Math 331-001
Final Examination
December 19, 2008

1. (35pts) Use separation of variables to solve the following PDE:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} - 4u, \qquad 0 < x < 1 \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0 \\ u(x,0) = 0,\ \dfrac{\partial u}{\partial t}(x,0) = 5 + 3\cos(2\pi x) \end{cases}$$

a) Separate the variables to find the two ODEs. It is convenient to simplify the boundary value ODE as much as possible.
b) Solve the boundary value ODE; be careful to examine any zero and/or negative eigenvalues.
c) Complete the solution of this PDE, and determine all coefficients.
d) Check that your solution satisfies the PDE, and the initial and boundary conditions

2. (25pts) Consider the following boundary value problem:

$$\begin{cases} \dfrac{d}{dx}\left(x \dfrac{d\phi}{dx} \right) = -\lambda x \phi, \qquad 0 < x < 1 \\ \dfrac{d\phi}{dx}(0) = \phi(1) = 0 \end{cases}$$

a) Find the Rayleigh quotient
b) Use a simple polynomial test function to find an upper bound on the lowest λ
c) Find all eigenfunctions, and sketch the first two of them. Find an asymptotic expression for λ_n when n is large

3. (15pts) Use the Fourier Transform method to solve the following PDE (for γ=const):

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\gamma \dfrac{\partial u}{\partial x}, \quad -\infty < x < +\infty,\ t \ge 0 \\ u(x,0) = e^{-x^2/4} \end{cases}$$

a) Obtain a closed-form solution (use the shift theorem).
b) Sketch the solution $u(x,t)$ as a function of x, for several values of t.
c) Check that your answer satisfies the PDE and the boundary condition.

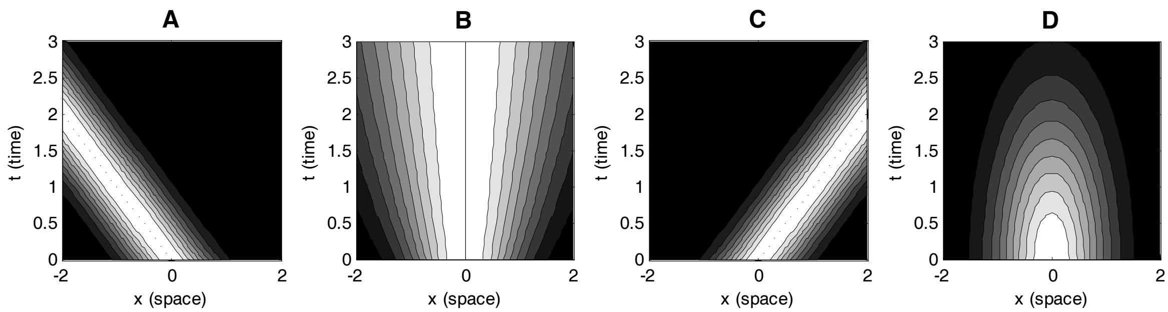
4. (15pts) Consider the heat equation for a 1D rod with thermal diffusivity k=1:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - \cos x, & 0 < x < 1 \\ \dfrac{\partial u}{\partial x}(0,t) = \pm u(0,t); \quad \dfrac{\partial u}{\partial x}(1,t) = 0; & u(x,0) = f(x) \end{cases}$$

a) Find and explain the correct sign in the boundary condition at x=0, in order for the boundary conditions to make sense physically [consider the direction of heat flux]

b) Find the equilibrium temperature distribution.

5. (10pts) Each of the following four panels shows a contour plot of a function of two variables, ***u*(*x*,*t*)**. The lighter shade corresponds to a higher value of this function. Which of these functions satisfy(ies) the following PDE: $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}$



[Hint: simply examine the signs of $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial x}$ at a couple different (x,y) points: is the function increasing or decreasing at this point?]

Some facts you may (or may not) find useful:

$f(x)$	F(ω)
$e^{-\alpha x^2}$	$\dfrac{1}{\sqrt{4\pi\alpha}} e^{-\omega^2/4\alpha}$
$\sqrt{\dfrac{\pi}{\beta}} e^{-x^2/4\beta}$	$e^{-\beta\omega^2}$
$\dfrac{1}{2\pi}\int_{-\infty}^{+\infty} f(\overline{x})g(x-\overline{x})d\overline{x}$	F(ω)G(ω)

Bessel equation: $z^2 \dfrac{d^2 f}{dz^2} + z\dfrac{df}{dz} + (z^2 - m^2) f = 0$

Large-z asymptotics: $J_m(z) \sim \dfrac{1}{\sqrt{z}} \cos\left(z - \dfrac{\pi}{4} - m\dfrac{\pi}{2}\right), \quad Y_m(z) \sim \dfrac{1}{\sqrt{z}} \sin\left(z - \dfrac{\pi}{4} - m\dfrac{\pi}{2}\right)$

Small-z asymptotics: $J_m(z) \sim z^m, \quad Y_m(z) \sim z^{-m} \ (m \neq 0), \quad Y_0(z) \sim \ln z$

Series representation: $J_m(z) = \sum_{k=0}^{\infty} \dfrac{(-1)^k (z/2)^{2k+m}}{k!(k+m)!}$