Math 335-002
Homework #16
Due April 9

1. Problem 6.6 on page 113.

2. Finish the problem we started in class: verify the divergence theorem for a vector field $\vec{\mathbf{u}}=(x^2, 0, 0)$, given a spherical cone $r \leq 1$, $\theta\leq\pi/6$.

3. Consider a part of the sphere $x^2 + y^2 + z^2 \leq 1$ satisfying $0<\theta<\pi/6$, $0<\varphi<\pi/2$. Sketch (roughly) this object and use spherical coordinates for the following calculations:

   a) Find the volume of this object.
   b) Verify the divergence theorem for the vector field $\vec{\mathbf{u}}=(0, 0, z^2)$ (two of the four surface integrals are zero)
   c) Find the surface area, including both the flat and the curved boundaries of this object.

Note on converting vectors between different coordinate systems

A vector should not depend on a coordinate system we choose to use, so

$$\mathbf{v} = v_x\, \mathbf{e}_x + v_y\, \mathbf{e}_y + v_z\, \mathbf{e}_z = v_1\, \mathbf{e}_1 + v_2\, \mathbf{e}_2 + v_3\, \mathbf{e}_3$$

where $\mathbf{e}_{1,\,2,\,3}$ are the unit vectors of any curvilinear orthogonal coordinate system. We may re-write the above equation in component form as

$$\mathbf{v} = (v_x,\, v_y,\, v_z)_{x\,y\,z} = (v_1,\, v_2,\, v_3)_{u1\,u2\,u3}$$

where the subscripts indicate the coordinate system of the components. The vector components in brackets are found by projecting the vector onto each of the unit vectors:

$$v_{x,\,y,\,z} = \mathbf{v} \cdot \mathbf{e}_{x,\,y,\,z} \quad \text{and} \quad v_{1,\,2,\,3} = \mathbf{v} \cdot \mathbf{e}_{1,\,2,\,3}$$

where the relationship between the curvilinear basis vectors $\mathbf{e}_{1,\,2,\,3}$ and the cartesian basis vectors $\mathbf{e}_{x,\,y,\,z}$ is given by

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u_i} \Bigg/ \left| \frac{\partial \mathbf{r}}{\partial u_i} \right| = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial u_i} = \frac{1}{h_i} \frac{\partial}{\partial u_i} (x, y, z)_{x\,y\,z}, \quad i = 1,\, 2,\, 3$$

For cylindrical coordinates, we have (see page 108)

$$\mathbf{e}_R = (1, 0, 0)_{R\,\varphi\,z} = (\cos\varphi, \sin\varphi, 0)_{x\,y\,z} \qquad v_R = \mathbf{v} \cdot \mathbf{e}_R$$
$$\mathbf{e}_\varphi = (0, 1, 0)_{R\,\varphi\,z} = (-\sin\varphi, \cos\varphi, 0)_{x\,y\,z} \qquad v_\varphi = \mathbf{v} \cdot \mathbf{e}_\varphi$$
$$\mathbf{e}_z = (0, 0, 1)_{R\,\varphi\,z} = (0, 0, 1)_{x\,y\,z} \qquad v_z = \mathbf{v} \cdot \mathbf{e}_z$$

For spherical coordinates, we have (see page 111)

$$\mathbf{e}_r = (1, 0, 0)_{r\,\theta\,\varphi} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)_{x\,y\,z} \qquad v_r = \mathbf{v} \cdot \mathbf{e}_r$$
$$\mathbf{e}_\theta = (0, 1, 0)_{r\,\theta\,\varphi} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)_{x\,y\,z} \qquad v_\theta = \mathbf{v} \cdot \mathbf{e}_\theta$$
$$\mathbf{e}_\varphi = (0, 0, 1)_{r\,\theta\,\varphi} = (-\sin\varphi, \cos\varphi, 0)_{x\,y\,z} \qquad v_\varphi = \mathbf{v} \cdot \mathbf{e}_\varphi$$