Math 340 * Final Exam Solution * May 3, 2012 * Victor Matveev

1) **(12pts)** Use the known Taylor series of the exponential and cosine functions to find **two leading terms** in the Taylor approximation for the function $f(x)$ near x=0, and use your result to estimate $f(0.2)$:

$$f(x) = \frac{2x^3}{\exp(2x)-\cos(2x)} = \frac{2x^3}{\left(\not{1} + 2x + \frac{(2x)^2}{2} + O(x^3)\right) - \left(\not{1} - \frac{(2x)^2}{2} + O(x^4)\right)} = \frac{2x^3}{2x+4x^2+O(x^3)} =$$

$$= \frac{x^2}{1+2x+O(x^2)} = \mathbf{x^2\,(1-2x)} + O(x^4)$$ where in the last step we used geometric series: $\frac{1}{1+u} \approx 1-u$

We can now use this series to estimate $f(0.2) \approx 0.2^2(1-0.4) = 0.04*0.6 = \mathbf{0.024}$

This is fairly accurate: true value is $\doteq 0.028$

==

2) **(18pts)** Consider the function $f(x) = x^3 + x + 1$

a) Make a rough sketch of the function $f(x)$, find an interval of x containing its root, $f(x)$=0, and use two iterations of the Newton's method to approximate this root, starting with x_o=0.

b) Explain why this root may also be obtained by the iteration $x_{n+1} = x_n + c\, f(x_n)$, where ***c*** is a **constant**. Find at least one value of c for which this iteration does converge. What value of c would give the fastest convergence?

Solution:

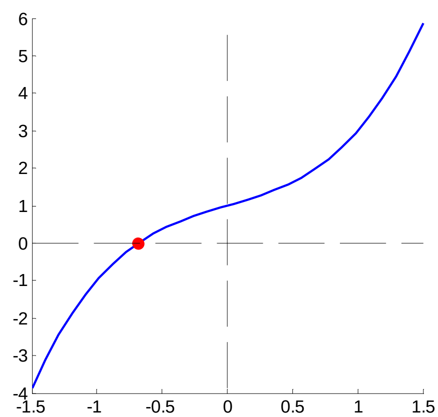
a) The root is enclosed within (–1, 0), since $f(-1) = -1 < 0$, and $f(0) = 1 > 0$

(A better/tighter interval is (–1, –1/2), since $f(-1/2) = -1/8 + 1/2 > 0$)

Two iterations of the Newton's method, starting with x_o=0:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n + 1}{3x_n^2+1}$$

$$x_1 = 0 - \frac{0+0+1}{0+1} = \mathbf{-1}; \qquad x_2 = -1 - \frac{-1-1+1}{3+1} = -1 + \frac{1}{4} = -\mathbf{\frac{3}{4}}$$



b) Consider the fixed point of this iteration, $x_{n+1} = g(x_n) = x_n + c\, f(x_n)$:

$x^* = x^* + c\, f(x^*) \;\Rightarrow\; c\, f(x^*) = 0 \;\Rightarrow\; f(x^*) = 0$: Therefore, if this iteration converges, it converges to the root x^*

Now, examine the convergence condition: $\left|g'(x^*)\right| = \left|1 + c\, f'(x^*)\right| = \left|1 + c\left(3(x^*)^2+1\right)\right| < 1$

$$-1 < 1 + c\left\{3(x^*)^2+1\right\} < 1 \;\Rightarrow\; -2 < c\left\{3(x^*)^2+1\right\} < 0 \;\Rightarrow\; -\frac{2}{3(x^*)^2+1} < c < 0 \;\Rightarrow\; \boxed{-2 < c < 0}$$

where the last inequality follows from the fact that the root x^* is within (–1, 0), and therefore $3(x^*)^2 + 1 > 1$

The simplest choice satisfying this convergence condition is $c = -1$: $x_{n+1} = g(x_n) = x_n - f(x_n) = -x_n^3 - 1$

For *fast* (better than linear) convergence we want $\left|g'(x^*)\right| = 0$, which requires $c = -\frac{1}{f'(x^*)} = -\frac{1}{3(x^*)^2+1}$

Note that this yields an iteration which is very close to the Newton's iteration!

3) **(18pts)** Find and compare the following approximations of the integral $\int_{-1}^{1} \frac{dx}{2+x}$ (it's exact value is $\ln(3) \doteq$ **1.0986**)

a) Find the midpoint approximation with 2 subintervals, M_2, and Simpson's approximation with 4 subintervals, S_4.

$n = 2,\ h = 1:$

$$M_2 = h\left[f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right)\right] = 1 \cdot \left[\frac{1}{3/2} + \frac{1}{5/2}\right] = \frac{2}{3} + \frac{2}{5} = \mathbf{\frac{16}{15}} \doteq \mathbf{1.067}$$

$n = 4,\ h = \frac{1}{2}:$

$$S_4 = \frac{h}{3}\left[f(-1) + 4f\left(-\frac{1}{2}\right) + 2f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right] = \frac{1}{6}\left[1 + 4\cdot\frac{2}{3} + \frac{2}{2} + 4\cdot\frac{2}{5} + \frac{1}{3}\right] = \frac{1}{6}\left[5 + \frac{8}{5}\right] = \mathbf{\frac{33}{30}} = \mathbf{1.1}$$

b) Find values of constants w_1, w_2 and x_1 so that the following integration rule has degree of precision of 5, and apply this integration rule to the integral given above (make sure to simplify the integration result):

$$\int_{-h}^{h} f(x)\,dx = w_1\ f(-x_1) + w_2\ f(0) + w_1 f(x_1)$$

Note that when $f(x)$ equals any odd power of x, we obtain 0=0, so we only need to check even powers:

Exact integral: Integration rule:

$f = 1: \quad \int_{-h}^{h} f(x)\,dx = \int_{-h}^{h} dx = 2h \qquad w_1\ f(-x_1) + w_2\ f(0) + w_1 f(x_1) = 2w_1 + w_2 = 2h \ \Rightarrow\ \boxed{w_2 = 2h - 2w_1 = \frac{8}{9}h}$

$f = x^2: \quad \int_{-h}^{h} f(x)\,dx = \int_{-h}^{h} x^2\,dx = \frac{2}{3}h^3 \qquad w_1\ f(-x_1) + w_2\ f(0) + w_1 f(x_1) = 2w_1 x_1^2 = \frac{2}{3}h^3$

$f = x^4: \quad \int_{-h}^{h} f(x)\,dx = \int_{-h}^{h} x^4\,dx = \frac{2}{5}h^5 \qquad w_1\ f(-x_1) + w_2\ f(0) + w_1 f(x_1) = 2w_1 x_1^4 = \frac{2}{5}h^5$

$\div \ \Rightarrow\ \boxed{x_1^2 = \frac{3h^2}{5} \Rightarrow x_1 = \sqrt{\frac{3}{5}}h, \quad w_1 = \frac{h^3}{3x_1^2} = \frac{5}{9}h}$

$$\Rightarrow \int_{-h}^{h} f(x)\,dx \approx \frac{h}{9}\left[5f\left(-\sqrt{\frac{3}{5}}h\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}h\right)\right]$$

$$\Rightarrow \int_{-1}^{1} \frac{dx}{2+x} \approx \frac{5}{9}\left[\frac{1}{2-\sqrt{3/5}} + \frac{1}{2+\sqrt{3/5}}\right] + \frac{8}{9\cdot 2} = \frac{5}{9}\left[\frac{4}{4-3/5}\right] + \frac{4}{9} = \frac{1}{9}\left[\frac{100}{17} + 4\right] = \frac{168}{9\cdot 17} = \mathbf{\frac{56}{51}} \doteq \mathbf{1.098}$$

4) (12pts) Find values of constants A, B and C so that the following finite difference approximates the second derivative of function $f(x)$ at x_o. Find also the error of this approximation. Check your answer by computing $D^{(2)}f(x_o)$ for $f(x)=x^2$:

$$D^{(2)}f(x_0) = Af(x_0-h) + Bf(x_0) + Cf(x_0+4h)$$

Expand the 1st and 3rd terms in Taylor series up to 3rd order, and sum the right-hand side:

$$A\times \quad f(x_0-h) \approx f(x_0) - h f'(x_0) + \frac{h^2}{2} f''(x_0) - \frac{h^3}{6} f'''(x_0)$$

$$\boxed{+} \quad B\times \quad f(x_0)$$

$$C\times \quad f(x+4h) \approx f(x_0) + 4h f'(x_0) + \frac{(4h)^2}{2} f''(x_0) + \frac{(4h)^3}{6} f'''(x_0)$$

$$f''(x) \quad \boxed{=} \quad \underbrace{(A+B+C)}_{=0} f(x) + \underbrace{(4C-A)}_{=0} h f'(x) + \underbrace{(A+16C)\frac{h^2}{2}}_{=1} f''(x) + \underbrace{(64C-A)\frac{h^3}{6} f'''(x)}_{\text{ERROR}}$$

$$\begin{cases} A+B+C=0 \\ A=4C \\ A+16C=20C=\dfrac{2}{h^2} \Rightarrow C=\dfrac{1}{10h^2} \end{cases} \Rightarrow \begin{cases} B=-A-C=-\dfrac{5}{10h^2} \\ A=4C=\dfrac{4}{10h^2} \\ C=\dfrac{1}{10h^2} \end{cases} \Rightarrow \boxed{f''(x_0) \approx \frac{4f(x_0-h)-5f(x_0)+f(x_0+4h)}{10h^2}}$$

The error of this numerical second derivative is given by the cubic term:

$$E \approx \underbrace{(64C-A)}_{6/h^2}\frac{h^3}{6} f'''(x_0) = \boxed{h f'''(x_0)}$$

More accurate error formula is $h f'''(c)$, where $c \in [x_0-h,\ x_0+4h]$

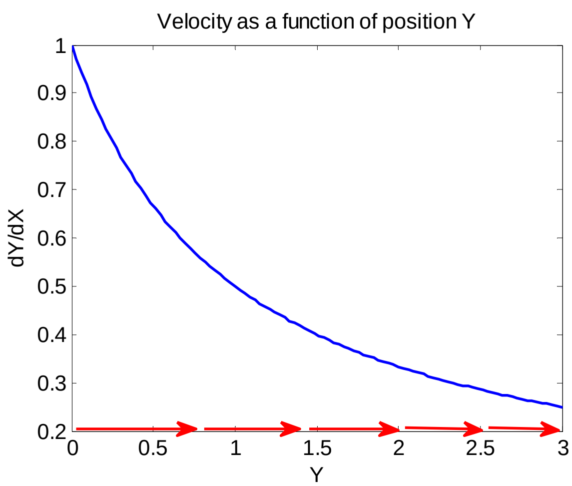
Check by applying this finite difference to $f(x)=x^2$: we obtain $f''(x_0)=2$, as we should:

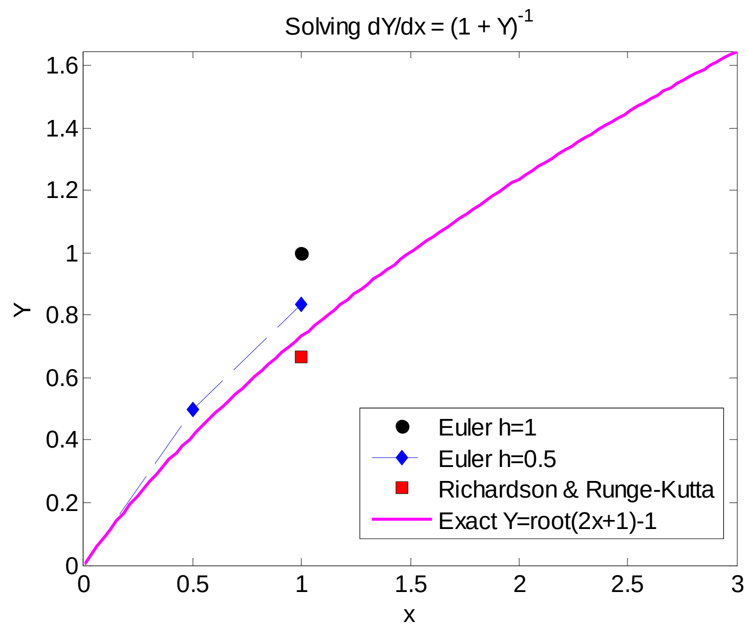
$$\frac{4f(x_0-h)-5f(x_0)+f(x_0+4h)}{10h^2} = \frac{(x_0-h)^2-5x_0^2+(x_0+4h)^2}{10h^2} = \frac{4x_0^2-8hx_0+4h^2-5x_0^2+x_0^2+8hx_0+16h^2}{10h^2} = \frac{20h^2}{10h^2} = \mathbf{2}$$

5) (18pts) Consider the following **autonomous** initial value problem: $\begin{cases} \dfrac{dY}{dx} = f(Y) = \dfrac{1}{1+Y} \\ Y(0) = 0 \end{cases}$

a) Without solving this equation, sketch the solution $Y(x)$ as a function of x (to do this, examine the graph $f(Y)$).

Since $f(Y) > 0$ when $Y > 0$, solution is monotonically increasing, but the rate of increase (velocity) is progressively smaller as Y grows, which results in downward concavity (see Figure below):





b) Euler Method $h = 0.5$: $Y(0.5) \approx \tilde{y}_1 = y_0 + h\, f(y_0) = 0 + 0.5 \cdot 1^2 = \mathbf{0.5}$

$$Y(1) \approx \tilde{y}_2 = \tilde{y}_1 + h\, f(\tilde{y}_1) = 0.5 + 0.5 \cdot \frac{1}{1+0.5} = \frac{1}{2} + \frac{1}{3} = \mathbf{\frac{5}{6}} \doteq 0.833$$

c) Euler Method $h = 0.1$: $Y(1) \approx y_1 = y_0 + h\, f(y_0) = 0 + 1 \cdot 1 = \mathbf{1}$

Richardson's Extrapolation: $Y(1) \approx 2\tilde{y}_2 - y_1 = \dfrac{10}{6} - 1 = \dfrac{4}{6} = \mathbf{\dfrac{2}{3}} \doteq 0.666$

d) One step of the midpoint Runge-Kutta method with $h = 1$:

$$Y(1) \approx y_1 = y_0 + h\, f\big(\, y_0 + 0.5h\, f(y_0)\big) = 0 + 1 \cdot \frac{1}{1 + 0 + 1 \cdot 0.5 \dfrac{1}{0+1}} = \mathbf{\frac{2}{3}} \doteq 0.666$$

(Same as Richardson's extrapolation of Euler results)

6) (12pts) Find the value of constant α for which the following method for solving $dY/dx = f(x, Y)$ has a local error of order h^3 (to do this, compare its Taylor expansion up to 2nd order in h with the Taylor expansion of exact solution):

$$y_{n+1} = y_n + \frac{h}{3}\left[f(x_n, y_n) + 2 f\left(x_n + \alpha h,\ y_n + \alpha h\, f(x_n, y_n)\right)\right]$$

$$= y_n + \frac{h}{3}\left[f(x_n\ y_n) + 2\left(f(x_n, y_n) + \alpha h \frac{\partial f}{\partial x}(x_n\ y_n) + \alpha h\, f(x_n, y_n)\frac{\partial f}{\partial Y}(x_n, y_n) + O(h^2)\right)\right]$$

$$= y_n + h\, f(x_n, y_n) + \frac{2\alpha h^2}{3}\left[\frac{\partial f}{\partial x}(x_n, y_n) + f(x_n, y_n)\frac{\partial f}{\partial Y}(x_n, y_n)\right] + O(h^3)$$

Compare this with the Taylor expansion of exact solution:

$$Y_{n+1} = Y_n + hY_n' + \frac{h^2}{2}Y_n'' + O(h^3) = Y_n + h\, f(x_n, Y_n) + \frac{h^2}{2}\frac{df}{dx}(x_n, Y_n) + O(h^3)$$

Now, use the chain rule to find the "full" x-derivative of $f(x_n, Y(x_n))$: $\frac{df}{dx}(x, Y(x)) = \frac{\partial f}{\partial x}(x, Y) + \frac{\partial f}{\partial Y}(x, Y)\underbrace{\frac{dY(x)}{dx}}_{f(x,Y)}$

$$\Rightarrow\ Y_{n+1} = Y_n + h\, f(x_n, Y_n) + \frac{h^2}{2}\left[\frac{\partial f}{\partial x}(x_n, Y_n) + \frac{\partial f}{\partial Y}(x_n, Y_n)\, f(x_n, Y_n)\right] + O(h^3)$$

The equations for y_n and Y_n agree up to terms of order h^3 if $\frac{2\alpha h^2}{3} = \frac{h^2}{2} \Rightarrow \boxed{\alpha = \frac{3}{4}}$

Thus, the 2nd order accurate method is:

$$\boxed{y_{n+1} = y_n + \frac{h}{3}\left[f(x_n, y_n) + 2 f\left(x_n + \frac{3}{4}h,\ y_n + \frac{3}{4}h\, f(x_n, y_n)\right)\right]}$$

7) (10pts) Find values of ***a*** and ***b*** that minimize the sum of squares of residuals (deviations) between the curve ***y(x)=a +b / x*** and the data poins (1/3, 2); (1/2, 0); (1, 1). Sketch the data points and the best-fit curve $y(x)$.

Find the sum of squares of residuals:

$$G(a,b) = \sum_{i=1}^{3}(y(x_i) - y_i)^2 = (a+3b-2)^2 + (a+2b)^2 + (a+b-1)^2$$

2) Minimize the sum of squares of residuals:

$$\begin{cases} \frac{1}{2}\frac{\partial G}{\partial a} = (a+3b-2) + (a+2b) + (a+b-1) = 3a+6b-3=0 \ \| \times 2 \\ \frac{1}{2}\frac{\partial G}{\partial b} = 3(a+3b-2)+2(a+2b)+(a+b-1) = 6a+14b-7=0 \end{cases} \Rightarrow \begin{cases} 6a+12b=6 \\ 6a+14b=7 \end{cases} \Rightarrow \boxed{\begin{array}{c} b=\frac{1}{2} \\ a=0 \end{array}}$$

Thus, the curve of given form that fits data the best is $\boxed{y(x) = \frac{1}{2x}}$