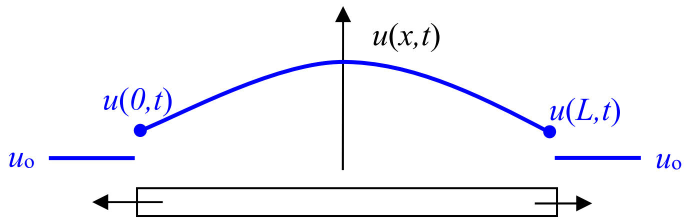
MATH 331-001 * Midterm Solution * October 12, 2010

1. Recall that $\phi = -K_0 \dfrac{\partial u}{\partial x}$, so **derivative sign is opposite to the sign of heat flux**. Consider for example the case $u(x,t) > u_o$ (rod is warmer than outside temperature):



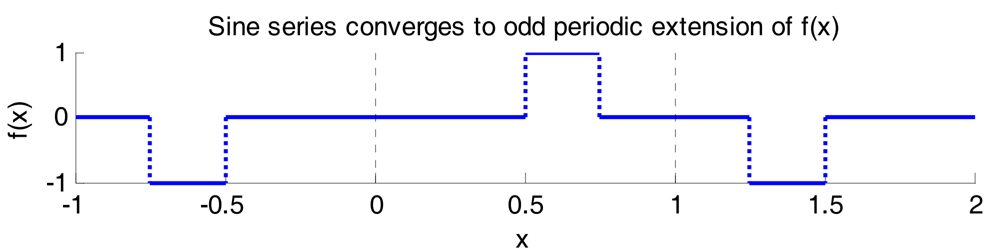
From above figure it is easy to see that the signs should be as follows:

$$\frac{\partial u}{\partial x}(0,t) = +\gamma\left[u(0,t) - u_o\right] > 0; \quad \frac{\partial u}{\partial x}(L,t) = -\gamma\left[u(L,t) - u_o\right] < 0 \quad (\text{Recall } \boldsymbol{\gamma > 0})$$

Examine the shape of the $u(x,t)$ distribution (blue curve) to understand the sign of du/dx

---

2. Find the **first four** non-zero terms in the sine series for the function $f(x) = \begin{cases} 1, & \frac{1}{2} < x < \frac{3}{4} \\ 0 & \text{otherwise} \end{cases}$ on the interval [0, 1]. Sketch what the **full** sine series



$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin n\pi x$$

Note that L=1

$$B_n = \frac{2}{\underbrace{1}_{L}}\int_0^1 f(x)\sin n\pi x\,dx = 2\int_{1/2}^{3/4} \sin n\pi x\,dx = -\frac{2}{n\pi}\cos n\pi x\Big|_{1/2}^{3/4} = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos\frac{3n\pi}{4}\right)$$

$$B_1 = \frac{2}{\pi}\left(\underbrace{\cos\frac{\pi}{2}}_{0} - \underbrace{\cos\frac{3\pi}{4}}_{-1/\sqrt{2}}\right) = \frac{\sqrt{2}}{\pi}; \qquad B_2 = \frac{2}{2\pi}\left(\underbrace{\cos\pi}_{-1} - \underbrace{\cos\frac{3\pi}{2}}_{0}\right) = -\frac{1}{\pi}$$

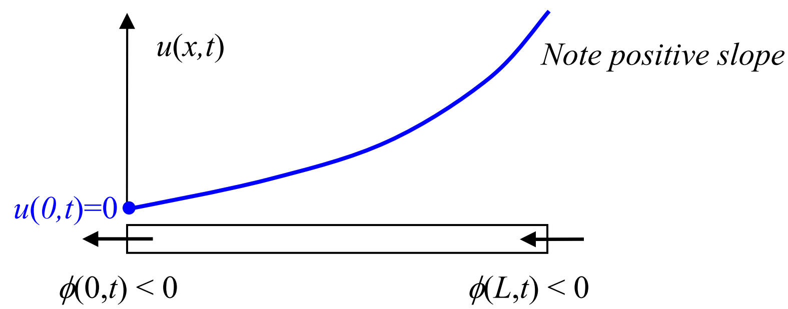
$$B_3 = \frac{2}{3\pi}\left(\underbrace{\cos\frac{3\pi}{2}}_{0} - \underbrace{\cos\frac{9\pi}{4}}_{1/\sqrt{2}}\right) = -\frac{\sqrt{2}}{3\pi}; \quad B_4 = \frac{2}{4\pi}\left(\underbrace{\cos 2\pi}_{1} - \underbrace{\cos 3\pi}_{-1}\right) = \frac{1}{\pi}$$

$$\boxed{\mathbf{f(x) = \frac{\sqrt{2}}{\pi}\sin\pi x - \frac{1}{\pi}\sin 2\pi x - \frac{\sqrt{2}}{3\pi}\sin 3\pi x + \frac{1}{\pi}\sin 4\pi x - \ldots}}$$

Note the slow convergence (discontinuous)

3. Consider the following heat equation: $$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - 4u \quad \left(0 < x < 1,\ \ t > 0\right) \\ u(0,t) = 0, \quad \dfrac{\partial u}{\partial x}\left(1,t\right) = \pm\mathbf{1} \\ u(x,0) = x \end{cases}$$

(a) Energy balance: Assuming temperature is positive, the sink term "-4*u*" cools the rod along its length, and the rod is also cooled to zero at its left end, so heat has to come into the right end to maintain heat balance



du/dx > 0: choose "+1"

Let's find the equilibrium temperature distribution (the blue curve above):

$$\begin{cases} \dfrac{d^2 u_{eq}}{dx^2} = 4u_{eq} \quad \mapsto \quad u_{eq}(x) = a\cosh 2x + b\sinh 2x \\ \text{Boundary conditions:} \\ u_{eq}(0) = 0 = a + 0 \quad \mapsto \quad a = 0 \quad \mapsto \quad u_{eq}(x) = b\sinh 2x \\ \dfrac{du_{eq}}{dx}\left(1\right) = 2b\cosh 2 = \mathbf{1} \quad \mapsto \quad b = \dfrac{\mathbf{1}}{2\cosh 2} \quad \mapsto \quad \boxed{\mathbf{u_{eq}(x) = \dfrac{\sinh 2x}{2\cosh 2}}} \end{cases}$$

Basically, $u_{eq} \propto \sinh(2x)$

(b) Does the heat energy of the rod/cable remain constant over time? (extra credit: 5 points)

No: although at equilibrium energy *is* constant, it is **not** constant initially (note the initial condition). There are two ways to show this:

Method 1: Compare total energy at t=0 and at t→∞ (equilibrium), to show they are **not equal**:

$$E(0) = \int_0^1 u(x,0)dx = \int_0^1 x\,dx = \frac{1}{2}$$

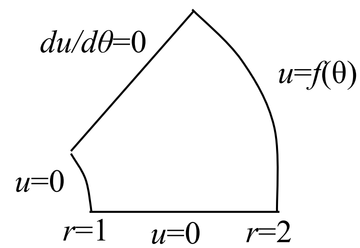
$$\text{but } E(\infty) = E_{eq} = \int_0^1 u_{eq}(x,0)dx = \frac{1}{2\cosh 2}\int_0^1 \sinh 2x\,dx = \frac{1}{4\cosh 2}\cosh 2x\Big|_0^1 = \boxed{\frac{\cosh 2 - 1}{4\cosh 2} \neq \frac{1}{2}}$$

Method 2: Integrate the PDE to see that the rate of change of energy in time is **not zero:**

$$\int_0^1 \frac{\partial u}{\partial t}dx = \int_0^1\left(\frac{\partial^2 u}{\partial x^2} - 4u\right)dx \mapsto \frac{d}{dt}\underbrace{\int_0^1 u\,dx}_{E(t)} = \underbrace{\left.\frac{\partial u}{\partial x}\right|_0^1}_{1-\frac{\partial u}{\partial x}(0,t)} - 4\underbrace{\int_0^1 u\,dx}_{E(t)} \mapsto \frac{dE(t)}{dt} = \boxed{\underbrace{1}_{\text{heat influx from right boundary}} - \underbrace{\frac{\partial u}{\partial x}(0,t)}_{\text{heat loss at left boundary}} - \underbrace{4E(t)}_{\text{heat loss along the length of rod } (Q=-4u)} \neq 0}$$

1. (52pts) Solve the Laplace's equation in a ring sector by following the steps below:

$$\begin{cases} \nabla^2 u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} & \left(\mathbf{0<\theta<\dfrac{\pi}{3},\ 1<r<2}\right) \quad \textbf{(i)} \\ u(r,0)=\dfrac{\partial u}{\partial \theta}\left(r,\dfrac{\pi}{3}\right)=0 \\ u(1,\theta)=0;\ u(2,\theta)=\sqrt{2}\sin(3\theta/2) \end{cases}$$



Example physical application: **equilibrium** temperature distribution

(ii)

$$u(r,\theta)=\varphi(\theta)G(r)$$

Multiply the Laplace's equation by r^2:

$$\frac{\partial^2}{\partial\theta^2}\left[\varphi(\theta)G(r)\right]+r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left[\varphi(\theta)G(r)\right]\right)=0$$

$$G(r)\frac{\partial^2\varphi}{\partial\theta^2}=-r\,\varphi(\theta)\frac{d}{dr}\left(r\frac{dG}{dr}\right)\quad\Big\|\div(G\varphi)$$

$$\frac{1}{\varphi(\theta)}\frac{d^2\varphi}{d\theta^2}=-\frac{r}{G(r)}\frac{d}{dr}\left(r\frac{dG}{dr}\right)=-\lambda$$

ODE 1 (BVP): $\varphi''(\theta)=-\lambda\varphi(\theta)$, $\quad \varphi(0)=\varphi'\left(\dfrac{\pi}{3}\right)=0$

ODE 2: $r\dfrac{d}{dr}\left(r\dfrac{dG}{dr}\right)=\lambda G$, $\quad G(1)=0$, $\quad G(2)\neq 0$

(iii)

$$\boxed{\begin{aligned}&\varphi''(\theta)=-\lambda\varphi(\theta)\\ &\varphi(0)=\varphi'\left(\frac{\pi}{3}\right)=0\end{aligned}} \mapsto \begin{cases} \boxed{\lambda>0} \mapsto \varphi(y)=C_1\sin(\sqrt{\lambda}\theta)+C_2\cos(\sqrt{\lambda}\theta) \\ \boxed{\lambda<0} \mapsto \varphi(y)=C_1\sinh(\sqrt{-\lambda}\theta)+C_2\cosh(\sqrt{-\lambda}\theta) \\ \boxed{\lambda=0} \mapsto \varphi(y)=C_1\theta+C_2 \end{cases}$$

$$\boxed{\lambda>0} \mapsto \begin{cases} \varphi(0)=C_2=0 \mapsto \boxed{\varphi_n(\theta)=C\sin(\sqrt{\lambda_n}\theta)} \mapsto \varphi'_n(\theta)=C\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}\theta) \\ \varphi'\left(\dfrac{\pi}{3}\right)=C\sqrt{\lambda_n}\cos\dfrac{\sqrt{\lambda}\pi}{3}=0 \mapsto \dfrac{\sqrt{\lambda}\pi}{3}=\pi\left(n-\dfrac{1}{2}\right) \mapsto \boxed{\sqrt{\lambda_n}=3\left(n-\dfrac{1}{2}\right)} \end{cases}$$

$$\boxed{\lambda<0} \mapsto \begin{cases} \varphi(0)=C_2=0 \mapsto \boxed{\varphi_n(\theta)=C\sinh(\sqrt{-\lambda_n}\theta)} \mapsto \varphi'_n(\theta)=C\sqrt{-\lambda_n}\cosh(\sqrt{-\lambda_n}\theta) \\ \varphi'\left(\dfrac{\pi}{3}\right)=C\sqrt{-\lambda_n}\underbrace{\cosh\dfrac{\sqrt{-\lambda}\pi}{3}}_{\neq 0}=0 \mapsto \boxed{\text{only trivial solution}} \end{cases}$$

$$\boxed{\lambda_0=0} \mapsto \varphi(0)=C_2=0 \mapsto \varphi'\left(\frac{\pi}{3}\right)=C_1=0 \mapsto C_1=0 \mapsto \boxed{\text{trivial solution}}$$

(iv) $\boxed{\begin{array}{l} r\dfrac{d}{dr}\left(r\dfrac{dG}{dr}\right)=\lambda G \\ G(1)=0 \\ G(2)\neq 0 \end{array}} \mapsto$

$$\boxed{\begin{array}{l}\lambda>0 \\ \sqrt{\lambda_n}=3\left(n-\dfrac{1}{2}\right)\end{array}} \quad r^2\frac{d^2G}{dr^2}+r\frac{dG}{dr}=\lambda G$$

Equidimensional equation, look for solutions of form $G(r)=C\,r^p$

$$p(p-1)C\,r^p+p\,C\,r^p=\lambda\,C\,r^p$$

$$\mapsto\ p(p-1)+p=\lambda \quad \mapsto \quad \boxed{p^2=\lambda} \quad \mapsto \quad G(r)=C_1r^{\sqrt{\lambda}}+C_2r^{-\sqrt{\lambda}}$$

$$G(1)=C_1+C_2=0 \ \mapsto\ C_1=-C_2 \ \mapsto\ G_n(r)=r^{\sqrt{\lambda_n}}-r^{-\sqrt{\lambda_n}}=\boxed{r^{3\left(n-\frac{1}{2}\right)}-r^{-3\left(n-\frac{1}{2}\right)}}$$

The general solution is: $u(r,\theta)=\sum_{n=1}^{\infty}B_nG_n(r)\varphi_n(\theta)=\boxed{\sum_{n=1}^{\infty}B_n\left[r^{3\left(n-\frac{1}{2}\right)}-r^{-3\left(n-\frac{1}{2}\right)}\right]\sin\left\{3\left(n-\frac{1}{2}\right)\theta\right\}}$

(v) Find B_n from boundary condition: $u(2,\theta)=\sum_{n=1}^{\infty}\underbrace{B_n\left[2^{3\left(n-\frac{1}{2}\right)}-2^{-3\left(n-\frac{1}{2}\right)}\right]}_{\text{Fourier Coef.}}\sin\left\{3\left(n-\frac{1}{2}\right)\theta\right\}=\sqrt{2}\sin\dfrac{3\theta}{2}$

$\mapsto\ B_n=0$ unless $n=1$ for which we have $B_1\left(2^{3/2}-2^{-3/2}\right)=\sqrt{2}\ \mapsto B_1=\dfrac{\sqrt{2}}{2\sqrt{2}-\dfrac{1}{2\sqrt{2}}}\times\dfrac{2\sqrt{2}}{2\sqrt{2}}=\dfrac{4}{8-1}=\dfrac{4}{7}$

Final answer: $\boxed{\mathbf{u(r,\theta)=\dfrac{4}{7}\left[r^{3/2}-r^{-3/2}\right]\sin\dfrac{3\theta}{2}}}$