Math 340 * Exam 2 * Solution * April 5, 2012 * Victor Matveev

Problem 1 Compute the following approximations of $\int_{-1}^{1} x^4 dx$

(**a**) Simpson's approximations S_2 and S_4 : compare with exact value $= \frac{2}{5} = 0.4$

$n=2:\ h=1:$

$$\mathbf{S_2} = \frac{h}{3}\left[f(-1)+4f(0)+f(1)\right] = \frac{1}{3}[1+0+1] = \frac{2}{3} \doteq 0.6666 \quad (\textbf{Error} \doteq 0.27)$$

$n=4:\ h=\frac{1}{2}:$

$$\mathbf{S_4} = \frac{h}{3}\left[f(-1)+4f\left(-\frac{1}{2}\right)+2f(0)+4f\left(\frac{1}{2}\right)+f(1)\right] = \frac{1}{6}\left[1+\frac{1}{4}+0+\frac{1}{4}+1\right] = \frac{5}{12} \doteq 0.4166 \quad (\textbf{Error} \doteq 0.017)$$

(**b**) Correction using asymptotic error formula:

$$\mathbf{CS}_2 = S_2 - \frac{h^4}{180}\left[f'''(1)-f'''(-1)\right] = \frac{2}{3}-\frac{48}{180} = \frac{2}{3}-\frac{12}{45} = \frac{18}{45} = \frac{2}{5} = 0.4 \quad (\textbf{Exact!})$$

(**c**) Correction using Richardson's extrapolation:

$$\mathbf{RS}_4 = \frac{2^4 S_4 - S_2}{2^4-1} = \frac{16S_4 - S_2}{15} = \frac{\frac{16\cdot 5}{12}-\frac{2}{3}}{15} = \frac{\frac{20}{3}-\frac{2}{3}}{15} = \frac{18}{3\cdot 15} = \frac{2}{5} = 0.4 \quad (\textbf{Exact!})$$

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Problem 2

$$\int_0^h \frac{f(x)}{\sqrt{x}}dx \approx w_1 f(0) + w_2 f(x_2) = \boxed{\frac{\sqrt{h}}{9}\left(8f(0)+10f\left(\frac{3h}{5}\right)\right)}$$ Derivation is below:

Exact integral: | Integration rule:

$f=1:\quad \int_0^h \frac{f(x)}{\sqrt{x}}dx = \int_0^h x^{-1/2}dx = 2h^{1/2} \qquad w_1 f(0)+w_2 f(x_2) = w_1 + w_2 = 2h^{1/2} \Rightarrow \boxed{w_1 = 2\sqrt{h} - w_2 = \frac{8}{9}\sqrt{h}}$

$f=x:\quad \int_0^h \frac{f(x)}{\sqrt{x}}dx = \int_0^h x^{1/2}dx = \frac{2}{3}h^{3/2} \qquad w_1 f(0)+w_2 f(x_2) = w_2 x_2 = \frac{2}{3}h^{3/2}$

$f=x^2:\quad \int_0^h \frac{f(x)}{\sqrt{x}}dx = \int_0^h x^{3/2}dx = \frac{2}{5}h^{5/2} \qquad w_1 f(0)+w_2 f(x_2) = w_2 x_2^2 = \frac{2}{5}h^{5/2}$

$\div \Rightarrow \boxed{x_2 = \frac{3h}{5}, \quad w_2 = \frac{2h^{3/2}}{3x_2} = \frac{10}{9}\sqrt{h}}$

$$\Rightarrow \int_0^{5\pi/9} \frac{\cos(x)}{\sqrt{x}}dx = \frac{\sqrt{5\pi/9}}{9}\left(8+10\cos\left(\frac{3}{5}\cdot\frac{5\pi}{9}\right)\right) = \frac{\sqrt{5\pi}}{27}\left(8+10\cos\left(\frac{\pi}{3}\right)\right) = \boxed{\frac{13\sqrt{5\pi}}{27}}$$

Problem 3 Examine $Df(x_0) = \dfrac{f(x_0 - 3h) - 4f(x_0) + 3f(x_0 + h)}{6h^2}$

Expand the 1st and 3rd terms in Taylor series up to 3rd order, and sum the numerator:

$$\begin{array}{rl} 1\times & f(x_0 - 3h) \approx f(x_0) - 3h\, f'(x_0) + \dfrac{(3h)^2}{2} f''(x_0) - \dfrac{(3h)^3}{6} f'''(x_0) \\ + \quad -4\times & f(x_0) \\ 3\times & f(x_0 + h) \approx f(x_0) + h\, f'(x_0) + \dfrac{h^2}{2} f''(x_0) + \dfrac{h^3}{6} f'''(x_0) \end{array}$$

$$= \underbrace{(1-4+3)}_{=0} f(x_0) + \underbrace{(-3+3)}_{=0} h\, f'(x_0) + \underbrace{(9+3)\frac{h^2}{2}}_{=6h^2} f''(x_0) + \underbrace{(-27+3)\frac{h^3}{6} f'''(x_0)}_{-4h^3 f'''(x_0)}$$

Divide by $6h^2$: $\boxed{Df(x_0) \approx \underbrace{f''(x_0)}_{\text{Second Derivative}} \underbrace{- \frac{2}{3} h\, f'''(x_0)}_{\text{ERROR}}}$ More accurate error: $-\dfrac{2h}{3} f'''(c)$, where $c \in [x - 3h, x + h]$

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Problem 4

(a) Find interpolating polynomial to $f(x) = \dfrac{1}{1+x}$ using Newton's divided differences:

$$f(x_0) = \frac{1}{1 - 1/2} = 2, \quad f(x_1) = 1, \quad f(x_2) = \frac{1}{1 + 1/2} = \frac{2}{3}$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1-2}{1/2} = -2$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{2/3 - 1}{1/2} = -\frac{2}{3}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{-2/3 + 2}{1} = \boxed{\frac{4}{3}}$$

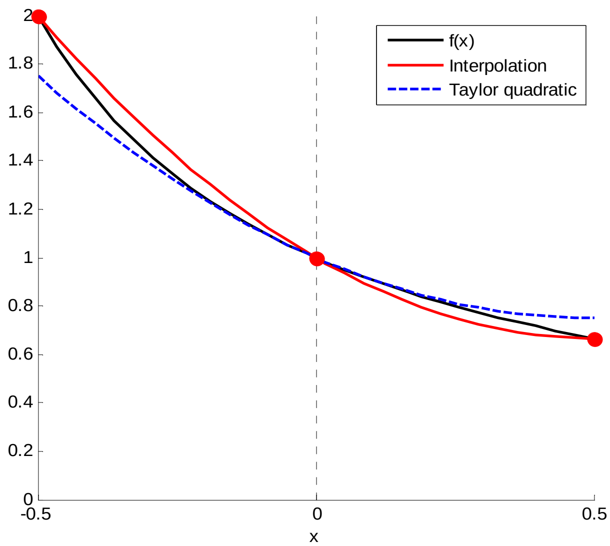
$$P_2(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$= 2 - 2\left(x + \frac{1}{2}\right) + \frac{4}{3}x\left(x + \frac{1}{2}\right) = 1 - 2x + \frac{4}{3}x^2 + \frac{2}{3}x = \boxed{1 - \frac{4}{3}x + \frac{4}{3}x^2 = 1 + \frac{4}{3}x(x-1)}$$ See Figure on next page

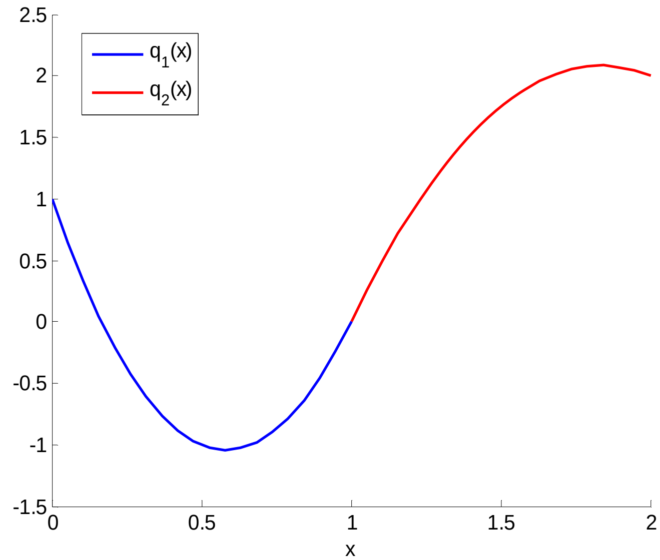
(**b**) Quadratic Taylor approximation (geometric series): $\dfrac{1}{1+x} \approx \boxed{1 - x + x^2}$ Coefficients equal 1 as compared to $\dfrac{4}{3}$ in (a)

(**c**) The interpolating polynomial is more accurate than the Taylor polynomial at $x = 0.47$, since it yields exact values of $f(x)$ at the nodes, while Taylor polynomial is strictly a local approximation, which works best in the neighborhood of the expansion point (in this case, close to $x_1 = 0$). This is clear from the plot:

Problem 4: Interpolating quadratic polynomial and Taylor polynomial for *f*(*x*)=1/(1+x)



Problem 5: Piece-wise quadratic spline



Problem 5 Find and sketch the piece-wise **quadratic** spline $q(x)$ on the interval [0, 2] that satisfies the conditions:

$$q(0)=1,\ q(1)=0,\ q(2)=2,\ \frac{dq}{dx}(2)=-1$$

$$q(x)=\begin{cases}0\le x\le 1:\ q_1(x)=ax^2+bx+c\\ 1\le x\le 2:\ q_2(x)=A(x-2)^2+B(x-2)+C\end{cases}$$

We have 6 unknowns for 6 constraints:

$$\begin{cases}q_1(0)=1\\ q_1(1)=0\\ q_1'(1)=q_2'(1)\\ q_2(1)=0\\ q_2(2)=2\\ q_2'(2)=-1\end{cases}\Rightarrow\begin{cases}c=1\\ a+b+1=0\\ 2a+b=-2A+B\\ A-B+C=0\\ C=2\\ B=-1\end{cases}\Rightarrow\begin{cases}c=1\\ a+b=-1\\ 2a+b=5\\ A=B-C=-3\\ C=2\\ B=-1\end{cases}\Rightarrow\begin{cases}c=1\\ a=6\\ b=-7\\ A=-3\\ C=2\\ B=-1\end{cases}$$

$$\boxed{q(x)=\begin{cases}0\le x\le 1:\ q_1(x)=6x^2-7x+1\\ 1\le x\le 2:\ q_2(x)=-3(x-2)^2-(x-2)+2\end{cases}}$$

Problem 5' Since $h_{2n}=\dfrac{b-a}{2n}$ and $h_{3n}=\dfrac{b-a}{3n}$, we have $h_{3n}=\dfrac{2}{3}h_{2n}$. Use asymptotic error expression (see problem 1):

$$\left.\begin{aligned}&I\approx S_{2n}-c\,h_{2n}^4\\ &I\approx S_{3n}-c\,h_{3n}^4=S_{3n}-c\left(\frac{2}{3}h_{2n}\right)^4\ \Big\|\times\left(\frac{3}{2}\right)^4\end{aligned}\right\}\boxminus\Rightarrow\left[\left(\frac{3}{2}\right)^4-1\right]I\approx\left(\frac{3}{2}\right)^4S_{3n}-S_{2n}\Rightarrow\boxed{I\approx\frac{\left(\frac{3}{2}\right)^4S_{3n}-S_{2n}}{\left(\frac{3}{2}\right)^4-1}=\frac{3^4\,S_{3n}-2^4\,S_{2n}}{3^4-2^4}}$$