Problem #18(c)

Verify that $\nabla \bullet (\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \bullet \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \bullet \nabla \times \mathbf{F}_2$

First, find cross product:

$$\left.\begin{array}{l}\mathbf{F}_1 = \langle M_1, N_1, P_1 \rangle \\ \mathbf{F}_2 = \langle M_2, N_2, P_2 \rangle\end{array}\right\} \; \mathbf{F}_1 \times \mathbf{F}_2 = \langle N_1 P_2 - N_2 P_1,\; M_2 P_1 - M_1 P_2,\; M_1 N_2 - M_2 N_1 \rangle$$

Now, use definition of divergence:

$$\nabla \bullet \mathbf{F} = \nabla \bullet \langle M, N, P \rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Apply this to the cross product expression given above

$$\begin{aligned}\nabla \bullet (\mathbf{F}_1 \times \mathbf{F}_2) &= \nabla \bullet \langle N_1 P_2 - N_2 P_1,\; M_2 P_1 - M_1 P_2,\; M_1 N_2 - M_2 N_1 \rangle \\ &= \frac{\partial (N_1 P_2 - N_2 P_1)}{\partial x} + \frac{\partial (M_2 P_1 - M_1 P_2)}{\partial y} + \frac{\partial (M_1 N_2 - M_2 N_1)}{\partial z}\end{aligned}$$

Now use the product term for each of the six terms, which gives us the following twelve terms:

$$= P_2 \frac{\partial N_1}{\partial x} + N_1 \frac{\partial P_2}{\partial x} - P_1 \frac{\partial N_2}{\partial x} - N_2 \frac{\partial P_1}{\partial x} + M_2 \frac{\partial P_1}{\partial y} + P_1 \frac{\partial M_2}{\partial y} - M_1 \frac{\partial P_2}{\partial y} - P_2 \frac{\partial M_1}{\partial y} + M_1 \frac{\partial N_2}{\partial z} + N_2 \frac{\partial M_1}{\partial z} - M_2 \frac{\partial N_1}{\partial z} - N_1 \frac{\partial M_2}{\partial z}$$

Since we have to show that this equals $\mathbf{F}_2 \bullet \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \bullet \nabla \times \mathbf{F}_2$, collect terms that multiply $M_{1,2}$, $N_{1,2}$ and $P_{1,2}$:

$$= M_2 \left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z} \right) + N_2 \left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x} \right) + P_2 \left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y} \right) - M_1 \left(\frac{\partial P_2}{\partial y} - \frac{\partial N_2}{\partial z} \right) - N_1 \left(\frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x} \right) - P_1 \left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y} \right)$$

Now, recall that $\nabla \times \mathbf{F} = \begin{pmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{pmatrix} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z},\; \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x},\; \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$

This makes it clear that the expression we found above can be written as:

$$= \langle M_2, N_2, P_2 \rangle \bullet (\nabla \times \mathbf{F}_1) - \langle M_1, N_1, P_1 \rangle \bullet (\nabla \times \mathbf{F}_2) = \mathbf{F}_2 \bullet \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \bullet \nabla \times \mathbf{F}_2$$

This completes the derivation

Problem #19(b)

Verify that $\nabla\times\left(g\mathbf{F}\right)=g\,\nabla\times\mathbf{F}+\nabla g\times\mathbf{F}$

First, expand the left-hand side

$$\left.\begin{array}{l}\mathbf{F}=\langle M,N,P\rangle\\ g\mathbf{F}=\langle gM,gN,gP\rangle\end{array}\right\}\;\nabla\times\left(g\mathbf{F}\right)=\begin{vmatrix} i & j & k\\ \partial_x & \partial_y & \partial_z\\ gM & gN & gP\end{vmatrix}=\left\langle \frac{\partial(gP)}{\partial y}-\frac{\partial(gN)}{\partial z},\ \frac{\partial(gM)}{\partial z}-\frac{\partial(gP)}{\partial x},\ \frac{\partial(gN)}{\partial x}-\frac{\partial(gM)}{\partial y}\right\rangle$$

Now use the product term for each of the six terms, which gives us the following twelve terms:

$$=\left\langle g\frac{\partial P}{\partial y}+P\frac{\partial g}{\partial y}-g\frac{\partial N}{\partial z}-N\frac{\partial g}{\partial z},\ g\frac{\partial M}{\partial z}+M\frac{\partial g}{\partial z}-g\frac{\partial P}{\partial x}-P\frac{\partial g}{\partial x},\ g\frac{\partial N}{\partial x}+N\frac{\partial g}{\partial x}-g\frac{\partial M}{\partial y}-M\frac{\partial g}{\partial y}\right\rangle$$

Now let's collect all terms proportional to g, and break this vector apart as a sum of the following two vectors:

$$=\left\langle g\frac{\partial P}{\partial y}-g\frac{\partial N}{\partial z},\ g\frac{\partial M}{\partial z}-g\frac{\partial P}{\partial x},\ g\frac{\partial N}{\partial x}-g\frac{\partial M}{\partial y}\right\rangle+\left\langle P\frac{\partial g}{\partial y}-N\frac{\partial g}{\partial z},\ M\frac{\partial g}{\partial z}-P\frac{\partial g}{\partial x},\ N\frac{\partial g}{\partial x}-M\frac{\partial g}{\partial y}\right\rangle$$

Let's factor out g in the first vector field:

$$=g\underbrace{\left\langle \frac{\partial P}{\partial y}-\frac{\partial N}{\partial z},\ \frac{\partial M}{\partial z}-\frac{\partial P}{\partial x},\ \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right\rangle}_{\nabla\times\mathbf{F}}+\left\langle P\frac{\partial g}{\partial y}-N\frac{\partial g}{\partial z},\ M\frac{\partial g}{\partial z}-P\frac{\partial g}{\partial x},\ N\frac{\partial g}{\partial x}-M\frac{\partial g}{\partial y}\right\rangle$$

Now we only have to show that the second vector field equals $\nabla g\times\mathbf{F}$:

$$\nabla g\times\mathbf{F}=\begin{vmatrix} i & j & k\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z}\\ M & N & P\end{vmatrix}=\left\langle P\frac{\partial g}{\partial y}-N\frac{\partial g}{\partial z},\ M\frac{\partial g}{\partial z}-P\frac{\partial g}{\partial x},\ N\frac{\partial g}{\partial x}-M\frac{\partial g}{\partial y}\right\rangle$$

Thus, our calculation above equals $g\,\nabla\times\mathbf{F}+\nabla g\times\mathbf{F}$ *Q.E.D.*

Problem #27

Consider a harmonic function f : $\nabla^2 f = \nabla \bullet \nabla f = 0$

(**a**) Show that $\displaystyle\oiint_S \nabla f \bullet \mathbf{n}\, d\sigma = 0$

Since this is a flux of the vector field $\mathbf{F} \equiv \nabla f$ across a closed surface, we can apply the divergence theorem:

$$\oiint_{S=\partial D} \underbrace{\mathbf{F}}_{\nabla f} \bullet \mathbf{n}\, d\sigma = \iiint_D \nabla \bullet \underbrace{\mathbf{F}}_{\nabla f}\, dV = \iiint_D \underbrace{\nabla^2 f}_{=0}\, dV = 0$$

(**b**) Now show that $\displaystyle\oiint_S f\, \nabla f \bullet \mathbf{n}\, d\sigma = \iiint_D |\nabla f|^2\, dV$

Once again, the left-hand side is a flux across a closed surface, so we can apply the divergece theorem:

$$\oiint_{S=\partial D} \underbrace{\mathbf{F}}_{f\,\nabla f} \bullet \mathbf{n}\, d\sigma = \iiint_D \nabla \bullet \underbrace{\mathbf{F}}_{f\,\nabla f}\, dV = \iiint_D \nabla \bullet \left(f\, \nabla f \right) dV$$

Now use the product rule (which is very easy to prove): $\nabla \bullet \left(f\, \mathbf{F} \right) = f\; \nabla \bullet \mathbf{F} + \mathbf{F} \bullet \nabla f$

Applying this to $\mathbf{F} = \nabla f$, we obtain $\nabla \bullet \left(f\, \nabla f \right) = f\; \nabla \bullet \nabla f + \nabla f \bullet \nabla f = f\, \nabla^2 f + |\nabla f|^2$

In our case $\nabla^2 f = 0$, which gives us the final result we need:

$$\oiint_S f\, \nabla f \bullet \mathbf{n}\, d\sigma = \iiint_D f\, \underbrace{\nabla^2 f}_{=0}\, dV + \iiint_D |\nabla f|^2\, dV = \iiint_D |\nabla f|^2\, dV$$

Problem #28

We are asked to calculate $\iint_S \nabla f \bullet \mathbf{n}\, d\sigma$,

where S is part of the surface of the sphere of radius a in the first octant, and $f(\mathbf{r}) = \ln|\mathbf{r}| \equiv \ln r$

First, calculate the gradient of $f(\mathbf{r})$:

$$\left.\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\ln r = \frac{1}{r}\frac{\partial}{\partial x}\underbrace{\sqrt{x^2+y^2+z^2}}_{r} = \frac{x}{r\sqrt{x^2+y^2+z^2}} = \frac{x}{r^2} \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\ln r = \frac{1}{r}\frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2} = \frac{y}{r\sqrt{x^2+y^2+z^2}} = \frac{y}{r^2} \\
\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\ln r = \frac{1}{r}\frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2} = \frac{z}{r\sqrt{x^2+y^2+z^2}} = \frac{z}{r^2}
\end{aligned}\right\} \Rightarrow \nabla f = \frac{\langle x, y, z\rangle}{r^2} = \frac{\mathbf{r}}{r^2}$$

On the surface of the sphere, $\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|} \equiv \frac{\mathbf{r}}{r}$

Thus we have $\iint_S \nabla f \bullet \mathbf{n}\, d\sigma = \iint_S \left(\frac{\mathbf{r}}{r^2}\right)\bullet\left(\frac{\mathbf{r}}{r}\right) d\sigma = \iint_S \frac{r^2}{r^3}\, d\sigma = \iint_S \frac{1}{r}\, d\sigma$

Now, since $r = a = const$ on the surface of the sphere, we obtain (recall that S=1/8 of the sphere which is in 1st octant)

$$\iint_S \nabla f \bullet \mathbf{n}\, d\sigma = \frac{1}{a}\iint_S d\sigma = \frac{Area(S)}{a} = \frac{1}{8a}4\pi a^2 = \boxed{\frac{\pi a}{2}}$$

Problem #30

As shown in class, application of divergence theorem to vector field $\mathbf{F}=f\,\nabla g$ yields the Green's first identity:

$$\oiint_{S=\partial D} f\,\nabla g \bullet \mathbf{n}\; d\sigma = \iiint_D \underbrace{\left(f\,\nabla^2 g + \nabla f \bullet \nabla g\right)}_{\nabla \bullet (f\,\nabla g)} dV$$

Interchanging the scalar fields f and g yields an equivalent expression

$$\oiint_{S=\partial D} g\,\nabla f \bullet \mathbf{n}\; d\sigma = \iiint_D \left(g\,\nabla^2 f + \nabla g \bullet \nabla f\right) dV$$

Now let's subtract the two formulas above:

$$\oiint_{S=\partial D} f\,\nabla g \bullet \mathbf{n}\; d\sigma - \oiint_{S=\partial D} g\,\nabla f \bullet \mathbf{n}\; d\sigma = \iiint_D \left(f\,\nabla^2 g + \nabla f \bullet \nabla g\right) dV - \iiint_D \left(g\,\nabla^2 f + \nabla g \bullet \nabla f\right) dV$$

All operations above are linear (integration, differentiation, vector and scalar products), so we can combine the integrals:

$$\oiint_{S=\partial D} \left(f\,\nabla g - g\,\nabla f\right) \bullet \mathbf{n}\; d\sigma = \iiint_D \left(f\,\nabla^2 g + \cancel{\nabla f \bullet \nabla g} - g\,\nabla^2 f - \cancel{\nabla g \bullet \nabla f}\right) dV = \iiint_D \left(f\,\nabla^2 g - g\,\nabla^2 f\right) dV$$

We obtained an equality known as the second Green's identity