Math 213-H01 ● Victor Matveev ● Final exam ● December 15, 2011

Please show all work to receive full credit.

1. **(12pts)** Consider a scalar field $f(x,y) = \ln(x^3 + y)$
   (a) Find the domain and the range of $f(x,y)$; sketch the domain.
   (b) Sketch several different level curves of this function
   (c) Use linear approximation around point (1, 0) to estimate $f(0.99,\ 0.05)$

2. **(11pts)** Sketch the vector field $\vec{\mathbf{F}}(x,y,z) = \langle x - y,\ x,\ 0\rangle$ in the *x*-*y* plane.

3. **(30pts)** Consider vector field $\vec{\mathbf{F}}(x,y,z) = \langle 1,\ 0,\ xz\rangle$ and part of the paraboloid surface $x^2 + y^2 + z = 1$ enclosed in the first octant.
   (a) Verify the Stokes Theorem for this vector field and this surface, by calculating all integrals that appear in the Stokes Theorem. Sketch the surface, show the direction of the normal, and the direction of circulation integral.
   (b) Verify the Divergence Theorem for this vector field and the region enclosed in the first octant by the given surface, by calculating all integrals that appear in the Divergence theorem.

4. **(14pts)** Which of the following integrals equal zero for *any* differentiable field $\vec{\mathbf{F}}(x,y,z)$ or $f(x,y,z)$, and for *any* closed curve *C* or *closed* surface *S*? Give a brief explanation for each answer. Hint: it may help to apply either the divergence theorem or the Stokes theorem to some of these integrals

   a) $\oiint_S \vec{\nabla} f \cdot \hat{\mathbf{n}}\, d\sigma$ b) $\oint_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$ c) $\oiint_S \vec{\nabla} \times \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}\, d\sigma$ d) $\oint_C \vec{\nabla} \times \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$

You may drop one of the remaining four problems:

5. **(11pts)** Calculate $\vec{\nabla} \cdot (\vec{\mathbf{r}} \ln r)$, where $\vec{\mathbf{r}} = \langle x,\ y,\ z\rangle$ is the position vector, and $r = |\vec{\mathbf{r}}|$.

6. **(11pts)** Sketch and use an appropriate coordinate transformation to find the area in the 1st quadrant bounded by two lines *y*=*x* and *y*=2*x*, and by two hyperbolas *xy*=1 and *xy*=2 [Hint: it is easier to calculate the Jacobian of the inverse coordinate transformation, and then reciprocate it].

7. **(11pts)** Find the limit, or show that it does not exist:

   (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x y^2}{x^2 + y^4}$ **(b)** $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y^3}{x^2 + y^2}$ **(c)** $\lim\limits_{(x,y)\to(1,1)} \dfrac{x \ln(xy)}{xy - 1}$

8. (**11pts**) Consider a sphere of radius R_0 with *charge density* decreasing quadratically with distance from the center as $\rho(r) = c / r^2$, where *c* is a constant, and *r* is the distance from the origin (i.e., *r* is the spherical radial variable)

a) Express constant *c* as a function of total charge of the sphere, *Q*.

b) Find the electric field $\vec{\mathbf{E}}(r)$ *inside* the sphere by applying the divergence theorem to the Gauss equation $\vec{\nabla} \bullet \vec{\mathbf{E}}(r) = \rho(r)$. Note that spherical symmetry requires that $\vec{\mathbf{E}}(r) = E(r)\,\hat{\mathbf{r}}, \text{ where } \hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r}, \text{ and } E(r) = \left|\vec{\mathbf{E}}(r)\right|$