Math 335-002 * Victor Matveev
Final examination * May 4, 2007

Please show all work to receive full credit. Notes and calculators are *not* allowed.

1. (10) Sketch the isocurves of a scalar field $f = \ln(x^2 - y)$ and indicate the direction of the gradient. Show the region in (x, y) space which is outside of the domain of f (i.e. where f is undefined).

2. (16) Use suffix notation in the following two problems:

a) Simplify $(\vec{b} \times \vec{a}) \cdot (\vec{c} \times \vec{a})$

b) Differentiate $\vec{\nabla} \times \left(\vec{\mathbf{r}} \ln r\right)$, where $\vec{\mathbf{r}}$ is the position vector, and $r = |\vec{\mathbf{r}}|$

3. (18) Verify the divergence theorem for a hemisphere ($z \geq 0$) of radius R=**2**, for a vector field $\vec{\mathbf{u}} = (0, 0, z^2)$, by calculating both the surface integral and the corresponding volume integral. Use spherical coordinates for integration (you may use Cartesian coordinates to calculate the divergence).

4. (16) Which of the following integrals is/are zero for any differentiable field **u** or f ? Apply either the divergence theorem or the Stokes theorem to each integral to answer this question.

a) $\oint_C \vec{\nabla} \times \vec{\mathbf{u}} \cdot d\vec{\mathbf{r}}$ b) $\oint_C \vec{\nabla} f \cdot d\vec{\mathbf{r}}$ c) $\oiint_S \vec{\nabla} \times \vec{\mathbf{u}} \cdot \vec{\mathbf{n}}\, dS$ d) $\oiint_S \vec{\mathbf{u}} \cdot \vec{\mathbf{n}}\, dS$

5. (10) Find the inertia tensor of a cube of side length 2L and uniform density ρ with respect to its **center** of mass. Express the result in terms of the mass of the cube. Is the tensor isotropic?

6. (20) Consider a sphere of radius R with the *charge density* increasing linearly with distance from the center as $\rho(r) = a\, r$, where a is a constant.

a) Calculate the total charge of this sphere, Q.

b) Find the electric potential Φ^{in} inside of the sphere, by solving $\nabla^2 \Phi^{in} = -\dfrac{\rho(r)}{\varepsilon_0}$

c) Calculate potential Φ^{out} outside of the sphere, by solving $\nabla^2 \Phi^{out} = 0$. Express Φ^{out} in terms of the total charge Q.

d) Fix the integration constants by matching the value of the electric field on the surface of the sphere, and use the constraint $\Phi^{out}(r) \to 0$ as $r \to \infty$.

7. Do one of the following two problems:

a) Use the line integral to calculate the work done by force $\vec{\mathbf{F}}=(x^2, y^{1/3}, z)$ along the path given by $y=e^{3x}$, $z=e^{2x}$, for x varying from 0 to 1. Is $\vec{\mathbf{F}}$ a conservative force?

b) Each picture given below either violates or illustrates one of the Maxwell's equations. Write down the equation for each picture, and indicate whether it is violated or not.

[See solutions for the pictures]

Unit vectors for cylindrical coordinates:

$\mathbf{e}_R = (1, 0, 0)_{R\,\varphi\,z} = (\cos\varphi, \sin\varphi, 0)_{x\,y\,z}$
$\mathbf{e}_\varphi = (0, 1, 0)_{R\,\varphi\,z} = (-\sin\varphi, \cos\varphi, 0)_{x\,y\,z}$
$\mathbf{e}_z = (0, 0, 1)_{R\,\varphi\,z} = (0, 0, 1)_{x\,y\,z}$

Unit vectors for spherical coordinates:

$\mathbf{e}_r = (1, 0, 0)_{r\,\theta\,\varphi} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)_{x\,y\,z}$
$\mathbf{e}_\theta = (0, 1, 0)_{r\,\theta\,\varphi} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)_{x\,y\,z}$
$\mathbf{e}_\varphi = (0, 0, 1)_{r\,\theta\,\varphi} = (-\sin\varphi, \cos\varphi, 0)_{x\,y\,z}$

Partial differentiation in curvilinear coordinates:

$$\nabla\cdot\vec{u} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3u_1)}{\partial w_1} + \frac{\partial(h_1h_3u_2)}{\partial w_2} + \frac{\partial(h_1h_2u_3)}{\partial w_3}\right]$$

$$\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial w_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial w_1}\right) + \frac{\partial}{\partial w_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial w_2}\right) + \frac{\partial}{\partial w_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial w_3}\right)\right]$$

Tensor of Inertia: $I_{ij} = \iiint_V \rho(\vec{r})\left(r^2\delta_{ij} - r_ir_j\right)dV$