } (x_0, \ldots, x_n) \in \bar{\Delta}^i_m,\ \bar{r}_i \leq m \leq 2\bar{r}_i - 1 \end{cases}.$$

(**C***Si*) $r_i = r$ or $b_i = b_0$.

The conditions which are labeled with a prime (i.e., "′") are ones used to characterize simple failure biasing. If there is no prime, then we use the condition to show a result about standard simulation. Also, if a condition is labeled with a "0," then it is an assumption needed to establish a result on the performance measure estimate. If it is labeled with an "i," then the condition is used to characterize the derivative with respect to the failure rate of component type i.

Basically, we use Conditions **C**$0'$ and **C**i' to ensure that the probabilities under simple failure biasing of certain sample paths are not too small. As we shall see, these conditions are necessary and sufficient for achieving bounded relative error. The first condition of **C**i and **C**Si states that a component of type i can trigger a failure transition on a most likely path to failure; the second part stipulates that the components of type i have one of the largest failure rates. The last part of **C**i is a technical condition. For more details and insights on the conditions, see Sections 5 and 6.3.

We let RE_i denote the relative error of the estimator of the derivative of γ with respect to the failure rate of component type i when using standard simulation, and RE'_i denote the same when using simple failure biasing. See Sections 6.2 and 6.3 for the precise definitions.

Table I summarizes the results and examples that are presented in this article and gives a fairly complete characterization of the simple failure-biasing method. The interpretation of Table I is as follows. Each row and column heading states some condition. We use "bdd" as an abbreviation for "bounded." For each off-diagonal entry, there is one of the following symbols: "$\Leftrightarrow$," which means "if and only if"; "$\Rightarrow$," which means "implies"; and "$\not\Rightarrow$," which means "does not imply." The symbol gives the relationship between the particular row and column headings, where the "$\Rightarrow$ " means the row heading implies the column heading and similarly for the "$\not\Rightarrow$." For example, **C**$0'$ holds if and only if RE' is bounded; RE'_i being bounded for one value of i does not imply that RE' is bounded; and RE_k bounded for all k implies that **C**$0'$ holds.

Table II gives the justification for all of the entries in Table I. If an entry in Table I is either "$\Rightarrow$ " or "$\Leftrightarrow$," then the corresponding entry in Table II either refers to a theorem or the result holds trivially. If an entry in Table I is "$\not\Rightarrow$," then the corresponding entry in Table II refers to the appropriate counterexample.

Some of the most interesting results contained in Table I are the following. First, the Condition **C**$0'$ is a necessary and sufficient condition for the performance measure to have bounded relative error when using simple failure biasing. Since **C**$0'$ imposes restrictions on all sample paths $(x_0, \cdots, x_n) \in \Delta_m$ with $r \le m \le 2r - 1$, we see that secondary paths to failure (i.e., $(x_0, \ldots, x_n) \in \Delta_m$, $m > r$) play an important role in determining the variance of the estimator; see Theorem 2 for further details. A similar observation holds for the derivatives and Condition **C**i'; see Theorem 5. Also, if RE'_i is bounded for some component type i, it does not necessarily follow that RE' remains bounded as $\epsilon \to 0$. In other words, it is possible to estimate a derivative more efficiently than the performance measure when using simple failure biasing; see Example 3. However, if we can estimate all derivatives with bounded relative error when using simple failure biasing, then we can also do so for the performance measure; see Theorem 7.

Table I Summary of Results

Condition	System balanced	C0′	RE' bdd	C_i	RE_i/RE bdd	RE_k/RE bdd $\forall k$	C_i'	RE_i' bdd	RE_k' bdd $\forall k$	C0′ CS_i	C0′ C_i	RE', RE_i' bdd
System balanced		$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$
C0′	$\not\Rightarrow$		$\Leftrightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
RE' bdd	$\not\Rightarrow$	$\Leftrightarrow$		$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
C_i	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$		$\Leftrightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
RE_i/RE bdd	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Leftrightarrow$		$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
RE_k/RE bdd $\forall k$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$		$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
C_i'	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$		$\Leftrightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
RE_i' bdd	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Leftrightarrow$		$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$
RE_k' bdd $\forall k$	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$		$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$
C0′, CS_i	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$		$\Rightarrow$	$\Rightarrow$
C0′, C_i	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$		$\not\Rightarrow$
RE', RE_i' bdd	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	$\not\Rightarrow$	

Table II Justification of Table I

Condition	System balanced	C0′	RE' bdd	C_i	RE_i/RE bdd	RE_k/RE bdd $\forall k$	C_i'	RE_i' bdd	RE_k' bdd $\forall k$	C0′ CS_i	C0′ C_i	RE', RE_i' bdd
System balanced		R2	R2	R1	R4	R4	R6	R6	R6	R2, R1	R2, R1	R2, R5
C0′	R12		R3	R19	R19	R19	R14	R14	R14	R19	R19	R14
RE' bdd	R12	R3		R19	R19	R19	R14	R14	R14	R19	R19	R14
C_i	R10	R16	R16		R4	R15	R10	R10	R15	R16	R16	R16
RE_i/RE bdd	R10	R16	R16	R4		R15	R10	R10	R15	R16	R16	R16
RE_k/RE bdd $\forall k$	R9	R9	R9	R1	R1		R10	R10	R11	R9	R9	R9
C_i'	R17	R17	R17	R19	R19	R19		R5	R15	R17	R17	R17
RE_i' bdd	R17	R17	R17	R19	R19	R19	R5		R15	R17	R17	R17
RE_k' bdd $\forall k$	R18	R7	R7	R19	R19	R19	R1	R1		R19	R19	R7
C0′, CS_i	R13	R1	R1	R1	R1	R15	R8	R8	R15		R1	R3, R8
C0′, C_i	R13	R1	R3	R1	R4	R15	R20	R20	R15	R20		R20
RE', RE_i' bdd	R13	R3	R1	R19	R19	R19	R5	R1	R15	R19	R19	

R1: Trivial.
R2: Proposition 1.
R3: Theorem 2.
R4: Theorem 4.
R5: Theorem 5.
R6: Theorem 6.
R7: Theorem 7.
R8: Theorem 8.
R9: Example 1.
R10: Example 1 with $i = 3$.
R11: Example 1 with $k = 3$.
R12: Example 2.
R13: Example 2 with $i = 1$.
R14: Example 2 with $i = 3$.
R15: Example 2 with $i = 1$ and $k = 3$
R16: Example 3 with $i = 1$.
R17: Example 3 with $i = 3$.
R18: Example 4.
R19: Example 4 with $i = 3$.
R20: Example 5 with $i = 3$.

5. ESTIMATING THE PERFORMANCE MEASURE USING SIMPLE FAILURE BIASING

We now investigate the estimation of γ using simple failure biasing. To accomplish this, we first must gain a better understanding of $f(x_0, \ldots, x_n)$, which we do by showing its connection to the likelihood ratio under simple failure biasing. Consider $(x, y) \in \Gamma$ with $x \in U$ and $x \neq 0$. Using (2) and

(11), we obtain

$$\frac{P(x, y)}{P'(x, y)} = \frac{\Theta(\epsilon^{b(x,y)})}{\Theta(\epsilon^{b(x,y)-s(x)})} = \Theta(\epsilon^{s(x)})$$

if $y \succ x$. If $y \prec x$, then both $P(x, y)$ and $P'(x, y)$ are $\Theta(1)$, and thus,

$$\frac{P(x, y)}{P'(x, y)} = \Theta(1)$$

when $y \prec x$. Since transition probabilities out of state 0 are not changed in simple failure biasing,

$$\frac{P(0, y)}{P'(0, y)} = 1$$

for all $(0, y) \in \Gamma$. It then follows that for $(x_0, \ldots, x_n) \in \Delta$, the likelihood ratio satisfies

$$L(x_0, \ldots, x_n) = \prod_{k=0}^{n-1} \frac{P(x_k, x_{k+1})}{P'(x_k, x_{k+1})} = \Theta(\epsilon^{\sum_{k=1}^{n-1} 1\{x_{k+1} \succ x_k\} s(x_k)})$$
$$= \Theta(\epsilon^{f(x_0, \ldots, x_n) - b_0}) \tag{14}$$

since both $s(0) = b_0$ and the sum in the definition of $f(x_0, \ldots, x_n)$ includes the initial failure transition which has no effect on the likelihood ratio when using simple failure biasing.

From (12) we see that the likelihood ratio is the key to determine how much simple failure biasing reduces the second moment. More specifically, consider $(x_0, \ldots, x_n) \in \Delta$. The smaller $L(x_0, \ldots, x_n)$ is, the less the path $(x_0, \ldots, x_n)$ will contribute to the second moment. From (14) the order of magnitude of $L(x_0, \ldots, x_n)$ is dictated by $f(x_0, \ldots, x_n)$. As we see in the following theorem, **C**0′ establishes the orders of magnitude for the likelihood ratio in terms of the $f(x_0, \ldots, x_n)$ for each path $(x_0, \ldots, x_n)$ that are needed to achieve bounded relative error.

THEOREM 2. *RE′ remains bounded as $\epsilon \to 0$ if and only if* **C**0′ *holds.*

We now give some insight into this result. Note that **C**0′ only imposes restrictions on paths of order ϵ^m with $r \le m \le 2r - 1$. The condition ensures that the contribution of each of these paths to the second moment is not greater than ϵ^{2r}, which is the order of magnitude for the variance required to achieve bounded relative error. It turns out that none of the paths of order ϵ^{2r} or higher will interfere with ensuring a second moment of order ϵ^{2r}, and paths of order ϵ^{r-1} or lower are not possible. The complete proof of Theorem 2 is given in the Appendix.

One interesting point about Theorem 2 is: it shows that to determine if simple failure biasing will result in bounded relative error for the performance measure estimate, we need to check not only the most likely paths to failure (i.e., $(x_0, \ldots, x_n) \in \Delta_r$) but also some secondary paths (i.e., $(x_0, \ldots, x_n) \in \Delta_m$, $m > r$) as well. The number of paths in each $|\Delta_m|$, $m \ge r$, is finite (see Lemma 9(ii) in the Appendix), and so the total number of paths that must be considered is finite since **C**0′ only restricts paths in Δ_m for $r \le m \le 2r - 1$. Furthermore, when $m \ge 2r$, we have $2r - m + b_0 \le b_0$, and by (5), the condition $f(x_0, \dots, x_n) \ge 2r - m + b_0$ used in **C**0′ becomes vacuous when $m \ge 2r$.

We now examine Condition **C**0′ more closely. To do so, we will use the following result, whose proof is in the Appendix.

PROPOSITION 3. *For all* $(x_0, \dots, x_n) \in \Delta_m$,

(i) $f(x_0, \dots, x_n) \le m + b_0$,
(ii) $b(x_0, x_1) - b_0 + \sum_{k=1}^{n-1} 1\{x_{k+1} \succ x_k\} b(x_k, x_{k+1}) = m$.

Now consider any path $(x_0, \dots, x_n) \in \Delta_m$, $m \ge r$. If **C**0′ is valid, then

$$2r - m + b_0 \le f(x_0, \dots, x_n) \le r + b_0 \tag{15}$$

by Proposition 3(i). Thus, from (14), we see that

$$L(x_0, \dots, x_n) = \underline{O}(\epsilon^m) \quad \text{and} \quad L(x_0, \dots, x_n) = O(\epsilon^{2r-m}).$$

Since $(x_0, \dots, x_n) \in \Delta_m$, it follows that

$$\frac{\Theta(\epsilon^m)}{P'\{(X_0, \dots, X_{\tau_F}) = (x_0, \dots, x_n)\}} = \underline{O}(\epsilon^m)$$

and

$$\frac{\Theta(\epsilon^m)}{P'\{(X_0, \dots, X_{\tau_F}) = (x_0, \dots, x_n)\}} = O(\epsilon^{2r-m}),$$

or equivalently,

$$P'\{(X_0, \dots, X_{\tau_F}) = (x_0, \dots, x_n)\} = O(1)$$

and

$$P'\{(X_0, \dots, X_{\tau_F}) = (x_0, \dots, x_n)\} = \underline{O}(\epsilon^{2m-2r}).$$

Hence, Condition **C**0′ ensures that a sample path $(x_0, \dots, x_n) \in \Delta_m$, $r \le m \le 2r - 1$, has probability under simple failure biasing at most of the order ϵ^{2m-2r}. In particular, this means that all of the most likely paths to failure under the original measure (i.e., $(x_0, \dots, x_n) \in \Delta_r$) have probability of order 1 under simple failure biasing. For example, if $r = 2$, then **C**0′ ensures that under simple failure biasing, (1) all paths in Δ_2 have probability of order 1 and (2) all paths in Δ_3 have probability at most of the order ϵ^2.

Also, it follows from (15) and Proposition 3(ii) that for $(x_0, \dots, x_n) \in \Delta_r$

$$\sum_{k=0}^{n-1} 1\{x_{k+1} \succ x_k\} b(x_k, x_{k+1}) = r + b_0 = \sum_{k=0}^{n-1} 1\{x_{k+1} \succ x_k\} s(x_k)$$

If **C**0′ holds. Recall that for any $(x, y) \in \Gamma$ with $y \succ x$, we have that $b(x, y) \ge s(x) \ge 1$. Thus, if **C**0′ is valid, then $b(x_k, x_{k+1}) = s(x_k)$ for all k

such that $x_{k+1} \succ x_k$. In other words, consider a failure transition (x, y) that occurs along any of the most likely paths to failure (under the original measure). If simple failure biasing gives bounded relative error, then $P(x, y)$ must be of the largest order of magnitude of any failure transition from state x.

In general, checking Condition **C**0′ for a given model may be tedious. However, as shown in Proposition 1, a simple sufficient condition for bounded relative error when using simple failure biasing is that the system is balanced. Developing more general sufficient conditions which can be easily verified may be difficult, and we have not done so.

Now we use Theorem 2 to reexamine Examples 1 and 2.

Example 1 (*continued*). Recall that $r = 1$, and so Theorem 2 establishes that we only have to consider the paths in Δ_1 to determine if γ can be estimated with bounded relative error using simple failure biasing. Note that

$$\Delta_1 = \{(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle), (\langle 0,0,0\rangle, \langle 0,1,0\rangle, \langle 0,2,0\rangle), (\langle 0,0,0\rangle, \langle 0,0,1\rangle)\},$$

and $f(\langle 0, 0, 0\rangle, \langle 0, 0, 1\rangle) = 1 < 2r - m + b_0 = 2$ since $m = 1$. Thus, **C**0′ does not hold.

Example 2 (*continued*). Since $r = 1$, we only need to examine Δ_1 to ascertain if simple failure biasing will result in bounded relative error. Note that

$$\Delta_1 = \{(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle), (\langle 0,0,0\rangle, \langle 0,1,0\rangle, \langle 0,2,0\rangle)\},$$

and $f(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle) = f(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle) = 2 \geq 2r - m + b_0 = 2$ since $m = 1$. Thus, **C**0′ holds.

6. ESTIMATING DERIVATIVES

In this section we discuss the estimation of the derivatives of γ with respect to component failure rates. We first analyze estimating these quantities using standard simulation. Then we examine Condition **C***i*′ from Section 4 which characterizes when simple failure biasing will give rise to bounded relative error for the derivative estimates.

6.1 Additional System Structure

To obtain results on derivative estimates, we will assume that the system has more structure. The main differences are that (1) we no longer allow for state-dependent failure rates and (2) we restrict the generality of failure propagation. The following modification was developed by Nakayama [1991].

We now assume that components of type i have failure rate λ_i which is independent of $x \in S$; i.e., $\lambda_i(x) = \lambda_i$ for all $x \in S$. We assume that $0 < \lambda_i < \bar{\lambda}_i$, where $\bar{\lambda}_i < \infty$. Furthermore, when parameterizing by the rarity parameter ϵ, we assume that $\lambda_i = \lambda_i(\epsilon) = \tilde{\lambda}_i \epsilon^{b_i}$, where $\tilde{\lambda}_i > 0$, $b_i \geq 1$, and b_i is integer valued. Note that b_i does not depend on the state x. Also, we no longer allow the $p(y; x, i)$ to depend on ϵ; i.e., $d_i(x, y) = 0$ for all $1 \leq i \leq C$ and $x, y \in S$. In this situation, we have that $b_0 = \min\{b_i : 1 \leq i \leq C\}$.

As in Nakayama [1991], we make the following additional assumptions to simplify the proofs.

(A4) *If* $p(y; x, i) > 0$ *and* $p(y; x, j) > 0$, *then* $b_i = b_j$.

(A5) *If there exists a component type* i *such that* $b_i = b_0$ *and* $p(y; 0, i) > 0$, *then there exists another component type* $j \neq i$ *such that* $b_j = b_0$ *and* $p(y; 0, j) \neq p(y; 0, i)$.

Assumption A4 stipulates that if the failure of either of two different types of components can trigger a transition from state x to y, then the failure rates of the two component types are of the same ϵ-order. This assumption also implies that for $(x, y) \in \Gamma$, $q(x, y)$ has a certain form:

$$q(x, y) = \begin{cases} c(x, y)\epsilon^{d(x,y)} & \text{if } y \succ x \\ \mu(x, y) & \text{if } y \prec x \\ 0 & \text{otherwise} \end{cases}$$

where $c(x, y) > 0$, $d(x, y) \geq 1$ are integer value, $\epsilon > 0$, and $\mu(x, y) > 0$. Hence, the transition rates for failure transitions consist of a single term rather than a sum as before. Assumption A4 is used to determine the order of magnitude of the derivative of γ with respect to λ_i; see the proof of Lemma 11 of Nakayama [1991] for further details.

Assumption A5 stipulates that if there is some component type i having one of the largest failure rates whose failure can cause a transition from state 0 to state y with some positive probability, then there must be some other component type j also having one of the largest failure rates which causes the same transition with a different probability. This condition is not unreasonable when we are considering large reliability systems. It should be noted that Assumption A5 holds if the component type j in Assumption A5 satisfies $b_j = b_0$ and $p(y; 0, j) = 0$. Assumption A5 is a technical assumption used to ensure that there is no cancellation in the calculations of certain expressions arising in the derivatives with respect to λ_i; see the proof of Lemma 11 of Nakayama [1991] for further details.

In the situation when there is no failure propagation, Assumption A4 is automatically satisfied, and Assumption A5 reduces to requiring that there are two different component types which both have failure rates of the order ϵ^{b_0}; i.e., there exists i and j such that $i \neq j$ and $b_i = b_j = b_0$. (Assumption A5 holds under this weaker condition for the following reason. Suppose that when component type i fails in state 0, the system then enters state y, and suppose that when component type $j \neq i$ fails in state 0, the system then enters state $z \neq y$. Then $p(y; 0, i) = 1$ and $p(x; 0, i) = 0$ for all $x \neq y$, and $p(z; 0, j) = 1$ and $p(x; 0, j) = 0$ for all $x \neq z$. Thus, $p(y; 0, i) = 1 \neq p(y; 0, j) = 0$, and so A5 holds.) If we allow for failure propagation but with the restriction that any given failure transition can be triggered by the failure of only one type of component (i.e., for $(x, y) \in \Gamma$, there is at most one component type i for which $p(y; x, i) > 0$), then again it is easy to verify that (1) Assumption A4 holds and (2) Assumption A5 reduces to requiring that there exist component types i and j such that $i \neq j$ and $b_i = b_j = b_0$.

6.2 Estimating Derivatives Using Standard Simulation

We now analyze likelihood ratio derivative estimators of the partial derivative of γ with respect to the failure rate of component type i obtained using standard simulation. The results in this section will be compared to those obtained in the following section when we consider estimating the derivatives using simple failure biasing. Throughout the rest of the article, we will employ the notation $\partial_i A(\lambda_1, \dots, \lambda_C) = (\partial/\partial\lambda_i) A(\lambda_1, \dots, \lambda_C)$ for some function $A(\lambda_1, \dots, \lambda_C)$.

Using the likelihood ratio method of estimating derivatives, we obtain

$$\partial_i \gamma = E[1\{\tau_F < \tau_0\} D_i],$$

where

$$D_i = \sum_{k=0}^{\tau_{\min}-1} \frac{\partial_i P(X_k, X_{k+1})}{P(X_k, X_{k+1})}.$$

See Nakayama et al. [1990], Glynn [1986], and Reiman and Weiss [1989] for further details.

We estimate $\partial_i \gamma$ using standard simulation in the following manner. Generate n i.i.d. samples $(\hat{I}_1, \hat{D}_{i,1}), \dots, (\hat{I}_n, \hat{D}_{i,n})$ of $(1\{\tau_F < \tau_0\}, D_i)$ using the probability measure P. We then form the point estimate

$$\hat{\gamma}_i(n) = \sum_{k=1}^{n} \hat{I}_k \hat{D}_{i,k}.$$

Nakayama [1991] established that for all ϵ sufficiently small,

$$\partial_i \gamma = \Theta(\epsilon^{\min\{r_i - b_i, \bar{r}_i - b_0\}}) \tag{16}$$

and

$$\sigma_i^2 \equiv \mathrm{Var}[1\{\tau_F < \tau_0\} D_i] = \Theta(\epsilon^{\min\{r_i - 2b_i, \bar{r}_i - 2b_0\}}), \tag{17}$$

where r_i and $\bar{r}_i$ are as defined in (6) and (7), respectively. In general, we cannot say whether $r_i - b_i \le \bar{r}_i - b_0$ or $r_i - b_i > \bar{r}_i - b_0$ or whether $r_i - 2b_i \le \bar{r}_i - 2b_0$ or $r_i - 2b_i > \bar{r}_i - 2b_0$. Also, the expression for $\partial_i \gamma$ is independent of whether we use standard simulation or importance sampling. However, the variance of the estimator depends on the simulation method being employed.

Let RE_i denote the relative error of the estimator of $\partial_i \gamma$ obtained using standard simulation, i.e.,

$$RE_i = z_\psi \frac{\sqrt{\sigma_i^2/n}}{\partial_i \gamma}.$$

Corollaries 4–6 of Nakayama [1991] established that certain derivatives can be estimated with the same relative accuracy as the performance measure itself when using standard simulation. (In particular, it was shown that this is the case if **C***Si* holds.) We now want to further characterize these derivatives. To do so, recall Condition **C***i* defined previously in Section 4. Then we have the following result, whose proof is given in the Appendix.

THEOREM 4. *RE_i/RE remains bounded as $\epsilon \to 0$ if and only if* **C***i holds.*

6.3 Estimating Derivatives Using Simple Failure Biasing

We can also use simple failure biasing to estimate derivatives. Note that

$$\partial_i \gamma = E[1\{\tau_F < \tau_0\} D_i] = E'[1\{\tau_F < \tau_0\} D_i L].$$

Thus, we estimate $\partial_i \gamma$ using simple failure biasing in the following manner. Generate n i.i.d. samples $(\tilde{I}_1, \tilde{D}_{i,1}, \tilde{L}_1), \ldots, (\tilde{I}_n, \tilde{D}_{i,n}, \tilde{L}_n)$ of $(1\{\tau_F < \tau_0\}, D_i, L)$ using the probability measure P'. We then form the point estimate

$$\tilde{\gamma}_i(n) = \sum_{k=1}^{n} \tilde{I}_k \tilde{D}_{i,k} \tilde{L}_k.$$

Let RE_i' denote the relative error of the estimator of $\partial_i \gamma$ obtained using simple failure biasing, i.e.,

$$RE_i' = z_\psi \frac{\sqrt{\sigma_i'^2 / n}}{\partial_i \gamma}, \tag{18}$$

where $\sigma_i'^2$ is the variance of $1\{\tau_F < \tau_0\} D_i L$ under the probability measure P'.

The next result establishes that Condition $\mathbf{C}i'$ from Section 4 is a necessary and sufficient condition for the derivative estimator with respect to λ_i to have bounded relative error when using simple failure biasing.

THEOREM 5. *RE_i' remains bounded as $\epsilon \to 0$ if and only if $\mathbf{C}i'$ holds.*

The basic idea of the proof of Theorem 5 is as follows. We first examine the orders of magnitude of the summands of D_i, and note that it depends on whether or not the transition (X_k, X_{k+1}) satisfies $n_i(X_{k-1})\lambda_i p(X_k; X_{k-1}, i) > 0$. Thus, the order of magnitude of D_i depends on whether $\tau_i \le \tau_F < \tau_0$ or $\tau_F < \min(\tau_0, \tau_i)$ occurs. We then derive bounds for D_i for these two situations. To take advantage of this, we express

$$\begin{aligned} E'&\big[1\{\tau_F < \tau_0\} D_i^2 L^2\big] \\ &= E\big[1\{\tau_F < \tau_0\} D_i^2 L\big] \\ &= E\big[1\{\tau_i \le \tau_F < \tau_0\} D_i^2 L\big] + E\big[1\{\tau_F < \min(\tau_0, \tau_i)\} D_i^2 L\big]. \end{aligned} \tag{19}$$

We separately analyze each expectation on the right-hand side of (19) and show that Condition $\mathbf{C}i'$ characterizes when simple failure biasing will give bounded relative error by using the bounds developed on D_i and some other bounds (Lemma 9 in the Appendix). The complete proof is given in the Appendix.

We now examine Condition $\mathbf{C}i'$ more closely. First suppose that $r_i - b_i \le \bar{r}_i - b_0$, and note that $\mathbf{C}i'$ only imposes restrictions on paths in Δ^i having probability of order ϵ^m with $r_i \le m \le 2r_i - 1$ and paths in $\bar{\Delta}^i$ having probability of order ϵ^m with $\bar{r}_i \le m \le 2r_i - 2b_i + 2b_0 - 1$. These conditions ensure that the contribution of each of these paths to the second moment is not greater than $\epsilon^{r_i - 2b_i}$, which is the order of magnitude for the variance required to achieve bounded relative error. It turns out that none of the paths of higher order will interfere with ensuring a second moment of order $\epsilon^{r_i - 2b_i}$, and paths of lower order are not possible. A similar situation occurs when $r_i - b_i > \bar{r}_i - b_0$.

In general, checking Condition $\mathbf{C}i'$ for a given model may be tedious. However, the following result shows that a simple sufficient condition for bounded relative error for the derivative estimators when using simple failure biasing is that the system is balanced. Developing more general sufficient conditions which can be easily verified may be difficult, and we have not done so.

THEOREM 6. *If the system is balanced, then when using simple failure biasing, RE_i' remains bounded as $\epsilon \to 0$.*

We now use Condition $\mathbf{C}i'$ to examine some examples, from which we will gain a better understanding of simple failure biasing. Our first example shows that $\mathbf{C}0'$ is not sufficient to ensure that we can estimate derivatives with bounded relative error when using simple failure biasing. This demonstrates the need for the additional Condition $\mathbf{C}i'$. Thus, the situation when using simple failure biasing differs somewhat from that when balanced failure biasing is utilized, in which the relative errors of the performance measure and all of the derivatives always remain bounded; see Nakayama [1991].

Example 2 (*continued*). We previously showed that $\mathbf{C}0'$ holds. Now consider component type 3. Note that $\Delta_1^3 = \emptyset$, $\Delta_2^3 = \{(\langle 0,0,0\rangle, \langle 0,0,1\rangle)\}$ and

$$\overline{\Delta}_1^3 = \{(\langle 0,0,0\rangle, \langle 1,0,0\rangle, \langle 2,0,0\rangle), (\langle 0,0,0\rangle, \langle 0,1,0\rangle, \langle 0,2,0\rangle)\},$$

and so $r_3 = 2$ and $\bar{r}_3 = 1$. (Similarly, we can show that $r_1 = \bar{r}_1 = r_2 = \bar{r}_2 = 1$.) Thus, $r_3 - b_3 = -1 \le \bar{r}_3 - b_0 = 0$. Consider $(\langle 0,0,0\rangle, \langle 0,0,1\rangle) \in \Delta_2^3$. Note that $f(\langle 0,0,0\rangle, \langle 0,0,1\rangle) = 1 < 2r_3 - m + b_0 = 3$ since $m = 2$. Hence, $\mathbf{C}3'$ does not hold. (Also, we can show that $\mathbf{C}3$ does not hold.)

The next example shows that $\mathbf{C}i'$ does not necessarily imply $\mathbf{C}0'$. Thus, it follows from Theorems 2 and 5 that when using simple failure biasing, we may be able to estimate a derivative with bounded relative error but not the performance measure itself. In other words, it is possible to obtain better estimates for a derivative than for the performance measure.

Example 3. Consider a system which has four types of components (i.e., $C = 4$), where each of the first two types has redundancy one (i.e., $n_1 = n_2 = 1$); the third type has redundancy two (i.e., $n_3 = 2$); and the fourth type has redundancy one (i.e., $n_4 = 1$). Also, the components of type 1 and 2 have failure rate ϵ, and so $b_1 = b_2 = 1$. The component of type 3 has failure rate ϵ^2, and the fourth type of component has failure rate ϵ^4; and so $b_3 = 2$ and $b_4 = 4$. Thus $b_0 = 1$. There is a single repairperson who repairs components at rate 1 using a processor-sharing discipline. It is then sufficient to define the state of the system to be $x = \langle x_1, x_2, x_3, x_4\rangle$, where x_i is the number of failed components of type i. Initially, all components are operational, and the system is considered to be failed if and only if either all components of types 1, 2, and 3 are failed or the component of type 4 is failed. When the component of type 1 fails in state $\langle 0, 0, 0, 0\rangle$, it causes the component of type 2 and one of the components of type 3 to fail, i.e., $p(\langle 1, 1, 1, 0\rangle; \langle 0, 0, 0, 0\rangle, 1) = 1$. Figure 3 is a state diagram of the model of this system with the original transition probabilities given on the arcs. We have lumped all of the states with $x_4 = 1$ into the single state $\langle x_1, x_2, x_3, 1\rangle \in F$ for all x_1, x_2, and x_3 since the system fails as soon as the component of type 4 fails. Also, we have omitted the arcs from state $\langle x_1, x_2, x_3, 1\rangle$ since these repair transitions do not play a role in our analysis. The set of failed states is given by $F = \{\langle 1, 1, 2, 0\rangle, \langle x_1, x_2, x_3, 1\rangle\}$.

Note that

$$\Delta_1 = \emptyset,$$
$$\Delta_2 = \{(\mathbf{a}, \mathbf{h}, \mathbf{l})\}$$
$$\Delta_3 = \{(\mathbf{a}, \mathbf{m}), (\mathbf{a}, \mathbf{h}, \mathbf{g}, \mathbf{k}, \mathbf{l}), (\mathbf{a}, \mathbf{h}, \mathbf{f}, \mathbf{j}, \mathbf{l}), (\mathbf{a}, \mathbf{h}, \mathbf{g}, \mathbf{h}, \mathbf{l}), (\mathbf{a}, \mathbf{h}, \mathbf{f}, \mathbf{h}, \mathbf{l})\}.$$

We can show that $\gamma = (5/6)\epsilon^2 + o(\epsilon^2)$, and so $r = 2$. Also, $r_1 = 2$, $r_2 = 3$, $r_3 = 2$, $r_4 = 3$, $\bar{r}_1 = 3$, $\bar{r}_2 = 2$, $\bar{r}_3 = 3$, and $\bar{r}_4 = 2$. Now consider the sample path $(\mathbf{a}, \mathbf{m}) \in \Delta_3$. Observe that $f(\mathbf{a}, \mathbf{m}) = 1 < 2r - m + b_0 = 2$ since $m = 2$, and so $\mathbf{C}0'$ does not hold.

Now consider component type 3. We have that $r_3 - b_3 = 0 < 2 = \bar{r}_3 - b_0$. Also, $\Delta_2^3 = \Delta_2$, and $f(\mathbf{a}, \mathbf{h}, \mathbf{l}) = 3 \geq 2r_3 - m + b_0 = 3$ since $m = 2$. Furthermore, $\Delta_3^3 = \Delta_3 - \{(\mathbf{a}, \mathbf{m})\}$, and we can establish that $f(x_0, \ldots, x_n) \geq 2 = 2r_3 - m + b_0$ for all $(x_0, \ldots, x_n) \in \Delta_3^3$. Additionally, $\bar{r}_3 = 3 > 2r_3 - 2b_3 + 2b_0 - 1 = 1$, and so the condition imposed by $\mathbf{C}3'$ on $f(x_0, \ldots, x_n)$ for $(x_0, \ldots, x_n) \in \overline{\Delta}_m^3$ is vacuous since there is no value of m for which $\bar{r}_i \leq m \leq 2r_i - 2b_i + 2b_0 - 1$. Therefore, $\mathbf{C}3'$ holds.

The previous example demonstrated that we may not be able to estimate the performance measure with bounded relative error even if we can do so for one of the derivatives. However, the following result shows that if we can estimate all of the derivatives with bounded relative error, then we can also achieve the same for the performance measure. See the Appendix for the proof.

THEOREM 7. *If* $\mathbf{C}i'$ *holds for all component types* $i = 1, \ldots, C$, *then* $\mathbf{C}0'$ *holds.*

The next example shows that being able to estimate a derivative with the same relative error as the performance measure using standard simulation does not necessarily imply that we can use simple failure biasing to estimate the derivative with bounded relative error. (In fact, even if we also assume that the performance measure can be estimated with bounded relative error when using simple failure biasing, this still does not guarantee that the derivative estimate using simple failure biasing will have bounded relative error. See Example 5 in the Appendix.)

Example 1 (*continued*). Note that $r_1 = r_2 = r_3 = 1$ and $\bar{r}_1 = \bar{r}_2 = \bar{r}_3 = 1$. Thus, $\mathbf{C}1$, $\mathbf{C}2$, and $\mathbf{C}3$ hold.

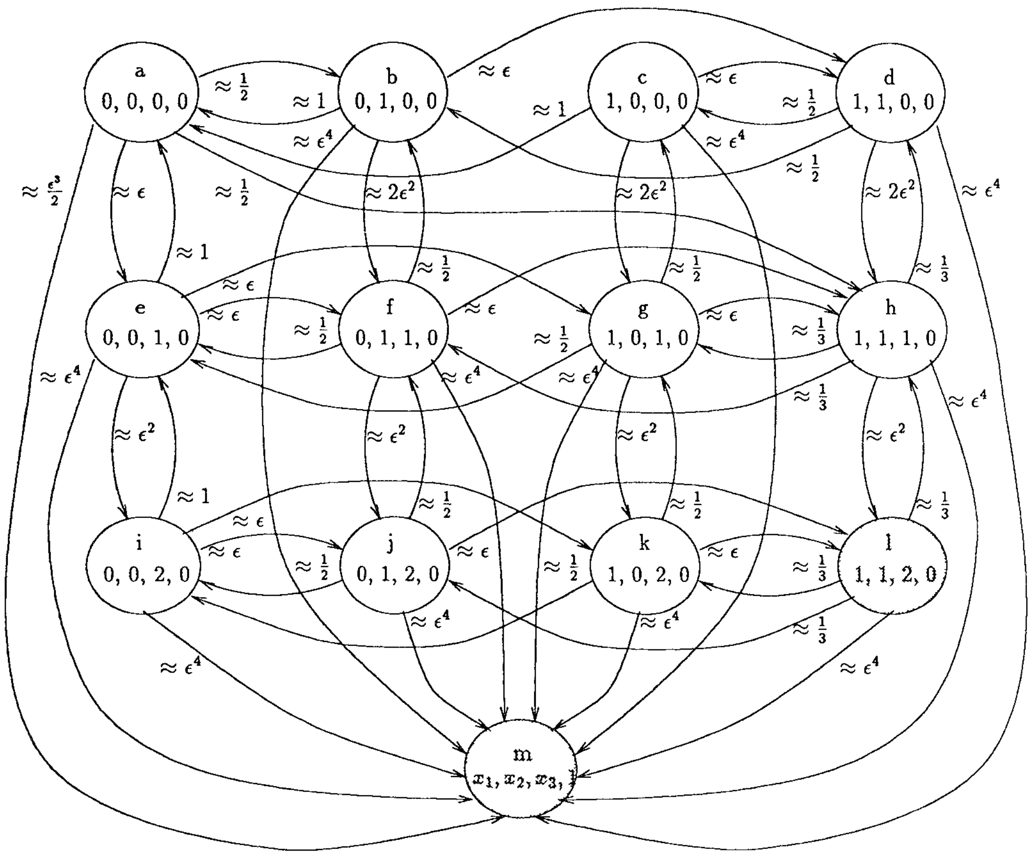

Fig. 3. Transition diagram for Example 3 with original transition probabilities.

Now consider component type 3. Observe that $r_3 - b_3 = -1 \leq \bar{r}_3 - b_0 = 0$. Consider the path $(\langle 0, 0, 0\rangle, \langle 0, 0, 1\rangle) \in \Delta_1^3$. We have that $f(\langle 0, 0, 0\rangle, \langle 0, 0, 1\rangle) = 1 < 2r_3 - m + b_0 = 2$ since $m = 1$. Thus, **C**3′ does not hold.

We now want to establish a simple sufficient condition that ensures that the derivative with respect to λ_i can be estimated with bounded relative error by using simple failure biasing. The next result shows that Condition **C**Si from Section 4 implies the desired conclusion; see the Appendix for the proof.

THEOREM 8. *If* **C**0′ *and* **C**Si *hold, then* **C**i' *holds.*

To gain a better understanding of Condition **C**Si, we define the *sensitivity* of γ with respect to λ_i to be $s_i \equiv s_i(\epsilon) = \lambda_i \cdot \partial_i \gamma$. We define s_i to have a *largest asymptotic magnitude* if $\liminf_{\epsilon \to 0} |s_i(\epsilon)/s_j(\epsilon)| > 0$ for all component types j. Then Corollary 4 of Nakayama [1991] establishes that

$$s_i \text{ has a largest asymptotic magnitude if and only if } \mathbf{C}Si \text{ holds.} \qquad (20)$$

Thus, if (1) we can estimate the performance measure with bounded relative error using simple failure biasing and (2) the sensitivity with respect to λ_i has a largest asymptotic magnitude, then simple failure biasing will result in bounded relative error for the estimate of $\partial_i \gamma$.

The following example shows that being able to estimate both γ and $\partial_i \gamma$ with bounded relative error by using simple failure biasing does not necessarily imply that we can estimate γ and $\partial_i \gamma$ with the same relative error when using standard simulation.

Example 4. Consider a system which has three types of components (i.e., $C = 3$), where each component type has a redundancy of two (i.e., $n_1 = n_2 = n_3 = 2$). Also, the components of type 1 and 2 have failure rate ϵ (i.e., $b_1 = b_2 = 1$), and the component of type 3 has failure rate ϵ^3 (i.e., $b_3 = 3$). Thus $b_0 = 1$. There is a single repairperson who repairs components at rate 1 using a processor-sharing discipline. It is then sufficient to define the state of the system to be $x = \langle x_1, x_2, x_3 \rangle$, where x_i is the number of failed components of type i. Initially, all components are operational, and the system is considered to be operational if and only if there is at least one component of each type operational. We assume there is no failure propagation. It is easy to see that $\gamma = \Theta(\epsilon)$, and so $r = 1$. Also, $r_1 = r_2 = \bar{r}_1 = \bar{r}_2 = \bar{r}_3 = 1$ and $r_3 = 4$. Note that $\Delta_1 = \{(\langle 0, 0, 0 \rangle, \langle 1, 0, 0 \rangle, \langle 2, 0, 0 \rangle), (\langle 0, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 2, 0 \rangle)\}$. It is easy to check that **C**0′ holds, and so Theorem 2 implies that RE' remains bounded as $\epsilon \to 0$.

Now consider component type 3. We have that $r_3 - b_3 = 1 > \bar{r}_3 - b_0 = 0$. We can easily show that $f(x_0, \ldots, x_n) \geq 2\bar{r}_3 - m - b_0 + 2b_3 = 7 - m$ for all $(x_0, \ldots, x_n) \in \Delta_m^3$ with $r_3 = 4 \leq m \leq 5 = 2\bar{r}_3 + 2b_3 - 2b_0 - 1$, and also that $f(x_0, \ldots, x_n) \geq 2\bar{r}_3 - m + b_0 = 3 - m$ for all $(x_0, \ldots, x_n) \in \bar{\Delta}_m^3$ with $\bar{r}_3 = 1 \leq m \leq 5 = 2\bar{r}_3 + 4b_0 - 1$. Thus, **C**3′ holds. (We also can show that **C**1′ and **C**2′ hold.) However, $r_3 > r$, $b_3 > b_0$, and $\bar{r}_3 - 2b_0 = -1 > r_3 - 2b_3 = -2$, and so **C**3 does not hold.

7. EXPERIMENTAL RESULTS

We now present some experimental results from running simulations of a large computing system to show that the asymptotic results developed in this article hold when simulating actual systems. Goyal et al. [1992], Nakayama et al. [1990], and Nakayama [1991] previously examined the same system but with different failure rates for some components. We used the SAVE package [1987] to obtain all of our results.

The system is composed of two types of processors, A and B, each having redundancy 2; two types of disk controllers, each having redundancy 2; and six sets of disk clusters, each having four disks. When a processor of one type fails, it causes a processor of another type to fail also with probability 0.01. Each type of component can fail in one of two modes which occur with equal probability. The repair rates of the all mode-1 failures are 1 per hour, and the repair rates of all mode-2 failures are 1/2 per hour. There is a single repairperson who fixes failed components in random-order service. The data is replicated on each of the disk clusters in such a way that all of the data in a cluster is accessible even if one of the disks in the cluster is failed. Each type of processor is paired with one of the types of disk controllers. The processors of a given type access the data on the disk clusters through one of its corresponding disk controllers. The system is operational if and only if (1) at least one processor of each type is operational and (2) all of the data is accessible. A block diagram of the system is given in Figure 4.

The failure rates of the processors of type A and B are $pAfr = 1/1000$ and $1/2000$ per hour, respectively. All of the disk controllers have failure rate $1/2000$ per hour. The disks in the first cluster have failure rate $d1fr = 1/12000$ per hour, and all other disks have failure rate $1/6000$ per hour. Thus the system is unbalanced.

We estimated γ and its sensitivities with respect to $pAfr$ and $d1fr$. Table III gives the results obtained using a numerical (nonsimulation) method, standard simulation, simple failure biasing, and balanced failure biasing. We obtained all simulation results from simulating one million events, where an event is either a component failure or repair. We give the relative half-width (i.e., confidence interval half-width divided by the point estimate) of the 99% confidence interval for all simulation estimates.

First note that the processors of type A have the largest failure rate of all the component types, and so it satisfies the second stipulation of Condition **C***i*. Theorem 4 then states that when using standard simulation, we should be able to estimate the sensitivity with respect to $pAfr$ with about the same relative error (as measured by the relative width of the 99% confidence interval) as the performance measure, and indeed this is the case. Also, the numerical value of the sensitivity with respect to $pAfr$ is much larger in magnitude than that with respect to $d1fr$, and our standard simulation estimate of the sensitivity with respect to $d1fr$ is worse than that of the other sensitivity. This agrees with Theorem 4 and (20). Using simple failure biasing, we obtained good estimates for both γ and its sensitivity with respect to $pAfr$ even though the system is unbalanced (see Theorems 2 and 8). The simple failure-biasing estimate of the sensitivity with respect to $d1fr$ is still not acceptable (see Theorem 5). Finally, balanced failure biasing produced reasonably stable estimates for all the quantities. We should also note that the confidence intervals for the estimates of γ and its sensitivity with respect to $pAfr$ are slightly better for simple failure biasing than for balanced failure biasing, thus demonstrating that simple failure biasing can yield better results than balanced failure biasing. For a more thorough empirical analysis of estimates of performance measures and sensitivities, see Goyal et al. [1992] and Nakayama et al. [1990].

8. CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

We have performed an extensive study of the simple failure-biasing method. In particular, we established necessary and sufficient conditions for obtaining performance measure and derivative estimators that have bounded relative error. One interesting aspect of these conditions is that it shows that in order to guarantee bounded relative error, we need to consider not only the most

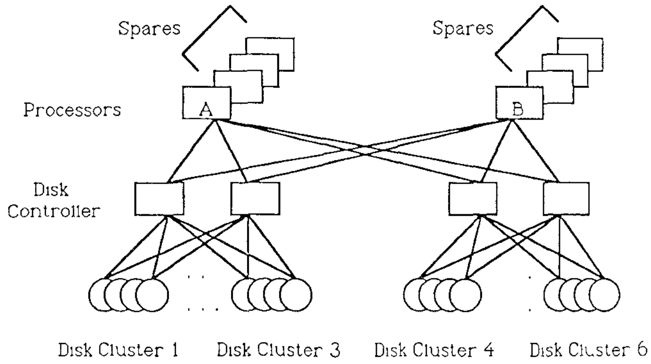

Fig. 4. Block diagram of a large computing system.

Table III. Estimates of γ and Sensitivities with Relative 99% Confidence Intervals

Performance Measure	Numerical Result	Standard Simulation	Simple Failure Biasing	Balanced Failure Biasing
γ	0.9421×10^{-3}	0.9945×10^{-3} $\pm$ 14.6%	0.9361×10^{-3} $\pm$ 3.1%	0.9471×10^{-3} $\pm$ 3.6%
$pAfr \cdot \frac{\partial}{\partial pAfr}\gamma$	0.5060×10^{-3}	0.5477×10^{-3} $\pm$ 40.5%	0.5000×10^{-3} $\pm$ 6.3%	0.4992×10^{-3} $\pm$ 10 6%
$d1fr \cdot \frac{\partial}{\partial d1fr}\gamma$	$-.5770 \times 10^{-5}$	$-.1013 \times 10^{-4}$ $\pm$ 106.2%	$-.5329 \times 10^{-5}$ $\pm$ 141.1%	$-.6030 \times 10^{-5}$ $\pm$ 46 3%

likely paths to system failure but also secondary paths leading to system failure. Also, we showed that the conditions for a derivative estimator to have bounded relative error do not imply those for the performance measure by constructing an example demonstrating that it is possible to estimate a derivative more efficiently than the performance measure when using simple failure biasing.

One area for future research is to develop easily implementable methods which will determine if simple failure biasing will or will not work well (i.e., if **C0**$'$ holds) for a given model. A simple sufficient condition for this is to check if the system is balanced. However, as we have seen, this condition is not necessary. Also, further work is required to characterize other importance-sampling schemes used for analyzing highly reliable systems. Finally, as we have shown in Example 2, when simple failure biasing yields bounded relative error, it may produce smaller confidence intervals then balanced failure biasing. Thus, it may be more appropriate to use simple failure biasing in certain contexts instead of balanced failure biasing, and additional research should be carried out to identify classes of models for which this is the case. Currently, the only method available for determining which method is most appropriate is experimentation.

APPENDIX

First we prove Proposition 3.

PROOF OF PROPOSITION 3. To prove our result, we first derive a representation for m in terms of the $b(x_k, x_{k+1})$. Consider any $(x_0, \dots, x_n) \in \Delta_m$. By definition we must have that

$$P\{(X_0, \dots, X_{\tau_F}) = (x_0, \dots, x_n)\} = \prod_{k=0}^{n-1} P(x_k, x_{k+1}) = \Theta(\epsilon^m).$$

The probability of the initial transition (x_0, x_1) is $\Theta(\epsilon^{b(x_0, x_1) - b_0})$ by Eq. (2). Also, $(x_0, \dots, x_n) \in \Delta_m$ implies that $x_k \neq 0$ for $k > 0$. Hence, the probability of any failure transition (x_k, x_{k+1}), $k > 0$, is $\Theta(\epsilon^{b(x_k, x_{k+1})})$ by Eq. (2). Each repair transition has probability of order 1. Thus, since $b(x, y) = 0$ if $y \prec x$,

$$b(x_0, x_1) - b_0 + \sum_{k=1}^{n-1} b(x_k, x_{k+1}) = m, \tag{21}$$

or equivalently,

$$b(x_0, x_1) - b_0 + \sum_{k=1}^{n-1} 1\{x_{k+1} \succ x_k\} b(x_k, x_{k+1}) = m, \tag{22}$$

which establishes part (ii).

To convert Eq. (22) into a bound on $f(x_0, \dots, x_n)$, note that $b(x, y) \geq s(x)$ for any $(x, y) \in \Gamma$ with $y \succ x$. It then follows that

$$s(x_0) - b_0 + \sum_{k=1}^{n-1} 1\{x_{k+1} \succ x_k\} s(x_k) \leq m,$$

which implies $f(x_0, \dots, x_n) \leq m + b_0$. □

To prove Theorem 2, we will need the next lemma.

LEMMA 9. *Consider* $(x_0, \dots, x_n) \in \Delta_m$, *where* $n > 0$ *and* $m \geq r$. *Then*

(*i*) $n \leq (m + 1)N$

(*ii*) $|\Delta_m| \leq |S|^{(m+1)N}$

(*iii*) $P\{(X_0, \dots, X_{\tau_F}) = (x_0, \dots, x_n)\} = \Theta(\epsilon^m)$ *and* $P\{(X_0, \dots, X_{\tau_F}) = (x_0, \dots, x_n)\} \leq \alpha\beta^m\epsilon^m$ *for all* $\epsilon > 0$ *sufficiently small, where* α *and* β *are constants which are independent of* $(x_0, \dots, x_n)$ *and* m

(*iv*) $L(x_0, \dots, x_n) = \Theta(\epsilon^{f(x_0, \dots, x_n) - b_0})$ *and* $L(x_0, \dots, x_n) \leq \nu\eta^m\epsilon^{f(x_0, \dots, x_n) - b_0}$ *for all* $\epsilon > 0$ *sufficiently small, where* ν *and* η *are constants which are independent of* $(x_0, \dots, x_n)$ *and* m.

PROOF. Parts (i), (ii), and the upper bound in (iii) are established in the proof of Theorem 1 of Nakayama [1991]. The first part of (iii) follows immediately from the definition of Δ_m. Also, we have shown the first part of (iv) in Eq. (14).

To prove the validity of the upper bound in part (iv), first note that for $(x_0, \dots, x_n) \in \Delta_m$,

$$L(x_0, \dots, x_n) = \prod_{k=0}^{n-1} \frac{P(x_k, x_{k+1})}{P'(x_k, x_{k+1})} \le \alpha\beta^m\epsilon^m \prod_{k=0}^{n-1} \frac{1}{P'(x_k, x_{k+1})} \tag{23}$$

by part (iii). From Eq. (11), we can express

$$P'(x, y) = \zeta(x, y)\epsilon^{1\{y \succ x\}(b(x,y)-s(x))} + o(\epsilon^{1\{y \succ x\}(b(x,y)-s(x))}),$$

where $\zeta(x, y)$ is independent of ϵ. Define $\zeta' = \min\{\zeta(x, y) : (x, y) \in \Gamma\}$ and $\zeta_. = \min\{1, \zeta'/2\}$. Note that $P'(x, y) \ge \zeta_. \epsilon^{1\{y \succ x\}(b(x,y)-s(x))}$ for all sufficiently small $\epsilon > 0$. Thus, for all sufficiently small $\epsilon > 0$,

$$\begin{aligned} \prod_{k=0}^{n-1} P'(x_k, x_{k+1}) &\ge \prod_{k=0}^{n-1} \zeta_. \epsilon^{1\{x_{k+1} \succ x_k\}(b(x_k,x_{k+1})-s(x_k))} \\ &= \zeta_.^n \epsilon^{\sum_{k=0}^{n-1} 1\{x_{k+1} \succ x_k\}(b(x_k,x_{k+1})-s(x_k))} \\ &\ge \zeta_.^{(m+1)N} \epsilon^{\sum_{k=0}^{n-1} 1\{x_{k-1} \succ x_k\}(b(x_k,x_{k+1})-s(x_k))} \end{aligned} \tag{24}$$

from part (i). We now examine the exponent of ϵ in Eq. (24).

First note that since $b(x, y) = 0$ if $y \prec x$,

$$\begin{aligned} &\sum_{k=0}^{n-1} 1\{x_{k+1} \succ x_k\}(b(x_k, x_{k+1}) - s(x_k)) \\ &\quad = \sum_{k=0}^{n-1} b(x_k, x_{k+1}) - \sum_{k=0}^{n-1} 1\{x_{k+1} \succ x_k\}s(x_k) \\ &\quad = \sum_{k=0}^{n-1} b(x_k, x_{k+1}) - f(x_0, \dots, x_n) = m + b_0 - f(x_0, \dots, x_n) \end{aligned}$$

by Eq. (21). It then follows from Eq. (24) that for all sufficiently small $\epsilon > 0$,

$$\prod_{k=0}^{n-1} P'(x_k, x_{k+1}) \ge \zeta_.^{(m+1)N} \epsilon^{m+b_0-f(x_0, \dots, x_n)}.$$

Thus, Eq. (23) implies that

$$L(x_0, \dots, x_n) \le \frac{\alpha}{\zeta_.^N} \left(\frac{\beta}{\zeta_.^N}\right)^m \epsilon^{f(x_0, \dots, x_n) - b_0},$$

and the proof is completed by letting $\nu = \alpha/\zeta_.^N$ and $\eta = \beta/\zeta_.^N$. □

Now we can prove Theorem 2.

PROOF OF THEOREM 2. Suppose **C0**$'$ holds. Since $\gamma = \Theta(\epsilon^r)$, it is clear from Eq. (13) that we need to prove that $E'[1\{\tau_F < \tau_0\}L^2] = O(\epsilon^{2r})$. Recall our expression for the second moment given in Eq. (12), and consider some

$(x_0, \ldots, x_n) \in \Delta_m$ where $r \leq m \leq 2r - 1$. Using Lemma 9, we see that

$$L(x_0, \ldots, x_n) P\{(X_0, \ldots, X_{\tau_F}) = (x_0, \ldots, x_n)\} = \Theta(\epsilon^{m + f(x_0, \ldots, x_n) - b_0}),$$

and so **C**0′ implies

$$L(x_0, \ldots, x_n) P\{(X_0, \ldots, X_{\tau_F}) = (x_0, \ldots, x_n)\} = O(\epsilon^{m + 2r - m}) = O(\epsilon^{2r}).$$

Hence

$$\sum_{m=r}^{2r-1} \sum_{\substack{(x_0, \ldots, x_n) \in \Delta_m \\ n > 0}} L(x_0, \ldots, x_n) P\{(X_0, \ldots, X_{\tau_F}) = (x_0, \ldots, x_n)\} = O(\epsilon^{2r}) \tag{25}$$

since $|\Delta_m| < \infty$ by Lemma 9.

Now consider $(x_0, \ldots, x_n) \in \Delta_m$, $n > 0$, with $m > 2r - 1$. By Lemma 9,

$$L(x_0, \ldots, x_n) \leq \nu \eta^m \epsilon^{f(x_0, \ldots, x_n) - b_0} \leq \nu \eta^m$$

for all $\epsilon > 0$ sufficiently small since $f(x_0, \ldots, x_n) \geq b_0$. Therefore, Lemma 9 implies that

$$\sum_{m=2r}^{\infty} \sum_{\substack{(x_0, \ldots, x_n) \in \Delta_m \\ n > 0}} L(x_0, \ldots, x_n) P\{(X_0, \ldots, X_{\tau_F}) = (x_0, \ldots, x_n)\}$$
$$\leq \sum_{m=2r}^{\infty} \sum_{\substack{(x_0, \ldots, x_n) \in \Delta_m \\ n > 0}} \nu \eta^m \alpha \beta^m \epsilon^m \leq \sum_{m=2r}^{\infty} |S|^{(m+1)N} \nu \eta^m \alpha \beta^m \epsilon^m \tag{26}$$
$$= \nu \alpha |S|^N \sum_{m=2r}^{\infty} \left(\eta \beta |S|^N \epsilon\right)^m = O(\epsilon^{2r})$$

for all sufficiently small ϵ since $\eta \beta |S|^N < \infty$. Using Eq. (25), we have that $E[1\{\tau_F < \tau_0\} L] = O(\epsilon^{2r})$, thereby showing RE' remains bounded as $\epsilon \to 0$.

Now suppose that **C**0′ does not hold. Then there exists some $(y_0, \ldots, y_k) \in \Delta_m$ such that $r \leq m \leq 2r - 1$ and $f(y_0, \ldots, y_k) \leq 2r - m + b_0 - 1$. Thus, Lemma 9 implies that

$$L(y_0, \ldots, y_k) P\{(X_0, \ldots, X_{\tau_F}) = (y_0, \ldots, y_k)\}$$
$$= \Theta(\epsilon^{f(y_0, \ldots, y_k) - b_0}) \Theta(\epsilon^m) = \underline{O}(\epsilon^{2r - m + b_0 - 1 - b_0}) \Theta(\epsilon^m) = \underline{O}(\epsilon^{2r-1}).$$

It then follows that

$$E[1\{\tau_F < \tau_0\} L] = \sum_{\substack{(x_0, \ldots, x_n) \in \Delta \\ n \geq 0}} L(x_0, \ldots, x_n) P\{(X_0, \ldots, X_{\tau_F}) = (x_0, \ldots, x_n)\}$$
$$\geq L(y_0, \ldots, y_k) P\{(X_0, \ldots, X_{\tau_F}) = (y_0, \ldots, y_k)\} = \underline{O}(\epsilon^{2r-1}),$$

and so $RE' \to \infty$ as $\epsilon \to 0$. □

We now prove Theorem 4.

PROOF OF THEOREM 4. Recall that $RE = \Theta(\epsilon^{-r/2})$. First suppose $\mathbf{C}i$ holds. Corollary 4 of Nakayama [1991] establishes that RE_i/RE remains bounded as $\epsilon \to 0$ if either $r_i = r$ or $b_i = b_0$. Hence, we may assume that $r_i > r$, $b_i > b_0$, and $\bar{r}_i - 2b_0 \le r_i - 2b_i$, and so $\sigma_i^2 = \Theta(\epsilon^{\bar{r}_i - 2b_0})$ by Eq. (17). Also, $\bar{r}_i - 2b_0 \le r_i - 2b_i$ implies that $\bar{r}_i - b_0 = \bar{r}_i - 2b_0 + b_0 \le r_i - 2b_i + b_0 < r_i - b_i$ since $b_i > b_0$, and so $\partial_i\gamma = \Theta(\epsilon^{\bar{r}_i - b_0})$. Furthermore, $r_i > r$ implies that $\bar{r}_i = r$ by Eq. (8), and so $RE_i = \Theta(\epsilon^{(r-2b_0)/2-(r-b_0)}) = \Theta(\epsilon^{-r/2})$. Thus, RE_i/RE remains bounded as $\epsilon \to 0$.

Now suppose that $\mathbf{C}i$ does not hold, i.e., assume $r_i > r$, $b_i > b_0$, and $\bar{r}_i - 2b_0 > r_i - 2b_i$. It then follows that $\bar{r}_i = r$ by Eq. (8) and $\sigma_i^2 = \Theta(\epsilon^{r_i - 2b_i})$ by Eq. (17). First suppose $r_i - b_i \le \bar{r}_i - b_0 = r - b_0$. Then $\partial_i\gamma = \Theta(\epsilon^{r_i - b_i})$, and so $RE_i = \Theta(\epsilon^{(r_i-2b_i)/2-(r_i-b_i)}) = \Theta(\epsilon^{-r_i/2})$. Hence, since $r_i > r$, we have $RE_i/RE \to \infty$ as $\epsilon \to 0$. Now suppose that $r_i - b_i > \bar{r}_i - b_0 = r - b_0$. Then $\partial_i\gamma = \Theta(\epsilon^{r-b_0})$, and $RE_i = \Theta(\epsilon^{(r_i-2b_i)/2-(r-b_0)})$. Note that $(r_i - 2b_i)/2 - (r - b_0) < (\bar{r}_i - 2b_0)/2 - (r - b_0) = -r/2$ since $\bar{r}_i = r$. Thus, we again see that $RE_i/RE \to \infty$ as $\epsilon \to 0$. □

We now prove Theorem 5 by first establishing some preliminary results. Our first lemma describes the forms of the summands in the expression for D_i; see Nakayama [1991] for the proof.

LEMMA 10. *Suppose* $(x, y) \in \Gamma$ *with* $x \in U$. *Then for all* ϵ *sufficiently small,*

(*i*) $\partial_i P(x, y)/P(x, y) = \rho(x, y)\epsilon^{-b_i} + o(\epsilon^{-b_i})$, *where* $\rho(x, y) \ne 0$ *is independent of* ϵ, *if* $y \succ x$ *with* $p(y; x, i) > 0$;

(*ii*) $\partial_i P(x, y)/P(x, y) = \rho(x, y) + o(1)$, *where* $\rho(x, y) < 0$ *is independent of* ϵ, *if either* $x \ne 0$ *and* $y \prec x$ *or* $x \ne 0$ *and* $y \succ x$ *with* $p(y; x, i) = 0$;

(*iii*) $\partial_i P(x, y)/P(x, y) = \rho(x, y)\epsilon^{-b_0} + o(\epsilon^{-b_0})$, *where* $\rho(x, y) < 0$ *is independent of* ϵ, *if* $x = 0$ *and* $y \succ 0$ *with* $p(y; 0, i) = 0$.

The next lemma establishes bounds on D_i.

LEMMA 11. *Consider* $(x_0, \dots, x_n) \in \Delta_m$, *where* $n > 0$ *and* $m \ge r$. *Then there exists a constant* ϕ *which is independent of* $(x_0, \dots, x_n)$ *and* ϵ *such that for all* $\epsilon > 0$ *sufficiently small,*

(*i*) $D_i(x_0, \dots, x_n) = \Theta(\epsilon^{-b_i})$ *and* $|D_i(x_0, \dots, x_n)| \le (m + 1)\phi\epsilon^{-b_i}$ *if* $(x_0, \dots, x_n) \in \Delta_m^i$;

(*ii*) $D_i(x_0, \dots, x_n) = \Theta(\epsilon^{-b_0})$ *and* $|D_i(x_0, \dots, x_n)| \le (m + 1)\phi\epsilon^{-b_0}$ *if* $(x_0, \dots, x_n) \in \bar{\Delta}_m^i$.

PROOF. The first part of (i) was established in the proof of Lemma 13 of Nakayama [1991]. Similarly, we can prove the first part of (ii). To show the other results, note that

$$|D_i(x_0, \dots, x_n)| \le \sum_{k=0}^{n-1} \left| \frac{\partial_i P(x_k, x_{k+1})}{P(x_k, x_{k+1})} \right|$$

by the triangle inequality. First suppose that $(x_0, \dots, x_n) \in \Delta_m^i$. Thus, the path $(x_0, \dots, x_n)$ has at least one transition that satisfies the conditions of

Lemma 10(i). Since $b_i \geq b_0$, we have that $\epsilon^{-b_0} = O(\epsilon^{-b_i})$ and $1 = o(\epsilon^{-b_i})$. Lemma 10 then implies that $\partial_i P(x, y)/P(x, y) = O(\epsilon^{-b_i})$ for all transitions $(x, y) \in \Gamma$ with $x \in U$. Now define $\rho_* = \max\{|\rho(x, y)| : (x, y) \in \Gamma\}$. Then $|\partial_i P(x, y)/P(x, y)| \leq 2\rho_* \epsilon^{-b_i}$ for all sufficiently small $\epsilon > 0$. It follows from Lemma 9(i) that for all sufficiently small ϵ,

$$|D_i(x_0, \dots, x_n)| \leq \sum_{k=0}^{n-1} 2\rho_* \epsilon^{-b_i} = 2n\rho_* \epsilon^{-b_i} \leq 2(m+1)N\rho_* \epsilon^{-b_i},$$

thereby proving (i), where $\phi = 2N\rho_*$.

Now suppose that $(x_0, \dots, x_n) \in \overline{\Delta}^i_m$. Then each of the transitions of $(x_0, \dots, x_n)$ satisfies either condition (ii) or (iii) of Lemma 10. Consequently, since $1 = o(\epsilon^{-b_0})$, we have that $|\partial_i P(x_k, x_{k+1})/P(x_k, x_{k+1})| \leq 2\rho_* \epsilon^{-b_0}$ for all sufficiently small $\epsilon > 0$. Then Lemma 9(i) implies that

$$|D_i(x_0, \dots, x_n)| \leq \sum_{k=0}^{n-1} 2\rho_* \epsilon^{-b_0} = 2n\rho_* \epsilon^{-b_0} \leq 2(m+1)N\rho_* \epsilon^{-b_0}$$

for all sufficiently small ϵ, which completes the proof. □

Now we are in a position to prove Theorem 5.

PROOF OF THEOREM 5. Assume $\mathbf{C}i'$ holds. First suppose that $r_i - b_i \leq \bar{r}_i - b_0$. Then Eq. (16) implies that $\partial_i \gamma = \Theta(\epsilon^{r_i - b_i})$. Thus, it is clear from Eq. (18) that to establish our result, we must show that $E'[1\{\tau_F < \tau_0\} D_i^2 L^2] = O(\epsilon^{2(r_i - b_i)})$.

Recall our decomposition in Eq. (19). First we work with $E[1\{\tau_i \leq \tau_F < \tau_0\} D_i^2 L]$. Consider any path $(x_0, \dots, x_n)$ that satisfies $\tau_i \leq \tau_F < \tau_0$. Thus, $(x_0, \dots, x_n) \in \Delta^i_m$ for some $m \geq r_i$, and so Lemmas 9 and 11 imply that

$$\begin{aligned} D_i^2(x_0, \dots, x_n) L(x_0, \dots, x_n) P\{(X_0, \dots, X_{\tau_F}) = (x_0, \dots, x_n)\} \\ \leq ((m+1)\phi\epsilon^{-b_i})^2 \nu\eta^m \epsilon^{f(x_0, \dots, x_n) - b_0} \alpha\beta^m \epsilon^m \\ = (m+1)^2 \alpha\nu\phi^2 (\beta\eta)^m \epsilon^{m - 2b_i + f(x_0, \dots, x_n) - b_0}. \end{aligned} \tag{27}$$

Now consider a path $(x_0, \dots, x_n) \in \Delta^i_m$, where $r_i \leq m \leq 2r_i - 1$. Condition $\mathbf{C}i'$ implies that $f(x_0, \dots, x_n) \geq 2r_i - m + b_0$, and so it follows from Eq. (27) that

$$\begin{aligned} D_i^2(x_0, \dots, x_n) L(x_0, \dots, x_n) P\{(X_0, \dots, X_{\tau_F}) = (x_0, \dots, x_n)\} \\ \leq (m+1)^2 \alpha\nu\phi^2 (\beta\eta)^m \epsilon^{2r_i - 2b_i} \end{aligned}$$

for all ϵ sufficiently small. Consequently,

$$\sum_{m=r_i}^{2r_i-1} \sum_{\substack{(x_0, \dots, x_n) \in \Delta^i_m \\ n>0}} D_i^2(x_0, \dots, x_n) L(x_0, \dots, x_n) \times P\{(X_0, \dots, X_{\tau_F}) = (x_0, \dots, x_n)\} = O(\epsilon^{2r_i - 2b_i})$$

since $|\Delta_m| < \infty$ for all m by Lemma 9. Using arguments similar to those employed to establish Eq. (26), we can show that

$$\sum_{m=2r_i}^{\infty} \sum_{\substack{(x_0,\ldots,x_n)\in\Delta_m^i \\ n>0}} D_i^2(x_0,\ldots,x_n)L(x_0,\ldots,x_n)$$
$$\times P\{(X_0,\ldots,X_{\tau_F}) = (x_0,\ldots,x_n)\} = O(\epsilon^{2r_i-2b_i})$$

using Lemmas 9 and 11. Thus,

$$E\left[1\{\tau_i \le \tau_F < \tau_0\}D_i^2L\right] = O(\epsilon^{2r_i-2b_i}). \tag{28}$$

Similarly, we can show that $\mathbf{C}i'$ implies that

$$E\left[1\{\tau_F < \min(\tau_0, \tau_i)\}D_i^2L\right] = O(\epsilon^{2r_i-2b_i}). \tag{29}$$

Therefore, if $r_i - b_i \le \bar{r}_i - b_0$, then $E[1\{\tau_F < \tau_0\}D_i^2L] = O(\epsilon^{2r_i-2b_i})$ from Eqs. (28) and (29). Hence, we have proven that $\mathbf{C}i'$ ensures RE_i' remains bounded as $\epsilon \to 0$ when $r_i - b_i \le \bar{r}_i - b_0$. Also, we can establish using similar arguments that $\mathbf{C}i'$ implies that RE_i' remains bounded as $\epsilon \to 0$ when $r_i - b_i > \bar{r}_i - b_0$.

By appropriately modifying the proof used to show that $RE' \to \infty$ as $\epsilon \to 0$ if $\mathbf{C}0'$ does not hold, we can prove that $RE_i' \to \infty$ as $\epsilon \to 0$ if any of the conditions of $\mathbf{C}i'$ is violated. □

We now prove Theorem 6.

Proof of Theorem 6. Since the system is balanced, $b_i = 1$ for all $1 \le i \le C$ and $b_0 = 1$. Thus, $s(x) = 1$ for all $x \in S$. Now consider any path $(x_0,\ldots,x_n) \in \Delta_m$. Note that $(x_0,\ldots,x_n)$ has exactly $m+1$ failure transitions (including the initial transition), and so $f(x_0,\ldots,x_n) = m+1$. To verify that $\mathbf{C}i'$ holds, first suppose that $r_i - b_i \le \bar{r}_i - b_0$, and so $r_i \le \bar{r}_i$. If $(x_0,\ldots,x_n) \in \Delta_m^i$, then we must have that $m \ge r_i$, and so $f(x_0,\ldots,x_n) = m + 1 \ge 2r_i - m + b_0 = 2r_i - m + 1$ holds. If $(x_0,\ldots,x_n) \in \bar{\Delta}_m^i$, then $m \ge \bar{r}_i \ge r_i$, and so $f(x_0,\ldots,x_n) = m+1 \ge 2r_i - m + 3b_0 - 2b_i = 2r_i - m + 1$ holds. We can similarly show that $\mathbf{C}i'$ holds if $r_i - b_i > \bar{r}_i - b_0$. □

We now prove Theorem 7.

Proof of Theorem 7. Suppose that $\mathbf{C}0'$ does not hold. Then there exists some $(y_0,\ldots,y_k) \in \Delta_m$ with $r \le m \le 2r - 1$ such that $f(y_0,\ldots,y_k) < 2r - m + b_0$. Now there must exist some component type i such that $(y_0,\ldots,y_k) \in \Delta_m^i$, and so $\Delta_m^i \ne \emptyset$. It then follows that $r_i \le m$.

First suppose $r_i - b_i \le \bar{r}_i - b_0$. From Eq. (8), we have that $r_i \ge r$. Thus, $r_i \le m$ and $m \le 2r - 1$ imply that $r_i \le m \le 2r_i - 1$. Also, we have that $f(y_0,\ldots,y_k) < 2r - m + b_0 \le 2r_i - m + b_0$, and so $\mathbf{C}i'$ does not hold.

Now suppose $r_i - b_i > \bar{r}_i - b_0$. Note that Eq. (8) implies that $r_i \ge r$ and $2\bar{r}_i + 2b_i - 2b_0 - 1 \ge 2r + 2b_i - 2b_0 - 1 \ge 2r - 1$ since $b_i \ge b_0$. Hence, $r_i \le m \le 2\bar{r}_i + 2b_i - 2b_0 - 1$. Also, $f(y_0,\ldots,y_k) < 2r - m + b_0 \le 2\bar{r}_i - m - b_0 + 2b_i$ since $b_i \ge b_0$. Therefore, $\mathbf{C}i'$ does not hold. □

Now we prove Theorem 8.

PROOF OF THEOREM 8. Assume $\mathbf{C}0'$ holds. First suppose $r_i = r$. Since $\bar{r}_i \geq r = r_i$, we have that $r_i - b_i \leq \bar{r}_i - b_0$ as $b_i \geq b_0$. Now consider any $(x_0, \ldots, x_n) \in \Delta_m^i$ with $r_i \leq m \leq 2r_i - 1$. Since $r_i = r$, we have that $r \leq m \leq 2r - 1$. Also, $\mathbf{C}0'$ and $r_i = r$ imply that $f(x_0, \ldots, x_n) \geq 2r - m + b_0 = 2r_i - m + b_0$. Now consider any $(x_0, \ldots, x_n) \in \bar{\Delta}_m^i$ with $\bar{r}_i \leq m \leq 2r_i - 2b_i + 2b_0 - 1$. It follows from $r_i = r$ and $b_i \geq b_0$ that $2r_i - 2b_i + 2b_0 - 1 \leq 2r - 1$. Thus, $\bar{r}_i \geq r$ implies that $r \leq m \leq 2r - 1$. Furthermore, $\mathbf{C}0'$ gives us that $f(x_0, \ldots, x_n) \geq 2r - m + b_0 \geq 2r_i - m + 3b_0 - 2b_i$ since $r_i = r$ and $b_i \geq b_0$. Therefore, $\mathbf{C}i'$ holds.

Now suppose that $b_i = b_0$. We may assume that $r_i > r$, and so $\bar{r}_i = r$ and $r_i - b_i > \bar{r}_i - b_0$. Consider any $(x_0, \ldots, x_n) \in \Delta_m^i$ with $r_i \leq m \leq 2\bar{r}_i + 2b_i - 2b_0 - 1$. Then, $2\bar{r}_i + 2b_i - 2b_0 - 1 = 2r - 1$, and so $r < m \leq 2r - 1$. Hence, $\mathbf{C}0'$ implies that $f(x_0, \ldots, x_n) \geq 2r - m + b_0 = 2\bar{r}_i - m - b_0 + 2b_i$. Now consider any $(x_0, \ldots, x_n) \in \bar{\Delta}_m^i$ with $\bar{r}_i \leq m \leq 2\bar{r}_i - 1$. Note that $r \leq m \leq 2r - 1$ since $\bar{r}_i = r$. Thus, $\mathbf{C}0'$ implies that $f(x_0, \ldots, x_n) \geq 2r - m + b_0 = 2\bar{r}_i - m + b_0$, and so $\mathbf{C}i'$ holds. □

The following example shows that it is possible that RE_i/RE remains bounded as $\epsilon \to \infty$, but RE_i'/RE' diverges as $\epsilon \to 0$. In other words, if standard simulation is used, we may be able to estimate some derivative and the performance measure with the same relative accuracy, but we cannot if simple failure biasing is used.

Example 5. Consider a system which has three types of components (i.e., $C = 3$), where each of the first two types has redundancy four (i.e., $n_1 = n_2 = 4$), and the third type has redundancy three (i.e., $n_3 = 3$). The components of type 1 and 2 have failure rate ϵ (i.e., $b_1 = b_2 = 1$), and the components of type 3 have failure rate ϵ^2 (i.e., $b_3 = 2$). Thus $b_0 = 1$. There is a single repairperson who repairs components at rate 1 using a processor-sharing discipline. It is then sufficient to define the state of the system to be $x = \langle x_1, x_2, x_3 \rangle$, where x_i is the number of failed components of type i. Initially, all components are operational, and the system is considered to be failed if and only if all of the components of all types are failed. When a component of type 1 fails in state $\langle 3, 0, 0\rangle$, it causes all four components of type 2 and all three components of type 3 to fail, i.e., $p(\langle 4, 4, 3\rangle; \langle 3, 0, 0\rangle, 1) = 1$. When a component of type 2 fails in state $\langle 0, 3, 0\rangle$, it causes all four components of type 1 and all three components of type 3 to fail, i.e., $p(\langle 4, 4, 3\rangle; \langle 0, 3, 0\rangle, 1) = 1$. When a component of type 3 fails in state $\langle 0, 0, 2\rangle$, it causes all of the components of types 1 and 2 to fail, i.e., $p(\langle 4, 4, 3\rangle; \langle 0, 0, 2\rangle, 1) = 1$.

It is easy to see that $\gamma = \Theta(\epsilon^3)$, and so $r = 2$. Also, $r_1 = 3$, $r_2 = 3$, $r_3 = 5$, $\bar{r}_1 = 3$, $\bar{r}_2 = 3$, and $\bar{r}_3 = 3$. We can show that $\mathbf{C}0'$ holds. Also, we have that $\bar{r}_3 - 2 = 1 \leq r_3 - 2b_3 = 1$, and so $\mathbf{C}3$ holds. (Also, $\mathbf{C}S3$ is not in force.) By considering the path $(\langle 0, 0, 0\rangle, \langle 0, 0, 1\rangle, \langle 0, 0, 2\rangle, \langle 4, 4, 3\rangle) \in \Delta_5^3$, we can show that $\mathbf{C}3'$ does not hold.

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