Fall 2008, Math 211, Common Exam 1 **Victor Matveev**

(1*a*) Find the point of intersection of the two lines:

METHOD 1:

$$\boxed{\begin{matrix} x-2=\frac{1}{2}(y+1)=\frac{1}{3}(z-3) \\ \frac{1}{3}(x-5)=\frac{1}{2}(y-1)=z-4 \end{matrix}} \mapsto \begin{cases} L_1(t)=<2,-1,3>+t<1,2,3> \\ L_2(s)=<5,1,4>+s<3,2,1> \end{cases} \mapsto \begin{cases} 2+t=5+3s \mapsto \boxed{t=3+3s} \\ -1+2t=1+2s \\ 3+3t=4+s \end{cases}$$

$$\mapsto \begin{cases} t=3+3s \\ -1+2(3+3s)=1+2s \mapsto 5+6s=1+2s \mapsto \boxed{s=-1} \mapsto \boxed{t=0} \mapsto \boxed{\mathbf{r}=\mathbf{r}_1=<2,-1,3>} \\ 3+3t=4+s \end{cases}$$

METHOD 2

$$\begin{cases} x-2=\frac{1}{2}(y+1)=\frac{1}{3}(z-3) \\ \frac{1}{3}(x-5)=\frac{1}{2}(y-1)=z-4 \end{cases} \mapsto \text{Eliminate } x \text{ and } z\text{:} \begin{cases} x-2=\frac{1}{2}(y+1) \mapsto \boxed{x=2+\frac{1}{2}(y+1)} \\ \frac{1}{2}(y+1)=\frac{1}{3}(z-3) \\ \frac{1}{3}(x-5)=z-4 \\ \frac{1}{2}(y-1)=z-4 \mapsto \boxed{z=4+\frac{1}{2}(y-1)} \end{cases}$$

$$\mapsto \begin{cases} x=2+\frac{1}{2}(y+1) \\ \frac{1}{2}(y+1)=\frac{1}{3}\left(1+\frac{1}{2}(y-1)\right) \mapsto y+1=\frac{2}{3}+\frac{1}{3}(y-1) \mapsto \boxed{y=-1} \\ \frac{1}{3}\left(\frac{1}{2}(y+1)-3\right)=z-4 \mapsto z=4+\frac{1}{3}\left(\frac{1}{2}(y+1)-3\right) \\ z=4+\frac{1}{2}(y-1) \end{cases} \mapsto \boxed{\begin{matrix} x=2 \\ y=-1 \\ z=3 \end{matrix}}$$

(1b) Determine the cosine of the angle between these two lines:

$$\cos\theta=\frac{\mathbf{v}_1\bullet\mathbf{v}_2}{|\mathbf{v}_1||\mathbf{v}_2|}=\frac{<1,2,3>\bullet<3,2,1>}{|<1,2,3>||<3,2,1>|}=\frac{3+4+3}{\sqrt{1+4+9}\sqrt{9+4+1}}=\frac{10}{14}=\frac{5}{7}$$

(2a) Determine the equation of the plane formed by the two parallel lines:

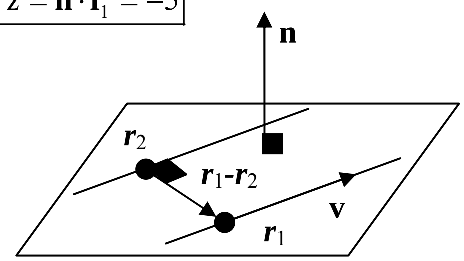
$x = 1 + 2t,\ y = -t,\ z = -1 + t$ and $x = -4t,\ y = 1 + 2t,\ z = 2 - 2t$

The two direction vectors are parallel, so we need another vector on the plane to find its normal:

$$\begin{cases} L_1 : \mathbf{r}_1 + t\,\mathbf{v}_1 = <1,0,-1> + t<2,-1,1> \\ L_2 : \mathbf{r}_2 + t\,\mathbf{v}_2 = <0,1,2> + t<-4,2,-2> \end{cases} \mapsto \mathbf{r}_1 - \mathbf{r}_2 = <1,-1,-3>$$

$$\mathbf{n} = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{v}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -3 \\ 2 & -1 & 1 \end{vmatrix} = <-4,-7,1> \quad \mapsto \quad \boxed{-4x - 7y + z = \mathbf{n}\cdot\mathbf{r}_1 = -5}$$

(2b) Determine the shortest distance between the two planes:



Note that vector (r_1-r_2) is orthogonal to the two lines:

$(\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathbf{v} = <1,-1,-3> \cdot <2, -1, 1> = 0$

Therefore, the shortest distance between the two lines is simply $|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{1+1+9} = \sqrt{11}$

(3a) Sketch and identify the surface $x^2 = y^2 + z^2$

The *x*-traces are circles with radius increasing linearly with *x*, therefore it's a cone with the main axis directed along the *x*-axis

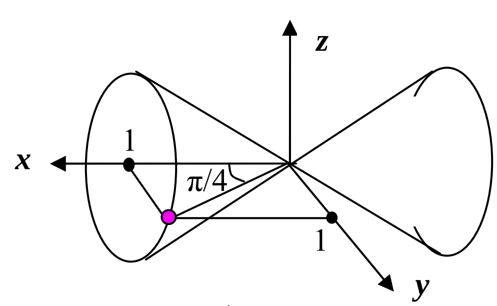
(3b) Write this equation in cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\cos\theta \\ z = z \end{cases} \mapsto \boxed{r^2\cos^2\theta = r^2\sin^2\theta + z^2} \mapsto \boxed{r^2(\cos^2\theta - \sin^2\theta) = z^2} \mapsto \boxed{r^2\cos(2\theta) = z^2}$$

(the last step is optional)

(3c) Show the location of the point $(\sqrt{2}, \frac{\pi}{4}, 0)$

Cartesian coordinates of this point are (1,1,0)



r = 1/cos(π/4)=√2

(4) For a particle moving along the space curve given by $\mathbf{r}(t)$=< $2t$, $1/t$, $(t\text{-}1)^3$ >

a) Evaluate $\frac{d^2r}{dt^2}(1)$:

$$\frac{dr}{dt} = \left\langle 2, -\frac{1}{t^2}, 3(t-1)^2 \right\rangle \mapsto \frac{d^2r}{dt^2} = \left\langle 0, \frac{2}{t^3}, 6(t-1) \right\rangle \mapsto \boxed{\frac{d^2r}{dt^2}(1) = \langle 0,2,0 \rangle}$$

b) Angle between $\frac{d^2r}{dt^2}$ and $\frac{dr}{dt}$ at time t=1:

$$\begin{cases} \frac{dr}{dt}(1) = \langle 2,-1,0 \rangle \\ \frac{d^2r}{dt^2}(1) = \langle 0,2,0 \rangle \end{cases} \mapsto \cos\theta = \frac{\langle 2,-1,0 \rangle \bullet \langle 0,2,0 \rangle}{|\langle 2,-1,0 \rangle| \, |\langle 0,2,0 \rangle|} = \boxed{-\frac{1}{\sqrt{5}}}$$

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(5) Find the velocity and the position vector, given the acceleration:

$$\frac{d^2r}{dt^2} = \langle 9\sin 3t, 9\cos 3t, 4 \rangle \xrightarrow{\text{integrate}} \frac{dr}{dt} = \langle -3\cos 3t, 3\sin 3t, 4t \rangle + \mathbf{C}$$

Match initial conditions: $\frac{dr}{dt}(0) = \langle -3,0,0 \rangle + \mathbf{C} =< 2,0,-7 > \mapsto \mathbf{C} =< 2,0,-7 > - < -3,0,0 >= \boxed{< 5,0,-7 >}$

$$\mapsto \boxed{\frac{dr}{dt} = \langle 5-3\cos 3t,\ 3\sin 3t,\ 4t-7 \rangle}$$

$$\xrightarrow{\text{integrate}} r = \left\langle 5t - \sin 3t, -\cos 3t, 2t^2 - 7t \right\rangle + \mathbf{C}$$

Match initial conditions: $r(0) = \langle 0,-1,0 \rangle + \mathbf{C} =< 3,4,0 > \mapsto \mathbf{C} =< 3,4,0 > - < 0,-1,0 >= \boxed{< 3,5,0 >}$

$$\mapsto \boxed{\frac{dr}{dt} = \left\langle 3+5t-\sin 3t,\ 5-\cos 3t,\ 2t^2-7t \right\rangle}$$