Bessel equation of order *m*: $z^2 f_{zz} + z\, f_z + (z^2 - m^2) f(z) = 0$

Solution: $f(z) = C_1 J_m(z) + C_2 Y_m(z)$

Jm **-- Bessel function of the first kind of order *m***

Ym **-- Bessel function of the second kind of order *m***

Small-z asymptotic behavior is found by neglecting the non-equidimensional term $z^2 f(z)$:

$$J_m(z) \sim \begin{cases} 1 & (m=0) \\ \dfrac{1}{2^m m!} z^m & (m>0) \end{cases} \qquad Y_m(z) \sim \begin{cases} \dfrac{2}{\pi} \ln z & (m=0) \\ -\dfrac{2^m (m-1)!}{\pi} z^{-m} & (m>0) \end{cases}$$

Large-z asymptotic behavior is found by analogy with a damped oscillator, or by substituting $f(z) = g(z)/\sqrt{z}$ into the Bessel equation:

$$J_m(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2} \right), \quad Y_m(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2} \right)$$

Note: all constant pre-factors above are arbitrary, so they are a matter of normalization convention. The widely adopted convention for Bessel functions of the 1st kind is that the unweighted integral over the semi-infinite line equals one.

Taylor series of the Bessel function of 1st kind of order *m*:

$$J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{z}{2}\right)^{2k+m} = \left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{z}{2}\right)^{2k}$$

Random fact: for the special value of *m*=1/2, it is easy to check by direct differentiation that the long-range asymptotics yield ***exact*** representation of the Bessel functions in terms of trigonometric functions:

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin(z), \quad Y_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos(z) \quad \text{Exact, not asymptotic!!}$$