Math 340 * Exam 1 * Solution
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Problem 1a) $\dfrac{\exp(2x^2)-(1+6x^2)^{1/3}}{x^4}$

In double precision, for very small x (see below) this expression will round off to $\dfrac{1-1}{x^4}=0$

Let's use Taylor polynomials to modify, recalling that $(1+u)^p \approx 1+pu+\dfrac{p(p-1)}{2}u^2+\ldots$

$$\left.\begin{array}{l} \bullet\ \exp(2x^2)\approx 1+2x^2+\dfrac{(2x^2)^2}{2}=1+2x^2+2x^4 \\ \bullet\ (1+6x^2)^{1/3}\approx 1+\dfrac{1}{3}6x^2-\dfrac{1}{2}\dfrac{1}{3}\dfrac{2}{3}\left(6x^2\right)^2=1+2x^2-4x^4 \end{array}\right\}$$

$$\frac{\exp(2x^2)-(1+6x^2)^{1/3}}{x^4}\approx\frac{\left(1+2x^2+2x^4\right)-\left(1+2x^2-4x^4\right)}{x^4}=\frac{6x^4}{x^4}=\boxed{6}\ \text{Not zero!}$$

Round-off in the numerator will be a problem when $4x^4<10^{-16}\Rightarrow$ problematic range is $\boxed{|x|<10^{-4}}$

Problem 1b) $\dfrac{x^2}{\sqrt{x^2+1}-\sqrt[3]{x^3+1}}$

Loss of significance will result for both very large *and* very small x, yielding a value of infinity:

• **Small** x: denominator rounded off to zero if $x^3<10^{-16}$ *and* $x^2<10^{-16}\Rightarrow$ problematic range is $\boxed{|x|<10^{-8}}$

Use $(1+x)^p\approx 1+px$, and $\dfrac{1}{1-u}\approx 1+u$

$$\frac{x^2}{\sqrt{x^2+1}-\sqrt[3]{x^3+1}}\approx\frac{x^2}{\left(1+\dfrac{x^2}{2}\right)-\left(1+\dfrac{x^3}{3}\right)}=\frac{x^2}{\dfrac{x^2}{2}-\dfrac{x^3}{3}}=2\frac{1}{1-\dfrac{2}{3}x}\approx 2\left(1+\frac{2}{3}x\right)=\boxed{2+\frac{4}{3}x}\ \text{Not unbounded!}$$

• **Large** x: denominator rounded off to zero if $x^3>10^{+16}$ *and* $x^2>10^{+16}\Rightarrow$ problematic range is $\boxed{|x|>10^{+8}}$

Use the same linear approximation $\left(1+\dfrac{1}{u}\right)^p\approx 1+\dfrac{p}{u}$

$$\frac{x^2}{\sqrt{x^2+1}-\sqrt[3]{x^3+1}}=\frac{x^2}{x\sqrt{1+\dfrac{1}{x^2}}-x\sqrt[3]{1+\dfrac{1}{x^3}}}\approx\frac{x}{\left(1+\dfrac{1}{2x^2}\right)-\left(1+\dfrac{1}{3x^3}\right)}\times\frac{2x^2}{2x^2}=\frac{2x^3}{1-\dfrac{2}{3x}}\approx\boxed{2x^3\left(1+\frac{2}{3x}\right)}$$

Problem 2(a):

Apply one iterations of the Newton's method to an appropriately chosen function to find x_1, starting with initial guess x_o=2. When estimating the error, recall that $\varepsilon_{n+1} = \frac{f''(c_n)}{2f'(x_n)}\varepsilon_n^2$, and use $\alpha \approx x_1$

$\sqrt[3]{5}$ is a root of equation $f(x) = x^3 - 5 = 0$, so

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 5}{3x_n^2} = x_n - \frac{x_n}{3} + \frac{5}{3x_n^2} = \frac{2x_n + 5/x_n^2}{3}$$

$$\begin{cases} x_0 = 2 \\ x_1 = \frac{4+5/4}{3} = \frac{21}{12} = \frac{7}{4} = \boxed{1.75} \end{cases}$$

Error bound: $\varepsilon_1 = \frac{f''(c_0)}{2f'(x_0)}\varepsilon_0^2 \approx \frac{6c_0}{6x_0^2}(x_0 - x_1)^2 \le \frac{2}{x_0^2}(1.75-2)^2 = \frac{0.25^2}{2} = \frac{1}{32} = 0.03125$

Problem 2(b):

Noting that $\sqrt[3]{5} = \sqrt[3]{8-3} = 2\sqrt[3]{1-3/8}$, use the linearization of the function $\sqrt[3]{1-x}$ to estimate $\sqrt[3]{5}$, and use the Taylor remainder formula to find the upper bound for the error of this linear approximation.

$$(1-x)^p = 1 - px + \underbrace{\frac{p(p-1)}{2}(1-x^*)^{p-2}x^2}_{R_1(x)}$$

$$\sqrt[3]{1-3/8} \approx 1 - \frac{1}{3}\frac{3}{8} = 1 - \frac{1}{8} = \frac{7}{8} \quad \Rightarrow \quad \sqrt[3]{5} = 2\sqrt[3]{1-3/8} \approx \frac{7}{4} = \boxed{1.75}$$ Same as using Newton's method!

$$R_1(x) = \frac{p(p-1)}{2}(1-x^*)^{p-2}x^2$$
$$= -\frac{1}{2}\frac{1}{3}\frac{2}{3}\underbrace{(1-x^*)^{-5/3}}_{\le(1-3/8)^{-5/3}}x^2 \le -\frac{1}{9}\left(\frac{8}{5}\right)^{5/3}\left(\frac{3}{8}\right)^2 = \frac{1}{5\cdot 8}\left(\frac{8}{5}\right)^{2/3} = \frac{1}{40}\left(1+\frac{3}{5}\right)^{2/3} \approx \frac{1}{40}\left(1+\frac{3}{5}\frac{2}{3}\right) \approx \boxed{\frac{1}{29}}$$

Multiplying by two, we find that the error bound on the approximation is about $\frac{1}{14} \doteq 0.07$

Problem 3(a): $x_{n+1} = g(x_n) = 1 - \dfrac{x_n^3}{3}$

The fixed point is bracketed by $0 < x^* < 1$, since $g(0) = 1 > 0$ but $g(1) = \dfrac{2}{3} < 1$ (see figure below)

The fixed point is stable since $g'(x^*) = -(x^*)^2 \;\Rightarrow\; |g'(x^*)| = (x^*)^2 < 1$ since $x^* < 1$

Convergence is linear and non-monotonic (oscillatory) since $g'(x^*) < 0$

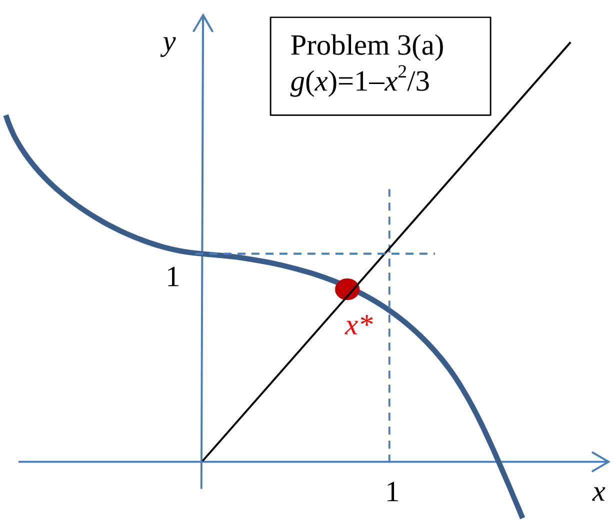
Problem 3(b): $x_{n+1} = g(x_n) = \dfrac{2}{3}x_n + \dfrac{1}{3x_n^2}$

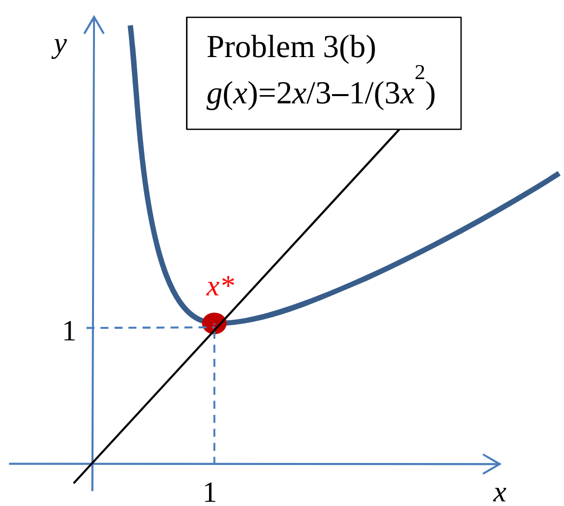
Fixed point is easily found algebraically: $x^* = g(x^*) = \dfrac{2}{3}x^* + \dfrac{1}{3(x^*)^2} \;\Rightarrow\; \dfrac{x^*}{3} = \dfrac{1}{3(x^*)^2} \;\Rightarrow\; (x^*)^3 = 1 \;\Rightarrow\; \boxed{x^* = 1}$

Let's examine stability:

$$\left.\begin{aligned} g'(x^*) &= g'(1) = \frac{2}{3} - \frac{2}{3(x^*)^3} = 0 \\ g''(x^*) &= \frac{2}{(x^*)^4} = 2 > 0 \end{aligned}\right\}\ \text{Convergence is quadratic and monotonic}$$

Monotonicity follows from $g''(x^*) > 0$, since Taylor expansion near fixed point gives $\varepsilon_{n+1} \approx \underbrace{g'(x^*)}_{=0}\varepsilon_n + \dfrac{g''(x^*)}{2}\varepsilon_n^2 > 0$





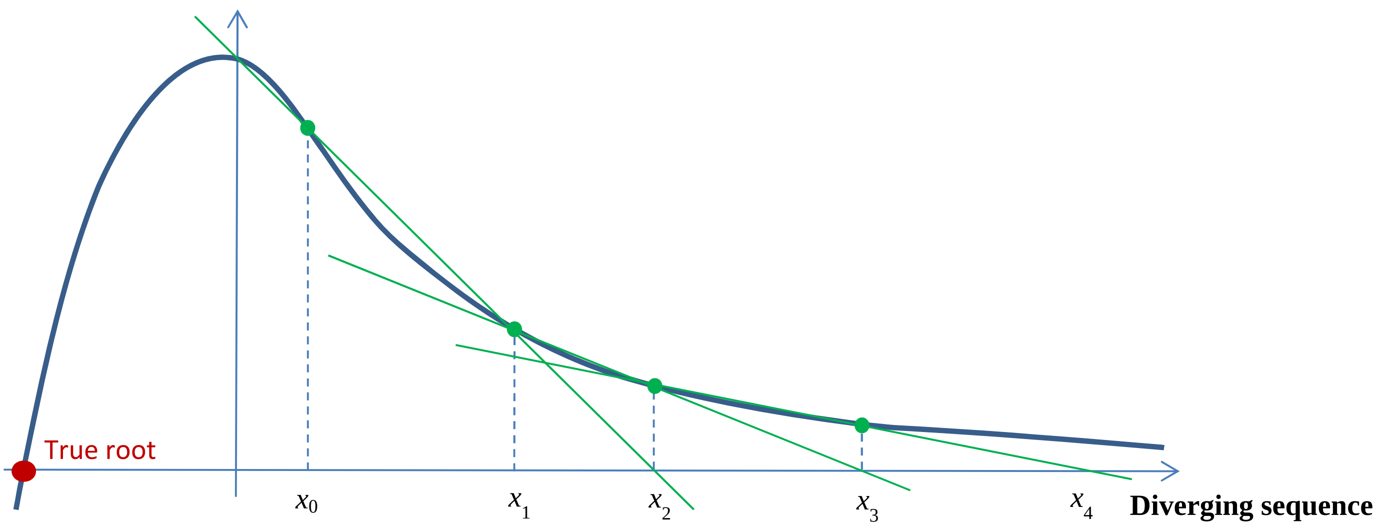
Problem 4: Equation of line connecting two points on the graph $y=f(x)$ corresponding to argument values x_o and x_1:

$$y - y_1 = m(x - x_1) \quad \text{where} \quad m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Denote the x-intercept of this line as $(x_2, \; y_2 = 0)$:

$$0 - y_1 = m(x_2 - x_1) \;\Rightarrow\; x_2 - x_1 = -\frac{y_1}{m} \;\Rightarrow\; x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)} \;\Rightarrow\; \boxed{x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}}$$

Secant method **does not** converge unconditionally; below is just one example of poor initial point choice:



Problem 5:

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F = function myFun(x)
x2 = x*x;
F = 1;
for k = 8 : -2 : 2
    F = 1 - x2 * F / (k * (k-1)) ;
end
```

After first loop iteration $(\mathrm{k}=8)$: $\quad F = 1 - \dfrac{x^2}{8\cdot 7}$

After second loop iteration $(\mathrm{k}=6)$: $\quad F = 1 - \dfrac{x^2}{6\cdot 5}\left(1 - \dfrac{x^2}{8\cdot 7}\right)$

After third loop iteration $(\mathrm{k}=4)$: $\quad F = 1 - \dfrac{x^2}{4\cdot 3}\left(1 - \dfrac{x^2}{6\cdot 5}\left(1 - \dfrac{x^2}{8\cdot 7}\right)\right)$

After last loop iteration $(\mathrm{k}=2)$: $\quad F = 1 - \dfrac{x^2}{2\cdot 1}\left(1 - \dfrac{x^2}{4\cdot 3}\left(1 - \dfrac{x^2}{6\cdot 5}\left(1 - \dfrac{x^2}{8\cdot 7}\right)\right)\right) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!}$

This is an 8-th order Taylor polynomial approximation for cos(x)