Math 335-002 * Victor Matveev
Final exam * May 12, 2008

Please show all work to receive full credit. Notes and calculators are *not* allowed.

1. (a, 6pts) Sketch the isocurves of f = exp(y)/x and indicate the direction of the gradient.
(b, 8pts) Use the linear approximation to estimate f(1.1,0.2)
(c, 6pts) Sketch vector field **u**=(x+y, x-y)

2. (10pts) Use suffix notation to differentiate $\vec{\nabla}\times\left(\vec{\mathbf{a}}\times\vec{\mathbf{r}}\right)$, where $\vec{\mathbf{r}}$ is the position vector, and **a** is a constant vector.

3. (8pts) Find work done by the force **F** = ($1/(x+y^2)$, $2y/(x+y^2)+1$, $2z$) in moving an object from point (2,0,0) to (1,1,1) along a straight line, using the potential function of this field

4. (10pts) Find the volume enclosed in the first octant by the surface $x^3 + y + z = 1$.

5. (12pts) Verify the divergence theorem for the spherical cone $\{0<r<2,\ 0<\theta<\pi/4\}$, for a vector field $\vec{\mathbf{u}}$ = (x, 0, 0), by calculating both the surface integral and the corresponding volume integral. Use spherical coordinates for integration (you may use Cartesian coordinates to calculate the divergence).

6. (10pts) Use Stokes's theorem to show that $\oint_C f\vec{\nabla}g\bullet d\vec{r} = -\oint_C g\vec{\nabla}f\bullet d\vec{r}$ for any differentiable scalar fields f and g [hint: convert to double integral and use product rules].

7. (8pts) Derive the scale factors ("stretch" factors) for the spherical coordinate system.

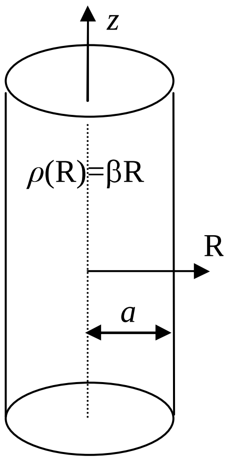
8. (10pts) Find the conductivity tensor of a material, σ_{ik}, given the following 3 measurements of current density at different values of electric field:

$\boldsymbol{j}$ = (4, 0, 0) A/m^2 when $\boldsymbol{E}$ = (2, 0, 0) V/m
$\boldsymbol{j}$ = (0, 4, 0) A/m^2 when $\boldsymbol{E}$ = (0, 2, 0) V/m
$\boldsymbol{j}$ = (2, 2, 1) A/m^2 when $\boldsymbol{E}$ = (1, 1, 1) V/m
[Note that units of σ_{ik} are (A/m^2)/(V/m)=A/(V·m)=S/m.]

Is this tensor isotropic? Re-calculate this tensor in a coordinate system rotated by $\pi/2$ around the z axis.

Do either problem 9 or problem 10:

9. (12pts) Consider the electromagnetic wave propagating in the y-direction, with the electric field polarized in the z-direction: $\boldsymbol{E}(y, t)=\{0, 0, A \cos k(y - c\,t)\}$, where A is a constant wave amplitude, k is the wave number, and c is the speed of light. Show that $\boldsymbol{E}$ satisfies the wave equation. Calculate the corresponding magnetic field $\boldsymbol{B}$ by integrating one of the Maxwell's equations. **Sketch** $\boldsymbol{E}(y)$ and $\boldsymbol{B}(y)$ at t=0.



10. (12pts) Consider a very long cable (cylinder) of radius a with a charge density inside the cable, ρ, increasing linearly with distance from the cable axis: $\rho(R)=\beta R$. Use **cylindrical** coordinates to find the electric potential Φ and electric field E both inside and outside of the cable, by solving $\nabla^2\Phi = -\rho/\varepsilon_0$ inside the cable, and $\nabla^2\Phi = 0$ outside of the cable. Assume that Φ depends only on the distance from the z-axis only: $\Phi=\Phi(R)$. Assume that E and Φ are continuous across the surface of the cable; this condition will fix the integration constants.

Unit vectors for cylindrical coordinates:

$\mathbf{e}_R = (1, 0, 0)_{R\,\varphi\,z} = (\cos\varphi, \sin\varphi, 0)_{x\,y\,z}$
$\mathbf{e}_\varphi = (0, 1, 0)_{R\,\varphi\,z} = (-\sin\varphi, \cos\varphi, 0)_{x\,y\,z}$
$\mathbf{e}_z = (0, 0, 1)_{R\,\varphi\,z} = (0, 0, 1)_{x\,y\,z}$

Unit vectors for spherical coordinates:

$\mathbf{e}_r = (1, 0, 0)_{r\,\theta\,\varphi} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)_{x\,y\,z}$
$\mathbf{e}_\theta = (0, 1, 0)_{r\,\theta\,\varphi} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)_{x\,y\,z}$
$\mathbf{e}_\varphi = (0, 0, 1)_{r\,\theta\,\varphi} = (-\sin\varphi, \cos\varphi, 0)_{x\,y\,z}$

Laplacian in cylindrical coordinates:

$$\nabla^2 f = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial f}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

Ohm's Law: $j_i = \sigma_{ik} E_k$