Math 335-002
Homework #17
Due April 14

1. Convert the Cartesian vector field $\vec{\mathbf{v}} = (y, 0, z^2)_{x\,y\,z}$ into the cylindrical coordinate system, $\vec{\mathbf{v}}=(v_R, v_\varphi, v_z)_{R\,\varphi\,z}$, and the spherical system, $\mathbf{v}=(v_r, v_\theta, v_\varphi)_{r\,\theta\,\varphi}$ (we did a similar problem in class; see note on the last page if in doubt)

2. Derive the expressions for the divergence in cylindrical coordinates, starting with the general expression
$$\vec{\nabla}\cdot\vec{\mathbf{v}} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(h_2 h_3 v_1)}{\partial u_1} + \frac{\partial(h_1 h_3 v_2)}{\partial u_2} + \frac{\partial(h_1 h_2 v_3)}{\partial u_3}\right]$$
where u_1=R, u_2=φ, u_3=z. Make sure to simplify whenever possible. Check your result against Eq. 6.15 on p. 109

3. Repeat for the spherical coordinate system. Check your result against Eq. (6.23) on p. 111.

4. Convert the Cartesian components of the vector field $\mathbf{v} = (x^2, 0, 0)_{x\,y\,z}$ into spherical components, $\mathbf{v} = (v_r, v_\theta, v_\varphi)_{r\,\theta\,\varphi}$. Then, find the divergence of this vector field using spherical coordinates (Eq. 6.23 on p. 111), and show that the result agrees with the simple Cartesian calculation, $\vec{\nabla}\cdot\vec{\mathbf{v}} = 2x$

5. In Cartesian coordinates, the basis vectors ($\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$) have constant direction everywhere, and therefore they have zero divergence (and curl). This is not always the case for the curvilinear coordinate basis vectors. Calculate the **divergence** of unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ in **spherical** coordinates, using the expression obtained in problem 3 (Eq. 6.23 on p. 111). Explain why two of the vectors have a non-zero divergence, using a simple sketch.

Note on converting vectors between different coordinate systems

A vector should not depend on a coordinate system we choose to use, so

$$\mathbf{v} = v_x\,\mathbf{e}_x + v_y\,\mathbf{e}_y + v_z\,\mathbf{e}_z = v_1\,\mathbf{e}_1 + v_2\,\mathbf{e}_2 + v_3\,\mathbf{e}_3$$

where $\mathbf{e}_{1,2,3}$ are the unit vectors of any curvilinear orthogonal coordinate system. We may re-write the above equation in component form as

$$\mathbf{v} = (v_x, v_y, v_z)_{x\,y\,z} = (v_1, v_2, v_3)_{u1\,u2\,u3}$$

where the subscripts indicate the coordinate system of the components. The vector components in brackets are found by projecting the vector onto each of the unit vectors:

$$v_{x,y,z} = \mathbf{v}\cdot\mathbf{e}_{x,y,z} \quad \text{and} \quad v_{1,2,3} = \mathbf{v}\cdot\mathbf{e}_{1,2,3}$$

where the relationship between the curvilinear basis vectors $\mathbf{e}_{1,2,3}$ and the cartesian basis vectors $\mathbf{e}_{x,y,z}$ is given by

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u_i}\Bigg/\left|\frac{\partial \mathbf{r}}{\partial u_i}\right| = \frac{1}{h_i}\frac{\partial \mathbf{r}}{\partial u_i} = \frac{1}{h_i}\frac{\partial}{\partial u_i}(x,y,z)_{x\,y\,z}, \quad i = 1, 2, 3$$

For cylindrical coordinates, we have (see page 108)

$$\begin{array}{ll}
\mathbf{e}_R = (1, 0, 0)_{R\,\varphi\,z} = (\cos\varphi, \sin\varphi, 0)_{x\,y\,z} & v_R = \mathbf{v}\cdot\mathbf{e}_R \\
\mathbf{e}_\varphi = (0, 1, 0)_{R\,\varphi\,z} = (-\sin\varphi, \cos\varphi, 0)_{x\,y\,z} & v_\varphi = \mathbf{v}\cdot\mathbf{e}_\varphi \\
\mathbf{e}_z = (0, 0, 1)_{R\,\varphi\,z} = (0, 0, 1)_{x\,y\,z} & v_z = \mathbf{v}\cdot\mathbf{e}_z
\end{array}$$

For spherical coordinates, we have (see page 111)

$$\begin{array}{ll}
\mathbf{e}_r = (1, 0, 0)_{r\,\theta\,\varphi} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)_{x\,y\,z} & v_r = \mathbf{v}\cdot\mathbf{e}_r \\
\mathbf{e}_\theta = (0, 1, 0)_{r\,\theta\,\varphi} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)_{x\,y\,z} & v_\theta = \mathbf{v}\cdot\mathbf{e}_\theta \\
\mathbf{e}_\varphi = (0, 0, 1)_{r\,\theta\,\varphi} = (-\sin\varphi, \cos\varphi, 0)_{x\,y\,z} & v_\varphi = \mathbf{v}\cdot\mathbf{e}_\varphi
\end{array}$$