Central Limit Theorems for Permuted Regenerative Estimators

James M. Calvin and Marvin K. Nakayama
Simulation and Modeling Laboratory
Department of Computer and Information Science
New Jersey Institute of Technology
Newark, NJ 07102-1982, U.S.A.
e-mail: calvin@cis.njit.edu, marvin@cis.njit.edu

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Abstract

We prove strong laws of large numbers and central limit theorems for some permuted estimators from regenerative simulations. These limit theorems provide the basis for constructing asymptotically valid confidence intervals for the permuted estimators.

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1 Introduction

In Calvin and Nakayama (1998), we introduced a new class of estimators for certain performance measures that can be estimated using the regenerative method for simulation. (For more details on the standard regenerative method, see, for example, Iglehart 1978 or Shedler 1993.) Our method applies to simulations of systems with two sequences of regeneration points. The basic idea for constructing the new estimator is to first generate a sample path of a fixed number of cycles from the first sequence of regeneration points and compute one estimate of the performance measure from the path. We can then divide up the path into segments based on the second sequence of regeneration points, and for each permutation of the segments create another sample path, from which we compute another estimate. Finally, we construct the new estimator, which we call the *permuted estimator*, as the average of the estimates over all permutations. Calvin and Nakayama (1998) derive explicit formulae for the permuted estimators, and so one does not actually have to compute all permutations to construct the permuted estimators. The permuted estimators only require the user to keep track of several extra accumulators during the simulation. The amount of extra storage is fixed and does not increase with the simulation run length, and so the permuted estimators are computationally efficient.

In Calvin and Nakayama (1998), we studied the small-sample behavior of the permuted estimators. It was proven that for any fixed number of cycles from the first sequence of regenerative cycles, the permuted estimators have no greater variance than and the same bias as the corresponding standard estimators; hence, the mean squared errors (MSEs) are no larger. While the permutation method has no effect (i.e., no MSE reduction) on ratio estimators for the long-run average reward, the MSE of estimators of other performance measures are typically strictly reduced by permuting.

In this paper we examine the asymptotic behavior of some permuted estimators, proving strong laws of large numbers and central limit theorems. These results are important because they establish the strong consistency of the permuted estimators and allow one to construct asymptotically valid confidence intervals. Since the permuted estimators are not expressed as the sum of quantities defined over a sequence of regenerative cycles, it was not originally clear how to exploit the regenerative structure to prove central limit theorems for the estimators. To overcome this difficulty, we will derive a new representation of our estimator that will allow us to exploit the regenerative structure to establish our asymptotic results. We consider here three performance measures analyzed in Calvin and Nakayama (1998): the second moment of a cumulative cycle reward, the time-average variance constant, and the expected cumulative reward until hitting a set of states F, of which the mean time to failure is a special case.

In some sense the permuted estimators are constructed by reusing the collected data, and as such, they are related to other statistical techniques. For example, the bootstrap (Efron 1979) takes a given sample and resamples the data with replacement. In contrast, one can think of our approach as resampling the data without replacement (i.e., a permutation), and then averaging over all possible resamples. Other related methods include U-statistics (Serfling 1980, Chapter 5), V-statistics (Sen 1977), and permutation tests (e.g., Conover 1980).

The rest of the paper has the following structure. We discuss the general assumptions in Section 2. In Sections 3, 4 and 5, we derive strong laws of large numbers and central limit theorems for the estimators of the second moment of a cumulative cycle reward, the time-average variance constant, and the expected cumulative reward until hitting F, respectively. In Section 6 we analyze a small model and derive the exact theoretical values for the second moment and the asymptotic variances associated with its standard and permuted estimators. We present empirical results in Section 7. We defer some of the more technical proofs to an appendix.

2 General Framework

Let $X = \{X(t) : t \geq 0\}$ be a continuous-time stochastic process having sample paths that are right continuous with left limits on a state space $S \subset \Re^d$. Note that we can handle discrete-time processes $\{X_n : n = 0, 1, 2, \ldots\}$ in this framework by letting $X(t) = X_{\lfloor t \rfloor}$ for all $t \geq 0$, where $\lfloor a \rfloor$ is the greatest integer less than or equal to a.

Let $T = \{T(i) : i = 0, 1, 2, \ldots\}$ be an increasing sequence of nonnegative finite stopping times. Consider the random pair (X, T) and for $i = 0, 1, 2, \ldots$, define the shift

$$\theta_{T(i)}(X, T) = ((X(T(i) + t)_{t \geq 0},\ (T(k) - T(i))_{k \geq i}) \,.$$

We define the pair (X, T) to be a delayed *regenerative process* (in the classic sense) if

(i) $\theta_{T(i)}(X, T)$, $i = 0, 1, 2, \ldots$, are identically distributed;

(ii) for each $i \geq 0$, $\theta_{T(i)}(X, T)$ does not depend on the "prehistory"

$$\Big((X(t))_{t < T(i)},\ T(0), T(1), \ldots, T(i)\Big) \,.$$

See p. 19 of Kalashnikov (1994), p. 52 of Shedler (1993), or Section 2.6 of Kingman (1972) for more details.

We assume that $T_1 = \{T_1(i) : i = 0, 1, 2, \ldots\}$ with $T_1(0) = 0$ and $T_2 = \{T_2(i) : i = 0, 1, 2, \ldots\}$ are two disjoint increasing sequences of nonnegative finite stopping times such that (X, T_1) and (X, T_2) are both regenerative processes. For example, if X is an irreducible, positive-recurrent Markov chain on a countable state space S, then we can define T_1 and T_2 to be the sequences of hitting times to the states $v \in S$ and $w \in S$, respectively, where we assume that $X(0) = v$ and $w \neq v$.

We want to estimate some performance measure α by generating a sample path segment $\vec{X}_m = \{X(t) : 0 \leq t < T_1(m)\}$ of a fixed number m of regenerative T_1-cycles of our regenerative process.

We use the terminology "T_l-cycles" to denote cycles determined by the sequence T_l for $l = 1, 2$; i.e., the i-th T_l-cycle is the path segment $\{X(t) : T_l(i-1) \leq t < T_l(i)\}$. Then the standard estimator of α, denoted by $\widehat{\alpha}(m)$, can be written in the form $\widehat{\alpha}(m) = h(\vec{X}_m)$, where h is a specified function that depends on m and α.

Calvin and Nakayama (1998) derive their permuted estimators as follows. First let $M_2 \equiv M_2(\vec{X}_m) = 1 + \sup\{i : T_2(i) \leq T_1(m)\}$ be the number of times that stopping times from sequence T_2 occur by the end of the mth T_1-cycle on the path $\vec{X}_m$. Divide the sample path $\vec{X}_m$ into segments, where the first segment is from time 0 to time $T_2(0)$, the last segment is from time $T_2(M_2)$ to time $T_1(m)$, and the other segments are the T_2-cycles in the path. Then permute the T_2-cycles in the path $\vec{X}_m$ to create a new path $\vec{X}'_m$, and compute an estimator $h(\vec{X}'_m)$ based on the permuted path $\vec{X}'_m$. There are $N(\vec{X}_m) \equiv (M_2 - 1)!$ possible paths that can be constructed by permuting the T_2-cycles. Let these paths be $\vec{X}_m^{(1)} \equiv \vec{X}_m, \vec{X}_m^{(2)}, \ldots, \vec{X}_m^{(N(\vec{X}_m))}$, and the new estimator is given by

$$\widetilde{\alpha}(m) = \frac{1}{N(\vec{X}_m)} \sum_{j=1}^{N(\vec{X}_m)} h(\vec{X}_m^{(j)}), \tag{1}$$

which we call the *permuted estimator*. Calvin and Nakayama (1998) derive explicit formulae for the permuted estimators for three different performance measures; e.g., see equations (6), (10), (14), (15) in the current paper. Hence, one does not actually have to carry out all permutations to compute the permuted estimators for these cases.

As noted in Calvin and Nakayama (1998), another way of looking at our permuted estimator is as follows. We first generate the original path $\vec{X}_m$ and based on this, we construct the $N(\vec{X}_m)$ new paths $\vec{X}_m^{(1)}, \ldots, \vec{X}_m^{(N(\vec{X}_m))}$. We then select one of the new paths at random uniformly from $\vec{X}_m^{(1)}, \ldots, \vec{X}_m^{(N(\vec{X}_m))}$, and let this be $\vec{X}'_m$. Calvin and Nakayama (1998) show that $\vec{X}'_m \stackrel{\mathcal{D}}{=} \vec{X}_m$, where "$\stackrel{\mathcal{D}}{=}$" denotes equality in distribution, and so we can think of $h(\vec{X}'_m)$ as a standard estimator of α since it has the same distribution as $\widehat{\alpha}(m) = h(\vec{X}_m)$. Then we construct our permuted estimator $\widetilde{\alpha}(m)$ to be the conditional expectation of $h(\vec{X}'_m)$ with respect to the uniform random choice of $\vec{X}'_m$ given the original path $\vec{X}_m$. In other words, if E_* denotes conditional expectation with respect to choosing $\vec{X}'_m$ from the uniform distribution on $\vec{X}_m^{(i)}$, $1 \leq i \leq N(\vec{X}_m)$, given $\vec{X}_m$, then we write

$$\widetilde{\alpha}(m) = E_*\left[h(\vec{X}'_m)\right].$$

Assuming that $E[|h(\vec{X}_m)|] < \infty$, the permuted estimator has the same mean as the standard one since for all m,

$$E[\widetilde{\alpha}(m)] = E\left[\, E_*[h(\vec{X}'_m)] \,\right] = E[h(\vec{X}'_m)] = E[\widehat{\alpha}(m)], \tag{2}$$

because $\vec{X}'_m \stackrel{\mathcal{D}}{=} \vec{X}_m$. Moreover, if $E[h(\vec{X}_m)^2] < \infty$, then decomposing the variance by conditioning on $\vec{X}_m$ gives us that for all m,

$$\mathrm{Var}\,(\widehat{\alpha}(m)) = \mathrm{Var}\left(h(\vec{X}'_m)\right) = \mathrm{Var}\left(E_*[h(\vec{X}'_m)]\right) + E\left[\,\mathrm{Var}(\, h(\vec{X}'_m) \,|\, \vec{X}_m)\,\right]. \tag{3}$$

Thus, since $E\left[\,\mathrm{Var}(\, h(\vec{X}'_m) \,|\, \vec{X}_m)\,\right] \geq 0$, the variance of the permuted estimator $\widetilde{\alpha}(m) = E_*\left[h(\vec{X}'_m)\right]$ is no greater than that of the standard estimator $\widehat{\alpha}(m)$.

3 Estimating the Second Moment of Cumulative Cycle Reward

Suppose that we want to estimate

$$\alpha = E\left[\left(\int_{T_1(0)}^{T_1(1)} g(X(t))\,dt\right)^2\right], \tag{4}$$

for some "reward" function $g : S \to \Re$. Under the regenerative method of simulation, the standard estimator of α is

$$\widehat{\alpha}(m) = \frac{1}{m}\sum_{k=1}^{m} Y(k)^2, \tag{5}$$

where

$$Y(k) = Y(g;k) = \int_{T_1(k-1)}^{T_1(k)} g(X(t))\,dt.$$

(We will suppress the dependence of Y on g in our notation unless it is needed for clarity.)

Performance measures α having the form in (4) arise in many practical contexts. For example, suppose X is the server-busy process of a queue, the T_1 sequence corresponds to a customer arriving to an empty system, and $g(x) = 1$ for $x > 0$ and $g(0) = 0$. Then α is the second moment of the length of a busy period. We also show in Section 4 how to estimate the time-average variance constant of a process by slightly modifying this framework.

Calvin and Nakayama (1998) explicitly calculate the permuted estimator in (1) for the performance measure in (4). To write an expression for the permuted estimator, we need some notation. For our two sequences of stopping times T_1 and T_2, let $H(1;2) \equiv H(1;2;m) \subset \{1, 2, \ldots, m\}$ denote the set of indices of the T_1-cycles in which at least one T_2 stopping time occurs, and define the complementary set $J(1;2) \equiv J(1;2;m) = \{1, 2, \ldots, m\} - H(1;2)$. More specifically,

$H(1;2) = \{i \le m : T_1(i-1) < T_2(j) < T_1(i) \text{ for some } j\}$. We analogously define the sets $H(2;1)$ and $J(2;1)$ with the roles of T_1 and T_2 reversed. Let $h_{12} \equiv h_{12}(m) = |H(1;2)|$. For $k \in H(1;2)$, define $T_2'(k) = \inf\{t > T_1(k-1) : T_2(l) = t \text{ for some } l\}$, which is the first occurrence of a stopping time from sequence T_2 after the $(k-1)$st stopping time from the sequence T_1. Similarly define $\widetilde{T}_2(k) = \sup\{t < T_1(k) : T_2(l) = t \text{ for some } l\}$, which is the last occurrence of a stopping time from sequence T_2 before the kth occurrence of the stopping-time sequence T_1. Then, for $k \in H(1;2)$, we let

$$Y_{12}(k) = \int_{T_1(k-1)}^{T_2'(k)} g(X(t))\,dt,$$

which is the contribution to $Y(k)$ until a stopping time from sequence T_2 occurs, and let

$$Y_{21}(k) = \int_{\widetilde{T}_2(k)}^{T_1(k)} g(X(t))\,dt,$$

which is the contribution to $Y(k)$ from the last occurrence of a stopping time from sequence T_2 in the kth T_1-cycle until the end of the cycle. Also, for $l \in J(2;1)$, let

$$Y_{22}(l) = \int_{T_2(l-1)}^{T_2(l)} g(X(t))\,dt,$$

which is the integral of $g(X(t))$ over the lth T_2-cycle in which there is no occurrence of a stopping time from sequence T_1. Also, define

$$\bar{Y}_{12}(m) = \frac{1}{h_{12}} \sum_{k \in H(1;2)} Y_{12}(k),$$

and

$$\bar{Y}_{21}(m) = \frac{1}{h_{12}} \sum_{k \in H(1;2)} Y_{21}(k).$$

Finally, define β_l, $l = 1, 2, \ldots, h_{12}$, such that $H(1;2) = \{\beta_1, \beta_2, \ldots, \beta_{h_{12}}\}$ and $\beta_1 < \beta_2 < \cdots < \beta_{h_{12}}$; i.e., β_l is the lth smallest element of the set $H(1;2)$. Also, define $\beta_0 = \beta_{h_{12}}$. For $k = \beta_l \in H(1;2)$ for some $l = 1, 2, \ldots, h_{12}$, define $\psi(k) = \beta_{l-1}$; i.e., $\psi(k)$ is the index in $H(1;2)$ that occurs just before k if k is not the first index, and $\psi(k)$ is the last element in $H(1;2)$ if k is the first element.

Then, our permuted estimator is given by $\widetilde{\alpha}(m) = \widehat{\alpha}(m)$ if $M_2 < 3$, and otherwise by

$$\begin{aligned} \widetilde{\alpha}(m) &= \frac{1}{m}\left(\sum_{k\in J(1;2)} Y(k)^2 + \sum_{k\in H(1;2)} \left[Y_{12}(k)^2 + Y_{21}(k)^2\right] \right. \\ &+ \frac{2}{h_{12}-1} \sum_{k\in H(1;2)} Y_{12}(k) \left(\sum_{j\in H(1;2)} Y_{21}(j) - Y_{21}(\psi(k)) \right) + \sum_{k\in J(2;1)} Y_{22}(k)^2 \\ &+ \left. 2(\bar{Y}_{12}(m) + \bar{Y}_{21}(m)) \sum_{k\in J(2;1)} Y_{22}(k) + \frac{2}{1+h_{12}} \sum_{\substack{j,l\in J(2;1)\\ j\neq l}} Y_{22}(j)Y_{22}(l) \right). \end{aligned} \tag{6}$$

As noted by Calvin and Nakayama (1998), it follows from (2) and (3) that if $E\left[Y(1)^4\right] < \infty$, then for any fixed number m of T_1-cycles, the estimator satisfies $E[\widetilde{\alpha}(m)] = \alpha$ and $\mathrm{Var}(\widetilde{\alpha}(m)) \le \mathrm{Var}(\widehat{\alpha}(m))$ when $\widehat{\alpha}(m)$ is the standard estimator of α as defined in (5).

Our goal now is to study the asymptotic properties of the permuted estimator. In particular, we want to prove a strong law of large numbers and a central limit theorem for the permuted estimator $\widetilde{\alpha}(m)$. Notice that part of the permuted estimator in (6) is a sum of quantities defined over regenerative cycles, but part of the estimator is a sum of products of terms defined over different regenerative cycles (for example in the last line of (6)). Thus, it was not originally clear how to establish limit theorems for the permuted estimator. Now we derive a new representation of our estimator that will allow us to exploit the regenerative structure to prove our asymptotic results. To do this, we need some more notation. Define $B_k \subset J(2;1)$ to be the set of indices of those T_2-cycles that do not contain any occurrences of the stopping times from the sequence T_1 and that are between $T_1(k-1)$ and $T_1(k)$. Let $1\{\cdot\}$ denote the indicator function of the event $\{\cdot\}$, and for T_1-cycle indices $k = 1, 2, \ldots$, define

$$\begin{aligned} Z_1(k) &= Y(k)^2\, 1\{k \notin H(1;2)\}, \\ Z_2(k) &= Y_{12}(k)^2\, 1\{k \in H(1;2)\}, \\ Z_3(k) &= Y_{21}(k)^2\, 1\{k \in H(1;2)\}, \\ Z_4(k) &= Y_{12}(k)\, 1\{k \in H(1;2)\}, \\ Z_5(k) &= Y_{21}(k)\, 1\{k \in H(1;2)\}, \\ Z_6(k) &= \sum_{l\in B_k} Y_{22}(l)\, 1\{k \in H(1;2)\}, \\ Z_7(k) &= \sum_{l\in B_k} Y_{22}(l)^2\, 1\{k \in H(1;2)\}, \end{aligned}$$

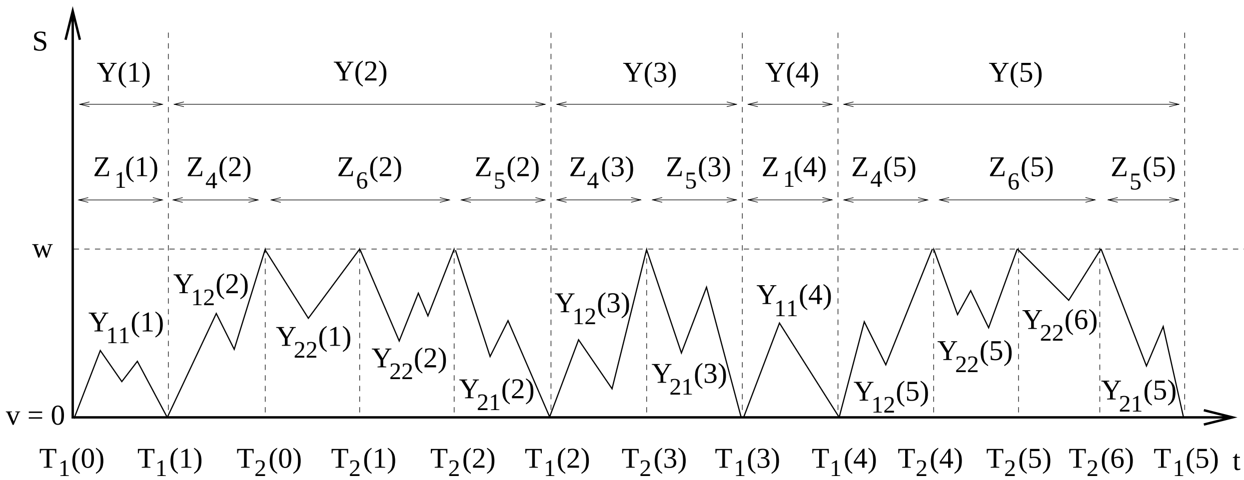

Figure 1: A sample path.

$$Z_8(k) \quad = \quad 1\{k \in H(1;2)\}.$$

Also, define $Z(k) = (Z_1(k), Z_2(k), \ldots, Z_8(k))$, $k = 1, 2, \ldots$, and note that $Z(k)$ is a random vector defined over the kth T_1-cycle. (Throughout this paper, all vectors are column vectors.) Hence, $Z(1), Z(2), \ldots$ are i.i.d. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_8) = E[Z(1)]$, and let $\Sigma = (\sigma_{i,j} : i, j = 1, 2, \ldots, 8)$ be the covariance matrix of $Z(1)$, where $\sigma_{i,j} = \mathrm{Cov}(Z_i(1), Z_j(1))$. We prove in Lemma 4 in the appendix that all of the means and covariances are finite if $E[Y(|g|;1)^4] < \infty$. For $i = 1, 2, \ldots, 8$, define the sample means

$$\bar{Z}_i(m) = \frac{1}{m} \sum_{k=1}^{m} Z_i(k),$$

and define $\bar{Z}(m) = (\bar{Z}_1(m), \bar{Z}_2(m), \ldots, \bar{Z}_8(m))$.

To illustrate some of our notation, we provide in Figure 1 an example of a sample path of a continuous-time process X on a continuous state space S. The sample path is the zigzag line in the figure. Assume that the reward function $g(x) = x$ for all $x \in S$. The T_1 (resp., T_2) sequence corresponds to hits to state $v = 0$ at the horizontal axis (resp., state w at the horizontal dashed line). Hence, the number of T_1-cycles is $m = 5$, and there are six T_2-cycles on the path. In the first and fourth (resp., second, third, and fifth) T_1-cycles, there are no occurrences (resp., at least one occurrence) of T_2 stopping times, and so $H(1;2) = \{2, 3, 5\}$, and $J(1;2) = \{1, 4\}$. Also, the third and fourth (resp., first, second, fifth, and sixth) T_2-cycles have (resp., do not have) occurrences of T_1 stopping times within them; thus, $H(2;1) = \{3, 4\}$, and $J(2;1) = \{1, 2, 5, 6\}$. Therefore, if $k = 1$ or 4, then $Y_{12}(k) = Y_{21}(k) = 0$, and so $Z_1(k) > 0$ and $Z_i(k) = 0$ for $i = 2, 3, \ldots, 8$. On the other hand, if $k = 2, 3, 5$, then $Z_1(k) = 0$, and $Z_i(k) > 0$ for $i = 2, 3, 4, 5, 8$. For $i = 6$ or 7, $Z_i(k) > 0$ when $k = 2$ and 5, but $Z_i(k) = 0$ when $k = 3$.

In the proof of Theorem 1 below, we show that the asymptotic variance of $\sqrt{m}(\widetilde{\alpha}(m) - \alpha)$ is

$$\sigma_\alpha^2 = \nabla f_\alpha(\mu)^T \, \Sigma \, \nabla f_\alpha(\mu), \tag{7}$$

where the function $f_\alpha : \Re^8 \to \Re$ is defined for $z = (z_1, z_2, \ldots, z_8) \in \Re^8$ as

$$f_\alpha(z) = z_1 + z_2 + z_3 + \frac{2z_4 z_5}{z_8} + z_7 + \frac{2z_6(z_4 + z_5)}{z_8} + \frac{2z_6^2}{z_8},$$

$\nabla f_\alpha(z)$ is the column vector of partial derivatives of f_α evaluated at z, and the superscript T denotes transpose. (Henceforth, we sometimes refer to σ_α^2 as the asymptotic variance *associated with* $\widetilde{\alpha}(m)$.) Thus, applying the following approach, we can estimate σ_α^2 from the same simulation that is used to construct the permuted estimator $\widetilde{\alpha}(m)$. First generate a sample path of m T_1-cycles and compute $\widetilde{\alpha}(m)$. From the same sample path, compute the vectors $(Z_1(k), Z_2(k), \ldots, Z_8(k))$, $k = 1, 2, \ldots, m$, and the sample means $\bar{Z}_i(m)$, $i = 1, 2, \ldots, 8$. For example, to compute $\bar{Z}_2(m)$, we take the average of the $Z_2(k)$ over all m T_1-cycle indices k, where $Z_2(k)$ is the square of the contribution to $Y(k)$ from the beginning of the k-th T_1-cycle until the first occurrence of a T_2-cycle if a T_2-cycle occurs during the k-th T_1-cycle, and $Z_2(k) = 0$ otherwise. Also, we estimate the covariance $\sigma_{i,j}$, $i, j = 1, 2, \ldots, 8$, by

$$\hat{\sigma}_{i,j}(m) = \frac{1}{m-1} \sum_{k=1}^{m} \left(Z_i(k) - \bar{Z}_i(m)\right) \left(Z_j(k) - \bar{Z}_j(m)\right),$$

and let $\hat{\Sigma}(m) = (\hat{\sigma}_{i,j}(m) : i, j = 1, 2, \ldots, 8)$. Then our estimator of σ_α^2 is given by $\hat{\sigma}_\alpha^2(m) = \nabla f_\alpha(\bar{Z}(m))^T \, \hat{\Sigma}(m) \, \nabla f_\alpha(\bar{Z}(m))$. To compute $\hat{\sigma}_\alpha^2(m)$ in a simulation, we only need accumulators to keep track of the sums of the $Z_i(k)$, $Z_i(k)^2$, and $Z_i(k)Z_j(k)$ for $i, j = 1, 2, \ldots, 8$. Thus, our estimator can be updated "on the fly" as the simulation progresses; i.e., the user does not have to store all of the $(Z_1(k), Z_2(k), \ldots, Z_8(k))$, $k = 1, 2, \ldots, m$, and use a second pass through the data.

Let "$\Rightarrow$" denote convergence in distribution, and let $N(x, y)$ denote a normal random variable with mean x and variance y. Then the following holds.

Theorem 1 *Consider estimating α defined in (4). Assume that $E\left[Y(|g|; 1)^4\right] < \infty$. Then as $m \to \infty$,*

(i) $\widetilde{\alpha}(m) \to \alpha$ *a.s.;*

(ii) $\sqrt{m}\,\left(\widetilde{\alpha}(m)-\alpha\right)/\hat{\sigma}_\alpha(m) \Rightarrow N(0,1)$.

Theorem 1 allows us to construct asymptotically valid confidence intervals. Specifically, an asymptotically valid $100(1-\delta)\%$ confidence interval for α is

$$\left[\widetilde{\alpha}(m)-\frac{\kappa_\delta\hat{\sigma}_\alpha(m)}{\sqrt{m}},\ \widetilde{\alpha}(m)+\frac{\kappa_\delta\hat{\sigma}_\alpha(m)}{\sqrt{m}}\right],$$

where κ_δ is the upper $\delta/2$ critical point of a standard normal distribution; i.e., $P\{N(0,1)\le\kappa_\delta\}=1-\delta/2$. Also, note that in Theorem 1, it is natural to assume a finite fourth moment for our central limit theorem since we are estimating a second moment.

Proof of Theorem 1. Observe that

$$\sum_{\substack{j,l\in J(2;1)\\ j\neq l}} Y_{22}(j)Y_{22}(l)=\left(\sum_{l\in J(2;1)}Y_{22}(l)\right)^2-\sum_{l\in J(2;1)}Y_{22}(l)^2=\left(\sum_{k=1}^m Z_6(k)\right)^2-\sum_{k=1}^m Z_7(k). \quad (8)$$

Thus,

$$\widetilde{\alpha}(m)=f_\alpha(\bar{Z}(m))+R_1(m)+R_2(m)+R_3(m)+R_4(m),$$

where

$$\begin{aligned}
R_1(m) &= \left(\frac{2m}{\sum_{k=1}^m Z_8(k)-1}-\frac{2m}{\sum_{k=1}^m Z_8(k)}\right)\bar{Z}_4(m)\bar{Z}_5(m),\\
R_2(m) &= \frac{-2}{m(h_{12}-1)}\sum_{k\in H(1;2)} Y_{12}(k)Y_{21}(\psi(k)),\\
R_3(m) &= \left(\frac{2m}{1+\sum_{k=1}^m Z_8(k)}-\frac{2m}{\sum_{k=1}^m Z_8(k)}\right)\bar{Z}_6(m)^2,\\
R_4(m) &= \frac{-2}{1+\sum_{k=1}^m Z_8(k)}\bar{Z}_7(m).
\end{aligned}$$

To establish the theorem, it suffices to prove that as $m\to\infty$, $f_\alpha(\bar{Z}(m))\to\alpha$ a.s., $\sqrt{m}\,R_i(m)\to 0$ a.s. for each i, $\sqrt{m}\,\left(f_\alpha(\bar{Z}(m))-\alpha\right)/\sigma_\alpha \Rightarrow N(0,1)$, and $\hat{\sigma}_\alpha(m)\to\sigma_\alpha$ a.s.; see Sections 1.3.1 and 1.5.4 of Serfling (1980).

Lemma 4 in the appendix establishes that all of the means μ_i and covariances $\sigma_{i,j}$ are finite and that $\mu_8>0$. In Lemma 5 in the appendix, we prove that $f_\alpha(\mu)=\alpha$. Thus, since f_α is continuous at μ and because the strong law of large numbers (SLLN) implies that $\bar{Z}(m)\to\mu$ a.s., as $m\to\infty$, we have that $f_\alpha(\bar{Z}(m))\to\alpha$ a.s., as $m\to\infty$. Moreover, it follows from the SLLN and the continuous differentiability of f_α at μ that $\hat{\sigma}_\alpha(m)\to\sigma_\alpha$ a.s. Also, the finiteness of Σ from Lemma 4 implies that $\sqrt{m}\,(\bar{Z}(m)-\mu) \Rightarrow N_8(0,\Sigma)$, as $m\to\infty$, where $N_d(a,A)$ is a d-dimensional normal random vector with mean vector a and covariance matrix A. Thus, since f_α has a non-zero differential at μ, applying the corollary on p. 124 of Serfling (1980) yields $\sqrt{m}\,(f_\alpha(\bar{Z}(m))-\alpha)/\sigma_\alpha \Rightarrow N\,(0,1)\,$, as $m\to\infty$.

We now show that for $i=1,\ldots,4$, $\sqrt{m}\,R_i(m)\to 0$ a.s., as $m\to\infty$. Observe that with probability 1,

$$\sqrt{m}\;R_1(m) \;=\; \left(\frac{2}{\sqrt{m}}\right)\left(\frac{1}{m}\sum_{k=1}^{m} Z_8(k)-\frac{1}{m}\right)^{-1}\frac{\bar{Z}_4(m)\bar{Z}_5(m)}{\bar{Z}_8(m)} \;\to\; 0\,\cdot\,\mu_8^{-1}\frac{\mu_4\mu_5}{\mu_8} \;=\; 0,$$

as $m\to\infty$ (since $\mu_8>0$). Similarly, we can prove that $\sqrt{m}\,R_3(m)\to 0$ a.s. and $\sqrt{m}\,R_4(m)\to 0$ a.s., as $m\to\infty$.

To show that $\sqrt{m}\,R_2(m)\to 0$ a.s., as $m\to\infty$, we first define $V(l)=Y_{12}(\beta_l)Y_{21}(\beta_{l-1})$, and note that

$$R_2(m)=\frac{-2}{m(h_{12}-1)}\left(\sum_{l=2}^{h_{12}} V(l)+V(1)\right).$$

For $l\geq 2$, $V(l)$ is only a function of the lth T_2-cycle that contains an occurrence of a stopping time from sequence T_1, and so $V(l)$, $l=2,3,\ldots$, are i.i.d. Also, for any l, $Y_{12}(\beta_l)$ and $Y_{21}(\beta_{l-1})$ are independent since they are in different T_1-cycles. Note that $Y_{12}(\beta_l)\stackrel{\mathcal{D}}{=}Z_4(\beta_1)$ and $Y_{21}(\beta_{l-1})\stackrel{\mathcal{D}}{=}Z_5(\beta_1)$. Thus,

$$\begin{aligned}
|\,E[V(2)]\,| \;&=\; |\,E[Z_4(\beta_1)]\,E[Z_5(\beta_1)]\,| \;=\; |\,E[Z_4(1)|Z_8(1)=1]\,E[Z_5(1)|Z_8(1)=1]\,| \\
&\leq\; \left|\frac{E[Z_4(1)]}{P\{Z_8(1)=1\}}\,\frac{E[Z_5(1)]}{P\{Z_8(1)=1\}}\right| \;<\; \infty,
\end{aligned}$$

since $P\{Z_8(1)=1\}=P\{T_2(0)<T_1(1)\}>0$. Moreover, $h_{12}\to\infty$ a.s., as $m\to\infty$, since $P\{T_2(0)<T_1(1)\}>0$. Therefore, the strong law of large numbers implies that

$$\frac{1}{h_{12}-1}\sum_{l=2}^{h_{12}} V(l)\to E[V(2)] \;\text{ a.s.,} \tag{9}$$

as $m\to\infty$. Since $V(1)=Y_{12}(\beta_1)Y_{21}(\beta_{h_{12}})$,

$$\frac{V(1)}{h_{12}-1}=Y_{12}(\beta_1)\frac{Y_{21}(\beta_{h_{12}})}{h_{12}-1}\to 0 \;\text{ a.s.,}$$

as $m\to\infty$, because $E[Y_{21}(\beta_{h_{12}})]<\infty$ and $h_{12}\to\infty$ a.s. (see Example C on p. 12 of Serfling 1980). Thus,

$$\sqrt{m}\,R_2(m)=\left(\frac{-2}{\sqrt{m}}\right)\frac{1}{h_{12}-1}\left(\sum_{l=2}^{h_{12}} V(l)+V(1)\right)\to 0 \;\text{ a.s.,}$$

as $m \to \infty$. ■

4 Time-Average Variance Constant

In Section 5.1 of Calvin and Nakayama (1998), we briefly described how to modify the framework of Section 3 to estimate the time-average variance σ_f^2 of the process $f(X)$ for some reward function $f : S \to \Re$. We now explain this more fully. To do this, define $r_t = (1/t)\int_0^t f(X(s))\,ds$ and $r = E[Y(f;1)]/E[\tau(1)]$, where $\tau(k) = T_1(k) - T_1(k-1)$ for $k \geq 1$. Assuming that $E[Y(f;k)^2] < \infty$ and $E[\tau(k)^2] < \infty$, then $t^{1/2}(r_t - r)/\sigma_f \Rightarrow N(0,1)$, as $t \to \infty$, where $\sigma_f^2 = (E[Y(f;1)^2] + r^2 E[\tau(1)^2] - 2rE[Y(f;1)\tau(1)])/E[\tau(1)]$; see Theorem 2.3 of Shedler (1993). The standard estimator of σ_f^2 is

$$\hat{\sigma}_f^2(m) = \frac{\sum_{k=1}^m Y(f;k)^2}{\sum_{k=1}^m \tau(k)} + \hat{r}(m)^2 \frac{\sum_{k=1}^m \tau(k)^2}{\sum_{k=1}^m \tau(k)} - 2\hat{r}(m)\frac{\sum_{k=1}^m Y(f;k)\tau(k)}{\sum_{k=1}^m \tau(k)},$$

where $\hat{r}(m) = \sum_{k=1}^m Y(f;k)/\sum_{k=1}^m \tau(k)$.

The permuted estimator of the time-average variance constant is

$$\begin{aligned}
\widetilde{\sigma}_f^2(m) \;=\; & \left(\sum_{k=1}^m \tau(k)\right)^{-1} \Bigg[\sum_{k\in J(1;2)} Y(k)^2 + \widehat{r}(m)^2 \sum_{k\in J(1;2)} \tau(k)^2 - 2\widehat{r}(m) \sum_{k\in J(1;2)} Y(k)\tau(k) \\
& + \sum_{k\in H(1;2)} Y_{12}(k)^2 + \widehat{r}(m)^2 \sum_{k\in H(1;2)} \tau_{12}(k)^2 - 2\widehat{r}(m) \sum_{k\in H(1;2)} Y_{12}(k)\tau_{12}(k) \\
& + \sum_{k\in H(1;2)} Y_{21}(k)^2 + \widehat{r}(m)^2 \sum_{k\in H(1;2)} \tau_{21}(k)^2 - 2\widehat{r}(m) \sum_{k\in H(1;2)} Y_{21}(k)\tau_{21}(k) \\
& + \frac{2}{h_{12}-1}\Bigg(\sum_{k\in H(1;2)} Y_{12}(k) \sum_{j\in H(1;2)} Y_{21}(j) - \widehat{r} \sum_{k\in H(1;2)} Y_{12}(k) \sum_{j\in H(1;2)} \tau_{21}(j) \\
& \quad - \widehat{r} \sum_{k\in H(1;2)} \tau_{12}(k) \sum_{j\in H(1;2)} Y_{21}(j) + \widehat{r}^2 \sum_{k\in H(1;2)} \tau_{12}(k) \sum_{j\in H(1;2)} \tau_{21}(j) \Bigg) \\
& - \frac{2}{h_{12}-1}\Bigg(\sum_{k\in H(1;2)} Y_{12}(k) Y_{21}(\psi(k)) - \widehat{r} \sum_{k\in H(1;2)} Y_{12}(k)\tau_{21}(\psi(k)) \\
& \quad - \widehat{r} \sum_{k\in H(1;2)} \tau_{12}(k) Y_{21}(\psi(k)) + \widehat{r}^2 \sum_{k\in H(1;2)} \tau_{12}(k)\tau_{21}(\psi(k)) \Bigg) \\
& + \sum_{k\in J(2;1)} Y_{22}(k)^2 + \widehat{r}(m)^2 \sum_{k\in J(2;1)} \tau_{22}(k)^2 - 2\widehat{r}(m) \sum_{k\in J(2;1)} Y_{22}(k)\tau_{22}(k) \\
& + 2(\bar{Y}_{12}(m) - \widehat{r}(m)\bar{\tau}_{12}(m) + \bar{Y}_{21}(m) - \widehat{r}(m)\bar{\tau}_{21}(m)) \left(\sum_{k\in J(2;1)} Y_{22}(k) - \widehat{r}(m) \sum_{k\in J(2;1)} \tau_{22}(k) \right)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{2}{1+h_{12}} \left(\sum_{k \in J(2;1)} Y_{22}(k) - \widehat{r}(m) \sum_{k \in J(2;1)} \tau_{22}(k) \right)^2 \\
&- \quad \frac{2}{1+h_{12}} \left(\sum_{k \in J(2;1)} Y_{22}(k)^2 + \widehat{r}(m)^2 \sum_{k \in J(2;1)} \tau_{22}(k)^2 - 2\widehat{r}(m) \sum_{k \in J(2;1)} Y_{22}(k)\tau_{22}(k) \right) \Bigg], \qquad (10)
\end{aligned}
$$

where for $k \in H(1;2)$ and for $l \in J(2;1)$,

$$
\begin{aligned}
\tau_{12}(k) &= T_2'(k) - T_1(k-1), \\
\tau_{21}(k) &= T_1(k) - \widetilde{T}_2(k), \\
\tau_{22}(l) &= T_2(l) - T_2(l-1), \\
\bar{\tau}_{12}(m) &= \frac{1}{h_{12}} \sum_{k \in H(1;2)} \tau_{12}(k), \\
\bar{\tau}_{21}(m) &= \frac{1}{h_{12}} \sum_{k \in H(1;2)} \tau_{21}(k).
\end{aligned}
$$

Now for T_1-cycle indices $k = 1, 2, \ldots$, define the variables $Z_1(k), Z_2(k), \ldots, Z_{21}(k)$, as follows. Define $Z_i(k)$ as before for $i = 1, 2, \ldots, 8$. For $i > 8$, we define

$$
\begin{aligned}
Z_9(k) &= Y(k), \\
Z_{10}(k) &= \tau(k), \\
Z_{11}(k) &= \tau(k)^2\, 1\{k \notin H(1;2)\}, \\
Z_{12}(k) &= \tau_{12}(k)^2\, 1\{k \in H(1;2)\}, \\
Z_{13}(k) &= \tau_{21}(k)^2\, 1\{k \in H(1;2)\}, \\
Z_{14}(k) &= \tau_{12}(k)\, 1\{k \in H(1;2)\}, \\
Z_{15}(k) &= \tau_{21}(k)\, 1\{k \in H(1;2)\}, \\
Z_{16}(k) &= 1\{k \in H(1;2)\} \sum_{l \in B_k} \tau_{22}(l), \\
Z_{17}(k) &= 1\{k \in H(1;2)\} \sum_{l \in B_k} \tau_{22}(l)^2, \\
Z_{18}(k) &= Y(k)\, \tau(k)\, 1\{k \notin H(1;2)\}, \\
Z_{19}(k) &= Y_{12}(k)\, \tau_{12}(k)\, 1\{k \in H(1;2)\}, \\
Z_{20}(k) &= Y_{21}(k)\, \tau_{21}(k)\, 1\{k \in H(1;2)\}, \\
Z_{21}(k) &= 1\{k \in H(1;2)\} \sum_{l \in B_k} Y_{22}(l)\tau_{22}(l).
\end{aligned}
$$

For T_1-cycle indices $k = 1, 2, \ldots$, we define $Z'(k) = (Z_1(k), Z_2(k), \ldots, Z_{21}(k))$. Also, let $\mu' = (\mu_1, \mu_2, \ldots, \mu_{21}) = E[Z'(1)]$, and let $\Sigma' = (\sigma_{i,j} : i, j = 1, 2, \ldots, 21)$ be the covariance matrix of $Z'(1)$, where $\sigma_{i,j} = \mathrm{Cov}(Z_i(1), Z_j(1))$. For $i, j = 1, 2, \ldots, 21$, define $\bar{Z}_i(m)$ and $\hat{\sigma}_{i,j}(m)$ as before as the sample means and sample covariances, respectively. Define $\bar{Z}'(m) = (\bar{Z}_1(m), \bar{Z}_2(m), \ldots, \bar{Z}_{21}(m))$ and $\hat{\Sigma}'(m) = (\hat{\sigma}_{i,j}(m) : i, j = 1, 2, \ldots, 21)$.

Now define the function $f_{\sigma^2} : \Re^{21} \to \Re$ for $z = (z_1, z_2, \ldots, z_{21})$ as

$$\begin{aligned} f_{\sigma^2}(z) &= z_{10}^{-1} \left[z_1 + z_2 + z_3 + \left(\frac{z_9}{z_{10}} \right)^2 (z_{11} + z_{12} + z_{13}) - 2 \frac{z_9}{z_{10}} (z_{18} + z_{19} + z_{20}) \right. \\ &\quad + \frac{2}{z_8} \left(z_4 z_5 + \frac{z_9}{z_{10}} \left(-z_4 z_{15} - z_{14} z_5 + \frac{z_9}{z_{10}} z_{14} z_{15} \right) \right) + z_7 + \left(\frac{z_9}{z_{10}} \right)^2 z_{17} - 2 \frac{z_9}{z_{10}} z_{21} \\ &\quad \left. + \frac{2}{z_8} \left(z_4 + z_5 - \frac{z_9}{z_{10}} (z_{14} + z_{15}) \right) \left(z_6 - \frac{z_9}{z_{10}} z_{16} \right) + \frac{2}{z_8} \left(z_6 - \frac{z_9}{z_{10}} z_{16} \right)^2 \right]. \end{aligned}$$

By using arguments similar to those applied in the proof of Theorem 1, we can establish the following results for the permuted estimator of σ_f^2. (Glynn and Iglehart 1987 prove a central limit theorem for the standard estimator of σ_f^2.)

Theorem 2 *Consider estimating the time-average variance constant σ_f^2. Assume that $E[Y(|f|; k)^4] < \infty$ and $E[\tau(k)^4] < \infty$. Then as $m \to \infty$,*

(i) $\widetilde{\sigma}_f^2(m) \to \sigma_f^2$ *a.s.;*

(ii) $\sqrt{m} \left(\widetilde{\sigma}_f^2(m) - \sigma_f^2 \right) / \hat{\sigma}_\sigma(m) \Rightarrow N(0, 1)$, *where* $\hat{\sigma}_\sigma(m) = \left(\nabla f_{\sigma^2}(\bar{Z}'(m))^T \, \hat{\Sigma}'(m) \, \nabla f_{\sigma^2}(\bar{Z}'(m)) \right)^{1/2}$.

5 Expected Cumulative Reward Until Hitting a Set

Suppose we are interested in estimating

$$\eta = E \left[\int_0^{T_F} g(X(t)) \, dt \right], \tag{11}$$

where $T_F = \inf\{t > 0 : X(t) \in F\}$ for some set of states $F \subset S$ and $g : S \to \Re$ is some "reward" function. Thus, η is the expected cumulative reward until hitting F when $T_1(0) = 0$.

A special case of η is the mean time to failure of a reliability system. In this context, one is interested in computing the expected time to system failure given that the system starts with all components operational. This corresponds to having X represent the evolution of the system over time, letting $g \equiv 1$, and taking the set F to be the set of "failed" states for the system; e.g., see Goyal et al (1992) for more details on the mean time to failure. On the other hand, η is the expected time until a buffer overflow in a queue with a finite buffer if X is the queue-length process, F corresponds to states in which a buffer overflows, and $g \equiv 1$.

It can be shown that

$$\eta = \frac{\xi}{\gamma}, \tag{12}$$

where

$$\xi = E\left[\int_0^{T_F \wedge T_1(1)} g(X(t))\, dt\right],$$

and

$$\gamma = E\left[1\{T_F < T_1(1)\}\right],$$

with $a \wedge b = \min(a, b)$; e.g., see Goyal et al. (1992). To estimate η, we generate one sample path $\vec{X}_m$ consisting of m T_1-cycles, and we use it to estimate ξ and γ.

We examine the estimation of the numerator and denominator in (12) separately. First, if we want to estimate $\alpha = \xi$, then the standard estimator of ξ is

$$\widehat{\xi}(m) = \frac{1}{m}\sum_{k=1}^{m} D(k),$$

where

$$D(k) = D(g, k) = \int_{T_1(k-1)}^{T_1(k) \wedge T'_F(k)} g(X(t))\, dt,$$

with $T'_F(k) = \inf\{t > T_1(k-1) : X(t) \in F\}$. (We will suppress the dependence of D on g in our notation unless it is needed for clarity.) On the other hand, if we want to estimate $\alpha = \gamma$, then the standard estimator of γ is

$$\widehat{\gamma}(m) = \frac{1}{m}\sum_{k=1}^{m} I(k),$$

where

$$I(k) = 1\{T'_F(k) < T_1(k)\}.$$

Thus, the standard estimator of η is

$$\widehat{\eta}(m) = \frac{\widehat{\xi}(m)}{\widehat{\gamma}(m)}. \tag{13}$$

To define our permuted estimator for η, we need more notation. For $k \in H(1; 2)$, let

$$\begin{aligned}
I_{12}(k) &= 1\{T'_F(k) < T'_2(k)\}, \\
I_{22}(l) &= 1\{T^{(2)}_F(l) < T_2(l)\}, \\
I_{21}(k) &= 1\{T^{(1)}_F(k) < T_1(k)\},
\end{aligned}$$

with $T_2'(k) = \inf\{t > T_1(k) : T_2(i) = t \text{ for some } i\}$, $T_F^{(2)}(l) = \inf\{t > T_2(l-1) : X(t) \in F\}$ and $T_F^{(1)}(k) = \inf\{t > \widetilde{T}_1(k) : X(t) \in F\}$. Hence, $I_{12}(k)$ (resp., $I_{21}(k)$) is the indicator of whether the set F is hit in the initial 1-2 segment (resp., final 2-1 segment) of the T_1-cycle with index $k \in H(1;2)$. Similarly, $I_{22}(l)$ is the indicator whether the set F is hit in the T_2-cycle with index $l \in J(2;1)$. Also, define

$$
\begin{aligned}
k_{21} &= |\{k \in H(1;2) : I_{21}(k) = 0\}|, \\
k_{12} &= |\{k \in H(1;2) : I_{12}(k) = 0\}|, \\
k_c &= |\{k \in H(1;2) : I_{12}(k) = 0,\ I_{21}(\psi(k)) = 0\}|.
\end{aligned}
$$

Define

$$
\begin{aligned}
D_{12}(k) &= \int_{T_1(k-1)}^{T_F'(k) \wedge T_2'(k)} g(X(t))\,dt, \\
D_{22}(l) &= \int_{T_2(l-1)}^{T_F^{(2)}(l) \wedge T_2(l)} g(X(t))\,dt, \\
D_{21}(k) &= \int_{\widetilde{T}_2(k)}^{T_F^{(1)}(k) \wedge T_1(k)} g(X(t))\,dt.
\end{aligned}
$$

Finally, let

$$
\begin{aligned}
r &= \sum_{l \in J(2;1)} I_{22}(l), \\
d_0 &= \sum_{l \in J(2;1)} D_{22}(l)\left(1 - I_{22}(l)\right), \\
d_1 &= \sum_{l \in J(2;1)} D_{22}(l) I_{22}(l).
\end{aligned}
$$

Then Calvin and Nakayama (1998) derive the following permuted estimators for ξ and γ:

(i) $\widetilde{\xi}(m) = \widehat{\xi}(m)$ if $M_2(\vec{X}_m) < 3$, and otherwise

$$
\begin{aligned}
\widetilde{\xi}(m) &= \frac{1}{m}\left(\sum_{k \in J(1;2)} D(k) + \sum_{k \in H(1;2)} D_{12}(k) \right. \\
&+ \frac{1}{h_{12} - 1 + r}\left(k_{12} \sum_{j \in H(1;2)} D_{21}(j) - \sum_{k \in H(1;2)} (1 - I_{12}(k)) D_{21}(\psi(k)) \right) \\
&+ \left. k_{12}\left(\frac{d_0}{r + h_{12}} + \frac{d_1}{r + h_{12} - 1} \right) \right); \qquad (14)
\end{aligned}
$$

(ii) $\widetilde{\gamma}(m) = \widehat{\gamma}(m)$ if $M_2(\vec{X}_m) < 3$, and otherwise

$$\widetilde{\gamma}(m) = \frac{1}{m}\left(\sum_{k \in J(1;2)} I(k) + h_{12} - \frac{k_{12}k_{21} - k_c}{h_{12} - 1 + r}\right). \tag{15}$$

It follows from (2) and (3) that if $E[D(1)^2] < \infty$, then for any fixed number m of T_1-cycles, the permuted estimators $\widetilde{\xi}(m)$ and $\widetilde{\gamma}(m)$ satisfy $E[\widetilde{\xi}(m)] = \xi$, $E[\widetilde{\gamma}(m)] = \gamma$, $\mathrm{Var}(\widetilde{\xi}(m)) \leq \mathrm{Var}(\widehat{\xi}(m))$, and $\mathrm{Var}(\widetilde{\gamma}(m)) \leq \mathrm{Var}(\widehat{\gamma}(m))$, where $\widehat{\xi}(m)$ and $\widehat{\gamma}(m)$ are the standard estimators of ξ and γ, respectively. The permuted estimator for η is then $\widetilde{\eta}(m) = \widetilde{\xi}(m)/\widetilde{\gamma}(m)$.

We now want to prove a strong law of large numbers and a central limit theorem for the permuted estimator $\widetilde{\eta}(m)$. As in the previous section, to accomplish this, we need to derive new expressions for $\widetilde{\xi}(m)$ and $\widetilde{\gamma}(m)$ that will allow us to exploit the regenerative structure of X. To do this, we need some notation. For $k = 1, 2, \ldots$, define

$$\begin{aligned}
W_1(k) &= D(k)\, 1\{k \notin H(1;2)\}, \\
W_2(k) &= D_{12}(k)\, 1\{k \in H(1;2)\}, \\
W_3(k) &= D_{21}(k)\, 1\{k \in H(1;2)\}, \\
W_4(k) &= I(k)\, 1\{k \notin H(1;2)\}, \\
W_5(k) &= 1 - I_{21}(k)\, 1\{k \in H(1;2)\}, \\
W_6(k) &= 1 - I_{12}(k)\, 1\{k \in H(1;2)\}, \\
W_7(k) &= \sum_{l \in B_k} I_{22}(l)\, 1\{k \in H(1;2)\}, \\
W_8(k) &= \sum_{l \in B_k} D_{22}(l) I_{22}(l)\, 1\{k \in H(1;2)\}, \\
W_9(k) &= \sum_{l \in B_k} D_{22}(l)(1 - I_{22}(l))\, 1\{k \in H(1;2)\}, \\
W_{10}(k) &= 1\{k \in H(1;2)\}.
\end{aligned}$$

Also, define $W(k) = (W_1(k), W_2(k), \ldots, W_{10}(k))$, $k = 1, 2, \ldots$, and note that $W(k)$ is a random vector defined over the kth T_1-cycle. Hence, $W(1), W(2), \ldots$ are i.i.d. For $i = 1, 2, \ldots, 10$, define the sample means $\bar{W}_i(m) = (1/m)\sum_{k=1}^{m} W_i(k)$, and define $\bar{W}(m) = (\bar{W}_1(m), \bar{W}_2(m), \ldots, \bar{W}_{10}(m))$. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_{10}) = E[W(1)]$, and let $\Psi = (\Psi_{i,j} : i, j = 1, 2, \ldots, 10)$ be the covariance matrix of $W(1)$, where $\Psi_{i,j} = \mathrm{Cov}(W_i(1), W_j(1))$. We prove in Lemma 6 in the appendix that all of the means and covariances are finite under the assumption that $E[D(|g|, 1)^2] < \infty$.

It turns out that the asymptotic variance of $\sqrt{m}(\widetilde{\eta}(m) - \eta)$ is $\sigma_\eta^2 = \nabla f_\eta(\nu)^T \, \Psi \, \nabla f_\eta(\nu)$, where the function $f_\eta : \Re^{10} \to \Re$ is defined as

$$\begin{aligned} f_\eta(w_1, w_2, \ldots, w_{10}) &= \left(w_1 + w_2 + \frac{w_3 w_6}{w_{10} + w_7} + \frac{w_6 w_9}{w_7 + w_{10}} + \frac{w_6 w_8}{w_7 + w_{10}} \right) \left(w_4 + w_{10} - \frac{w_5 w_6}{w_{10} + w_7} \right)^{-1} \\ &= \frac{(w_1 + w_2)(w_7 + w_{10}) + w_6(w_3 + w_8 + w_9)}{(w_4 + w_{10})(w_7 + w_{10}) - w_5 w_6}. \end{aligned}$$

To estimate σ_η^2 from the same sample path $\vec{X}_m$ of m T_1-cycles used to construct $\widetilde{\eta}(m)$, we employ $\bar{W}_i(m)$ as an estimate of ν_i, $i = 1, 2, \ldots, 10$. Also, we estimate the covariance $\Psi_{i,j}$, $i, j = 1, 2, \ldots, 10$, by $\hat{\Psi}_{i,j}(m) = (1/(m-1)) \sum_{k=1}^m \left(W_i(k) - \bar{W}_i(m) \right) \left(W_j(k) - \bar{W}_j(m) \right)$, and let $\hat{\Psi}(m) = (\hat{\Psi}_{i,j}(m) : i, j = 1, 2, \ldots, 10)$. Then $\hat{\sigma}_\eta^2(m) = \nabla f_\eta(\bar{W}(m))^T \, \hat{\Psi}(m) \, \nabla f_\eta(\bar{W}(m))$ is our estimator of σ_η^2.

Theorem 3 *Consider estimating η in (11). Assume that $E\left[D(|g|, 1)^2\right] < \infty$ and $P\{T_F < T_1(1)\} > 0$. Then as $m \to \infty$,*

(i) $\widetilde{\eta}(m) \to \eta$ *a.s.;*

(ii) $\sqrt{m} \left(\widetilde{\eta}(m) - \eta \right) / \hat{\sigma}_\eta(m) \Rightarrow N(0, 1)$.

Proof. Observe that

$$\begin{aligned} \widetilde{\eta}(m) = & \Bigg(\bar{W}_1(m) + \bar{W}_2(m) + \frac{\bar{W}_3(m) \bar{W}_6(m)}{\bar{W}_{10}(m) + \bar{W}_7(m) - (1/m)} + \frac{\bar{W}_6(m) \bar{W}_9(m)}{\bar{W}_7(m) + \bar{W}_{10}(m)} \\ & + \frac{\bar{W}_6(m) \bar{W}_8(m)}{\bar{W}_7(m) + \bar{W}_{10}(m) - (1/m)} - \frac{1}{m(h_{12} - 1 + r)} \sum_{k \in H(1;2)} (1 - I_{12}(k)) D_{21}(\psi(k)) \Bigg) \\ & \times \left(\bar{W}_4(m) + \bar{W}_{10}(m) - \frac{\bar{W}_5(m) \bar{W}_6(m) - (k_c/m^2)}{\bar{W}_{10}(m) + \bar{W}_7(m) - (1/m)} \right)^{-1}. \end{aligned}$$

Thus,

$$\widetilde{\eta}(m) = f_\eta(\bar{W}(m)) \, U(m) + R_5(m) + R_6(m) + R_7(m), \tag{16}$$

where

$$\begin{aligned} U(m) = & \left(\bar{W}_4(m) + \bar{W}_{10}(m) - \frac{\bar{W}_5(m) \bar{W}_6(m)}{\bar{W}_{10}(m) + \bar{W}_7(m)} \right) \\ & \times \left(\bar{W}_4(m) + \bar{W}_{10}(m) - \frac{\bar{W}_5(m) \bar{W}_6(m) - (k_c/m^2)}{\bar{W}_{10}(m) + \bar{W}_7(m) - (1/m)} \right)^{-1}, \end{aligned}$$

$$
\begin{aligned}
R_5(m) &= \left(\frac{-1}{m(h_{12}-1+r)} \sum_{k \in H(1;2)} (1 - I_{12}(k)) D_{21}(\psi(k)) \right) \\
&\quad \times \left(\bar{W}_4(m) + \bar{W}_{10}(m) - \frac{\bar{W}_5(m)\bar{W}_6(m) - (k_c/m^2)}{\bar{W}_{10}(m) + \bar{W}_7(m) - (1/m)} \right)^{-1}, \\
R_6(m) &= \left(\frac{\bar{W}_3(m)\bar{W}_6(m)}{\bar{W}_{10}(m) + \bar{W}_7(m) - (1/m)} - \frac{\bar{W}_3(m)\bar{W}_6(m)}{\bar{W}_{10}(m) + \bar{W}_7(m)} \right) \\
&\quad \times \left(\bar{W}_4(m) + \bar{W}_{10}(m) - \frac{\bar{W}_5(m)\bar{W}_6(m) - (k_c/m^2)}{\bar{W}_{10}(m) + \bar{W}_7(m) - (1/m)} \right)^{-1}, \\
R_7(m) &= \left(\frac{\bar{W}_6(m)\bar{W}_8(m)}{\bar{W}_7(m) + \bar{W}_{10}(m) - (1/m)} - \frac{\bar{W}_6(m)\bar{W}_8(m)}{\bar{W}_7(m) + \bar{W}_{10}(m)} \right) \\
&\quad \times \left(\bar{W}_4(m) + \bar{W}_{10}(m) - \frac{\bar{W}_5(m)\bar{W}_6(m) - (k_c/m^2)}{\bar{W}_{10}(m) + \bar{W}_7(m) - (1/m)} \right)^{-1}.
\end{aligned}
$$

To establish the theorem, it then suffices to show that as $m \to \infty$, $f_\eta(\bar{W}(m)) \to \eta$ a.s., $\hat{\sigma_\eta}(m) \to \sigma_\eta$ a.s., $\sqrt{m}\,(f_\eta(\bar{W}(m)) - \eta)/\sigma_\eta \Rightarrow N(0,1)$, $U(m) \to 1$ a.s., and $\sqrt{m}\,R_i(m) \to 0$ a.s. for $i = 5, 6, 7$. Lemma 6 in the appendix establishes that all of the means ν_i and covariances $\Psi_{i,j}$ are finite. Hence, applying arguments similar to those used in the proof of Theorem 1, we can establish the current theorem. ■

6 Exact Analysis of a Small Model

We now provide an exact analysis of a small model to show that one can obtain strict reductions in the asymptotic variances by using permuted estimators rather than the standard ones. Also, we use our results to gain some insights into desirable properties of T_2-cycles.

The model is a 2-state discrete-time Markov chain with transition probabilities $P(1,1) = \lambda$, $P(1,2) = 1 - \lambda$, $P(2,1) = 1 - \beta$, and $P(2,2) = \beta$, with $0 < \lambda, \beta < 1$. We will examine the effect of different choices of λ and β. We let the T_1-sequence (resp., T_2-sequence) correspond to hits to state 1 (resp., state 2). We take the reward function $g \equiv 1$, and we focus on α from Section 3, which is then the second moment of the length of a T_1-cycle.

It is straightforward to show that for our small model, the exact theoretical value for $\sigma_s^2 \equiv E[Y(g;1)^4] - (E[Y(g;1)])^2$, which is the asymptotic variance of $\sqrt{m}(\widehat{\alpha}(m) - \alpha)$, is given by

$$
\sigma_s^2 = \lambda + \frac{(1-\lambda)}{(1-\beta)^4}\left(\beta^4 - 5\beta^3 + 11\beta^2 + \beta + 16\right) - \left(\lambda + (1-\lambda)\left(\frac{4 - 3\beta + \beta^2}{(1-\beta)^2}\right)\right)^2.
$$

The exact theoretical value for σ_α^2, which is the asymptotic variance associated with the permuted estimator of α and is defined in (7), is given by

$$\sigma_\alpha^2 = \lambda + \frac{(1-\lambda)}{(1-\beta)^4}\left(\beta^4 - 5\beta^3 + 7\beta^2 + \beta + 16\right) - \left(\lambda + (1-\lambda)\left(\frac{4-3\beta+\beta^2}{(1-\beta)^2}\right)\right)^2.$$

The absolute difference in the asymptotic variances is

$$\sigma_s^2 - \sigma_\alpha^2 = 4(1-\lambda)\frac{\beta^2}{(1-\beta)^4},$$

which goes to ∞ as $\beta \uparrow 1$.

Considering now the relative decrease in the asymptotic variance, we can show that

$$\begin{aligned}\frac{\sigma_s^2 - \sigma_\alpha^2}{\sigma_s^2} \\ = \frac{4\beta^2}{\frac{\lambda}{1-\lambda}(1-\beta)^4 + (\beta^4 - 5\beta^3 + 11\beta^2 + \beta + 16) - \frac{1}{1-\lambda}\left(\lambda(1-\beta)^2 + (1-\lambda)\left(4-3\beta+\beta^2\right)\right)^2}.\end{aligned}$$

The relative difference is maximized as $\lambda \downarrow 0$, and

$$\lim_{\lambda \downarrow 0} \frac{\sigma_s^2 - \sigma_\alpha^2}{\sigma_s^2} = \frac{4\beta}{\beta^2 - 6\beta + 25},$$

which is strictly increasing in β for $0 < \beta < 1$, and approaches $1/5$ as $\beta \uparrow 1$. Therefore, since the frequency of T_2-cycles increases as β increases, this suggests that one desirable characteristic of T_2-cycles is that they occur frequently.

Although the relative reduction in asymptotic variance for this small example is modest, in the next section we consider estimators of the time-average variance constant of a larger model, for which numerical experiments show significant reductions in the asymptotic variance.

7 Empirical Results

We now present some empirical results from estimating the time-average variance constant σ_f^2 and constructing confidence intervals for it based on our central limit theorem in Theorem 2. We consider the discrete-time Ehrenfest urn model, which has transition probabilities $P_{0,1} = P_{s,s-1} = 1$, and

$$P_{i,i+1} = \frac{s-i}{s} = 1 - P_{i,i-1}, \quad 0 < i < s.$$

In our experiments we take $s = 8$, and define the reward function $f(i) = i$. The regenerative sequences T_1 and T_2 correspond to hitting times to the states v and w, respectively, and so state v

w	Average HW			Coverage		
	$v = 0$	$v = 1$	$v = 4$	$v = 0$	$v = 1$	$v = 4$
0	5.66 *	2.73	1.15	0.779 *	0.860	0.893
1	2.76	2.73 *	1.13	0.808	0.860 *	0.888
2	1.70	1.73	1.10	0.778	0.864	0.890
3	1.28	1.28	1.09	0.736	0.886	0.888
4	1.15	1.14	1.14 *	0.730	0.894	0.892 *
5	1.29	1.23	1.08	0.764	0.899	0.895
6	1.74	1.57	1.09	0.788	0.898	0.899
7	2.72	2.12	1.12	0.822	0.892	0.898
8	4.79	2.67	1.14	0.821	0.866	0.896

Table 1: 50 T_1-cycles when $v = 0$.

is the return state for the regenerative simulation. We ran experiments with three different choices of v (0, 1, 4). In each experiment, we constructed the standard estimator and the permuted estimator for each possible choice of $w \neq v$. For each choice of v and w, we ran 1,000 independent replications, where in each replication we constructed a 90% confidence interval for σ_f^2.

The theoretical value of σ_f^2 is 14 and is independent of the choice of v. However, the expected T_1-cycle length varies considerably over the different choices of v. The expected T_1-cycle length for $v = 0$ is a factor of 8 (resp., 70) greater than that for $v = 1$ (resp., $v = 4$). Hence, to make the results somewhat comparable across the different values of v, we changed the number of simulated T_1-cycles for each case so that the total expected number of simulated transitions remains approximately the same.

Table 1 (resp., Table 2) contains the results for all three choices of v when the number of T_1-cycles for $v = 0$ is 50 (resp., 200). The tables give the sample average of the half-widths (HW) of the constructed 90% confidence intervals and the observed coverage over the 1,000 replications. The entries in the tables corresponding to $w = v$ are the results for the standard estimator, and we marked these with an asterisk so they can be easily identified.

Note that for $v = 0$ or 1, the reduction in the average half-width is greatest when $w = 4$. The reason for this seems to be two-fold. First, $w = 4$ has the shortest T_2-cycles and so this choice of w yields many T_2-cycles to permute, which agrees with the suggestion in Section 6. Secondly, in the fixed-sample setting, we see that by (3), the amount of reduction obtained in the (finite-sample) variance is $E\left[\mathrm{Var}(\, h(\vec{X}'_m) \,|\, \vec{X}_m)\,\right]$, which means that one obtains a large reduction in the (finite-sample) variance when the value of the sample performance varies a lot over the different permuted

	Average HW			Coverage		
w	$v=0$	$v=1$	$v=4$	$v=0$	$v=1$	$v=4$
0	3.30 *	1.48	0.57	0.824 *	0.878	0.907
1	1.46	1.48 *	0.56	0.865	0.878 *	0.904
2	0.90	0.90	0.55	0.864	0.908	0.901
3	0.65	0.65	0.54	0.848	0.913	0.914
4	0.57	0.56	0.57 *	0.854	0.898	0.917 *
5	0.64	0.61	0.54	0.874	0.906	0.911
6	0.87	0.78	0.55	0.867	0.898	0.910
7	1.38	1.07	0.56	0.858	0.911	0.910
8	2.47	1.38	0.57	0.874	0.893	0.910

Table 2: 200 T_1-cycles when $v = 0$.

paths. This suggests that a similar situation may also hold when considering the asymptotic variance. When $w = 4$ in our Ehrenfest model, there is a lot of variability in the paths of the T_2-cycles, and so permuting them leads to quite different overall sample paths.

When $v = 0$, the reduction in the average half-width is about a factor of 6 for $w = 4$. When $v = 4$, the reduction is relatively small. State 4 is the best return state in the sense of minimizing the asymptotic variance associated with the standard estimator of σ_f^2. Therefore, it appears that the permuted estimator is a significant improvement over the standard estimator if the standard estimator is based on a relatively "bad" return state. However, if one is able to choose a near-optimal return state to begin with, permuting yields a modest improvement. (Unfortunately, there seem to be no reliable rules for choosing *a priori* a good return state.) Comparing across the three values of v, we see that the minimum average half-width over the different choices of w does not change much across the different values of v, suggesting that it may be possible to compensate for a bad choice of return state by an appropriate choice of w.

Now we examine the coverages of the 90% confidence intervals. For the shorter run length (Table 1), there seems to be a slight degradation in the coverage when $v = 0$ for the choice of w that minimizes the average half-width (i.e., $w = 4$). However, for the longer run length (Table 2), all of the coverages are close to the nominal level. The reason for this may be as follows. The permuted estimators have smaller asymptotic variance, and so the confidence intervals based on the permuted estimators are shorter on average than those based on the standard estimator. The confidence intervals are centered roughly at the same location since the point estimators have the same mean (see (2)), but since the point estimators are biased when the number m of T_1-cycles is finite, the larger width of the interval based on the standard estimator somewhat compensates for the bias. This leads to slightly better coverage for the standard intervals in the small-sample context. As $m \to \infty$, the bias vanishes, and so asymptotically the relative advantage of the standard estimator disappears.

8 Appendix

Lemma 4 *Assume that $E\left[Y(|g|;1)^4\right] < \infty$. Then $|\mu_i| < \infty$ and $|\sigma_{i,j}| < \infty$ for all $i, j = 1, 2, \ldots, 8$.*

Proof. Assume first that g is non-negative; then so are the $Z_i(k)$. Note that

$$Y(1)^2 = Z_1(1) + Z_2(1) + Z_6(1)^2 + Z_3(1) + 2Z_4(1)Z_6(1) + 2Z_4(1)Z_5(1) + 2Z_6(1)Z_5(1). \qquad (17)$$

Hence, assuming that $E[Y(1)^4] < \infty$ implies that $E[Z_1(1)^2]$, $E[Z_2(1)^2]$, $E[Z_3(1)^2]$, $E[Z_6(1)^4]$, and $E[Z_4(1)^2 Z_5(1)^2]$ are all finite. Note that $E[Z_8(1)^2] = \mu_8 = P\{T_2(0) < T_1(1)\} < \infty$. Also, we must have that $\mu_8 > 0$. (If this were not the case, then $P\{T_2(0) > T_1(1)\} = 1$ implies that $P\{T_2(0) > T_1(k)\} = 1$ for $k \geq 0$ by the regenerative property. It then would follow that $P\{T_2(0) = \infty\} = 1$, and so T_2 would not be a sequence of finite regeneration points, contradicting our assumption.) Observe that given $Z_8(1) = 1$, $Z_4(1)$ and $Z_5(1)$ are (conditionally) independent since (X, T_2) is a regenerative process. Thus,

$$\begin{aligned} E[Z_4(1)^2 Z_5(1)^2] &= E[Z_4(1)^2 Z_5(1)^2 | Z_8(1) = 1]\, P\{Z_8(1) = 1\} \\ = E[Z_4(1)^2 | Z_8(1) = 1]\, E[Z_5(1)^2 | Z_8(1) = 1]\, P\{Z_8(1) = 1\} &= \frac{E[Z_4(1)^2]\, E[Z_5(1)^2]}{P\{Z_8(1) = 1\}}, \end{aligned} \qquad (18)$$

and so both $E[Z_4(1)^2]$ and $E[Z_5(1)^2]$ are finite since $\mu_8 > 0$. In addition, $E[Z_6(1)^4] < \infty$ implies that $E[Z_7(1)^2] < \infty$ since $Z_6(1)^4 = \left(Z_7(1) + \sum_{\substack{j,l \in B_1 \\ j \neq l}} Y_{22}(j) Y_{22}(l)\right)^2$. Hence, $E[Z_i(1)^2] < \infty$ for $i = 1, 2, \ldots, 8$, which implies that all of the $\mu_i < \infty$. Moreover, it follows that $E[Z_i(1) Z_j(1)] \leq \left(E[Z_i(1)^2]\, E[Z_j(1)^2]\right)^{1/2} < \infty$ from the Cauchy-Schwarz inequality, and so $|\sigma_{i,j}| < \infty$ for all i and j. If g is not non-negative, replace g by its absolute value in the previous calculations, and use the assumption that $E\left[Y(|g|;1)^4\right] < \infty$ to get the finiteness of the moments. ∎

Lemma 5 *$f_\alpha(\mu) = \alpha$.*

Proof. Note that by (17),

$$\alpha = \mu_1 + \mu_2 + E[Z_6(1)^2] + \mu_3 + 2E[Z_4(1)Z_6(1)] + 2E[Z_4(1)Z_5(1)] + 2E[Z_6(1)Z_5(1)].$$

Also, we have that

$$f_\alpha(\mu) = \mu_1 + \mu_2 + \mu_3 + \frac{2\mu_4\mu_5}{\mu_8} + \mu_7 + \frac{2\mu_6(\mu_4+\mu_5)}{\mu_8} + \frac{2\mu_6^2}{\mu_8}.$$

Arguing as in (18) we can prove that $E[Z_4(1)Z_5(1)] = \mu_4\mu_5/\mu_8$, $E[Z_4(1)Z_6(1)] = \mu_4\mu_6/\mu_8$, and $E[Z_5(1)Z_6(1)] = \mu_5\mu_6/\mu_8$, and so it remains to establish that $E[Z_6(1)^2] = \mu_7 + 2\mu_6^2/\mu_8$. By an equality analogous to (8), this is equivalent to proving that

$$E\left[\sum_{\substack{j,l\in B_1\\ j\neq l}} Y_{22}(j)Y_{22}(l)\right] = \frac{2\mu_6^2}{\mu_8}. \tag{19}$$

Let $K = |B_1|$, and given $Z_8(1) = 1$, K follows a geometric distribution with parameter $\rho = P\{T_1(1) < T_2(1) \,|\, Z_8(1) = 1\}$. Also, let $\phi_0 = E[Y_{22}(1) \,|\, Z_8(1) = 1]$, and note that given K and $Z_8(1) = 1$, $Y_{22}(j)$ and $Y_{22}(l)$ are independent for $j \neq l$ with $j, l \in B_1$. Then the left-hand side of (19) satisfies

$$\begin{aligned} E\left[\sum_{\substack{j,l\in B_1\\ j\neq l}} Y_{22}(j)Y_{22}(l)\right] &= E\left[E\left[\sum_{\substack{j,l\in B_1\\ j\neq l}} Y_{22}(j)Y_{22}(l)\,\middle|\, K, Z_8(1)=1\right]\middle|\, Z_8(1)=1\right]\mu_8 \\ &= E\left[K(K-1)\phi_0^2\,\middle|\, Z_8(1)=1\right]\mu_8 \;=\; 2\mu_8\phi_0^2\frac{(1-\rho)^2}{\rho^2}. \end{aligned} \tag{20}$$

We can similarly show that $2\mu_6^2/\mu_8 = 2\mu_8\phi_0^2(1-\rho)^2/\rho^2$, thereby establishing (19), and hence, we have proved that $f_\alpha(\mu) = \alpha$. ∎

Lemma 6 *Assume that $E\left[D(|g|,1)^2\right] < \infty$ and $P\{T_F < T_1(1)\} > 0$. Then $|\nu_i| < \infty$ and $|\Psi_{i,j}| < \infty$ for all $i, j = 1, 2, \ldots, 10$.*

Proof. Note that $D(1) = W_1(1) + W_2(1) + A_1(1) + A_2(1)$, where

$$A_1(1) = (1 - I_{12}(1)) \sum_{l\in B_1} D_{22}(l) \prod_{j\in B_1,\, j<l} (1 - I_{22}(j)),$$

and

$$A_2(1) = W_3(1)W_6(1) \prod_{l\in B_1} (1 - I_{22}(l)).$$

Assume first that g is non-negative; then so are the $W_i(m)$. Hence, assuming that $E[D(|g|,1)^2] < \infty$ implies that $E[W_1(1)^2]$, $E[W_2(1)^2]$, $E[A_1(1)^2]$, and $E[A_2(1)^2]$ are all finite. Also, note that $E[W_4(1)^2]$, $E[W_5(1)^2]$, $E[W_6(1)^2]$, and $E[W_{10}(1)^2]$ are all trivially finite, and as we argued in the proof of Lemma 4, we must have that $\nu_{10} = P\{T_2(0) < T_1(1)\} > 0$.

We now prove that $E[W_8(1)^2] < \infty$. We previously defined $K = |B_1|$, and given $W_{10}(1) = 1$, K follows a geometric distribution with parameter $\rho = P\{T_1(1) < T_2(1)|T_2(0) < T_1(1)\}$. First, observe that if $\rho = 1$, then $W_8(1) = 0$ a.s., and the result trivially holds. Now assume $\rho < 1$. Consider $A_1(1)$, and note that given $W_{10}(1) = 1$, the quantities $(1 - I_{21}(1))$ and $\sum_{l\in B_1} D_{22}(l) \prod_{j\in B_1, j<l}(1 - I_{22}(j))$ are independent. Therefore, applying arguments similar to those used to prove (18), we get that $E\left[A_1(1)^2\right] = E\left[1 - I_{21}(1)\right] \phi \,/\, \nu_{10}$, where $\phi \equiv E\left[\left(\sum_{l\in B_1} D_{22}(l) \prod_{j\in B_1,\ j<l}(1 - I_{22}(j))\right)^2\right]$. Now $E[A_1(1)^2] < \infty$ implies that $\phi < \infty$. Suppose that $B_1 = \{j_1, j_2, \ldots, j_K\}$, where $j_1 < j_2 < \cdots < j_K$. Then

$$\begin{aligned} \phi &= E\left[D_{22}(j_1)^2 \middle| K \geq 1\right] P\{K \geq 1\} + E\left[\left(\sum_{l\in B_1,\ l\neq j_1} D_{22}(l) \prod_{j\in B_1,\ j<l} (1 - I_{22}(j))\right)^2\right] \\ &\quad + E\left[D_{22}(j_1)\left(\sum_{l\in B_1,\ l\neq j_1} D_{22}(l) \prod_{j\in B_1,\ j<l} (1 - I_{22}(j))\right)\right]. \end{aligned}$$

Moreover, $P\{K \geq 1\} = P\{K \geq 1|W_{10}(1) = 1\}\nu_{10} = (1-\rho)\nu_{10} > 0$ since $\nu_{10} > 0$ and we assumed that $\rho < 1$. Then $\phi < \infty$ implies that $E\left[D_{22}(j_1)^2|K \geq 1\right] < \infty$. Now let $C_i = D_{22}(j_i)I_{22}(j_i)$, $i = 1, 2, \ldots$, and it follows that $\phi_2 \equiv E\left[C_1^2|K \geq 1\right] < \infty$ since $C_1^2 \leq D_{22}(j_1)^2$. Consequently, $\phi_1 \equiv E\left[C_1|K \geq 1\right] < \infty$. Also, given $W_{10}(1) = 1$, the C_i, $i = 1, 2, \ldots$, are i.i.d. and independent of K. Hence, arguing as in (20), we get that $E\left[W_8(1)^2\right] = \left(\phi_2\rho(1-\rho) + 2\phi_1^2(1-\rho)^2\right)\nu_{10}/\rho^2 < \infty$ since $\phi_1 < \infty$, $\phi_2 < \infty$, $0 < \rho < 1$, and $\nu_{10} < \infty$.

We can similarly establish that $E[W_3(1)^2]$, $E[W_7(1)^2]$, and $E[W_9(1)^2]$ are all finite. Hence, $E[W_i(1)^2] < \infty$ for $i = 1, 2, \ldots, 10$, which implies that all of the $\nu_i < \infty$. Also, $|\Psi_{i,j}| < \infty$ for all i and j, by the Cauchy-Schwarz inequality. If g is not non-negative, replace g by its absolute value in the previous calculations, and use the assumption that $E\left[D(|g|;1)^2\right] < \infty$ to get the finiteness of the moments. ∎

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