Math 332-001 Final Exam

December 18, 2008

Answer all questions in the booklet provided. Show all of your work.

1. (/16) Evaluate each of the expressions below. If the expression is multi-valued, find all values and indicate which corresponds to the principal value. Give your answers in Cartesian ($z = x + iy$) form, and simplify as much as possible.

   (a) $|\exp(2-i)|$ $\quad$ (b) i^i $\quad$ (c) $(1+i)^{1/3}$ $\quad$ (d) $\mathrm{Im}\,(\sin[(1-i)\pi/4])$

2. (/16) Verify using the Cauchy-Riemann equations that $f(z) = \cos(2z)$ is analytic, then use the Cauchy integral formula to calculate

$$\int_C \frac{f(z)}{[z-(\pi+i\pi/2)]^3}\,dz$$

   where C is the closed, positively oriented circular contour about the origin with radius 2π.

3. (/16) Show that the two power series, given by

$$\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$$

   are analytic continuations of each other by (a) identifying where each converges and (b) showing that they converge to the same function $f(z)$ in the intersection of the regions you found in part (a). Without calculating the series explicitly, identify where a Taylor series of $f(z)$ taken about $z_0 = -1 + i$ would converge.

4. (/16) For each of the functions below, identify all singularities and their types, including branch points. Give the order of any poles. If possible, calculate the residue at each singularity.

$$(a) \quad f(z) = \tfrac{z^2+2z+1}{z^2-1} \qquad (b) \quad f(z) = \frac{\csc z}{z}$$

$$(c) \quad f(z) = z e^{-1/z^2} \qquad (d) \quad f(z) = \frac{\mathrm{Log} z}{z-1}$$

5. (/16) Use the residue theorem to evaluate

$$\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 - 2x + 2}\, dx.$$

Show all steps, being sure to show explicitly what portions of the contour integral go to zero.

6. (/20) In class, we showed that

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(x^2+1)} = \pi/\sqrt{2}$$

by integrating $f(z) = \exp(-1/2 \log(z))/(z^2+1)$ around a certain closed contour with the positive real axis chosen as the branch cut for $\log z$. Verify this result by repeating the calculation two different ways:

(a) Integrate $f(z)$ along the contour formed by connecting $z = -R$ to $z = -\rho$ by a straight line, followed by a semicircular path (in the half-plane given by $\mathrm{Im}(z) \geq 0$) connecting $z = -\rho$ to $z = \rho$, followed by a straight line connecting $z = \rho$ to $z = R$, followed by a semicircular path (in the same half-plane) connecting $z = R$ to $z = -R$, taking the limit as $\rho \to 0$ and $R \to \infty$. This time, take the negative imaginary axis as the branch cut for $\log z$. Show all steps.

(b) Transform the original integral using $u = \sqrt{x}$ and solve the resulting integral using contour integration. Show all steps.

Useful identities:

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

$$\cos(iz) = \cosh z \qquad \sin(iz) = i \sinh z$$