Math 331 * Final Examination * December 18, 2018

Read carefully. Please show all work. This is a closed-book test: no notes or electronic devices allowed.

1. **(16pts)** Consider the function $f(x) = \cos(x/2)$ defined on the interval $0 < x < \boldsymbol{\pi}$ (note that L=π)
   a) Use the identity $2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$ to find the **cosine** series for $f(x)$ (assume $\boldsymbol{L=\pi}$)
   b) Write down the sum of first three non-zero terms in the **cosine** series.
   c) Plot the **even** periodic extension of $f(x)$ and the first terms in the cosine series, $A_0 + A_1\cos\frac{n\pi x}{L}$, on the interval −3π <x<3π (you don't have to know the answer to make a rough plot).
   d) Use your graph to check whether A_1 is positive or negative.

2. **(16pts)** Consider the following heat equation for a rod of length L=1 with constant thermal properties (assume k=1):

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) - xe^x \quad (0<x<1,\ \ t>0) \\ \frac{\partial u}{\partial x}(0,t) = 0 \\ u(1,t) = 1 \end{cases}$$

   a) Determine the **equilibrium** temperature distribution, and plot it on the interval [0, 1]
   b) Where does the energy enter, and where does it leave the rod? Explain your answers.

3. **(20pts)** Solve the following equation of a vibrating string of length $\boldsymbol{L=\pi}$. Explain each step in your solution, but you don't have to consider negative eigenvalues in the boundary value problem:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad (0<x<\boldsymbol{\pi},\ t>0) \\ \frac{\partial u}{\partial x}(0,t) = u(\boldsymbol{\pi},t) = 0 \\ u(x,0) = 0; \quad \frac{\partial u}{\partial t}(x,0) = 4\cos\frac{5x}{2} \end{cases}$$

4. **(16pts)** Consider the following Sturm-Liouville problem:

$$\begin{cases} \frac{d}{dx}\left(\sqrt{x}\,\frac{d\phi}{dx}\right) + \lambda\,\phi = 0 \ \ (0<x<1) \\ \frac{d\phi}{dx}(0) = 0 \\ \frac{d\phi}{dx}(1) + 2\phi(1) = 0 \end{cases}$$

   a) Make a rough plot of the first two eigenfunctions
   b) Derive the Rayleigh Quotient for this problem. Can one rule out λ<0?
   c) Find the value of constant p so that the test function $u_T(x)=1-px^2$ satisfies given boundary conditions. Use this test function to find an upper bound on the lowest eigenvalue

5. (12pts) Consider the following finite difference approximation of the derivative of a smooth function $u(x)$:

$$\frac{du}{dx}(x_n) \approx \frac{u_{n+1} - u_{n-1}}{2\Delta x} \quad \text{where } u_n \equiv u(x_n); \; \Delta x \equiv x_{n+1} - x_n$$

a) Derive the expression for the error of this approximation (hint: expand u_{n+1} and u_{n-1} in a Taylor series around point x_n)

b) Apply the given approximation to the special case $u(x) = x^3$, and show that the error in your result agrees with your answer to part "a".

<u>Choose one out of the remaining two problems:</u>

6. (20pts) Solve the following heat equation in a disk with rotationally-symmetric (angle-independent) boundary conditions. What is the dominant approximation of the solution for $t > 0$? Hint: Bessel equation will appear in some form after you separate variables: $z^2 f_{zz} + z\, f_z + (z^2 - m^2) f(z) = 0$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{k}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) & (0 \le r < R,\ t > 0) \\ u(R,t) = 0 \\ u(r,0) = f(r) \end{cases}$$

7. (20pts) Consider the Sturm-Liouville problem with type-III (Robin) boundary conditions at the right end:

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad (0 < x < 1) \\ \phi(0) = 0; \quad \dfrac{d\phi}{dx}(1) = 2\phi(1) \end{cases}$$

a) Use geometric methods to determine all solutions. Carefully consider *all* signs of λ.

b) Make a rough plot of the first two eigenfunctions ϕ_1 and ϕ_2, on the interval [0, 1].

c) Use your graphs (and simple trigonometry) to give an *approximate* value for the eigenvalue λ_4

d) Obtain an estimate of the ***lowest*** eigenvalue λ_1 by plugging a simple algebraic test function of your choice into the Rayleigh Quotient of this problem.