Math 340 * Final Exam * May 3, 2012 * Victor Matveev
Please read questions carefully, and show all work for each problem.

1) **(12pts)** Use the known Taylor series of the exponential and cosine functions to find **two leading terms** in the Taylor approximation for the function $f(x)$ near x=0, and use your result to estimate $f(0.2)$:

$$f(x) = \frac{2x^3}{\exp(2x) - \cos(2x)}$$

2) **(18pts)** Consider the function $f(x) = x^3 + x + 1$

a) Make a rough sketch of the function $f(x)$, find an interval of x containing its root, $f(x)$=0, and use two iterations of the Newton's method to approximate this root, starting with x_o=0.

b) Explain why this root may also be obtained by the iteration $x_{n+1} = x_n + c\, f(x_n)$, where $\boldsymbol{c}$ is a **constant**. Find at least one value of c for which this iteration does converge. What value of c would give the fastest convergence?

3) **(18pts)** Find and compare the following approximations of the integral $\int_{-1}^{1} \frac{dx}{2+x}$ (it's exact value is $\ln(3) \doteq 1.0986$)

a) Find the midpoint approximation with 2 subintervals, M_2, and Simpson's approximation with 4 subintervals, S_4.

b) Find values of constants w_1, w_2 and x_1 so that the following integration rule has degree of precision of 5, and apply this integration rule to the integral given above (make sure to simplify the integration result):

$$\int_{-h}^{h} f(x)\,dx = w_1\, f(-x_1) + w_2\, f(0) + w_1\, f(x_1)$$

4) **(12pts)** Find values of constants A, B and C so that the following finite difference approximates the second derivative of function $f(x)$ at x_o. Find also the error of this approximation. Check your answer by computing $D^{(2)}f(x_o)$ for $f(x)=x^2$:

$$D^{(2)} f(x_0) = Af(x_0 - h) + Bf(x_0) + Cf(x_0 + 4h)$$

5) **(18pts)** Consider the following **autonomous** initial value problem: $\begin{cases} \dfrac{dY}{dx} = f(Y) = \dfrac{1}{1+Y} \\ Y(0) = 0 \end{cases}$

a) Without solving this equation, sketch the solution $Y(x)$ as a function of x (to do this, examine the graph $f(Y)$).

b) Estimate $Y(0.5)$ and $Y(1)$ using Euler method with h=0.5 (check that these agree with your $Y(x)$ sketch in (a)).

c) Estimate $Y(1)$ using Euler method with step h=1, and use this value along with your result in (b) to find the Richardson's extrapolation for $Y(1)$.

d) Approximate $Y(1)$ using the midpoint Runge-Kutta method with step h=1: $y_{n+1} = y_n + h\, f\left(y_n + 0.5h\, f(y_n)\right)$

6) **(12pts)** Find the value of constant α for which the following method for solving $dY/dx = f(x, Y)$ has a local error of order h^3 (to do this, compare its Taylor expansion up to 2nd order in h with the Taylor expansion of exact solution):

$$y_{n+1} = y_n + \frac{h}{3}\left[f(x_n, y_n) + 2f\left(x_n + \alpha h,\ y_n + \alpha h\, f(x_n, y_n)\right)\right]$$

7) **(10pts)** Find values of $\boldsymbol{a}$ and $\boldsymbol{b}$ that minimize the sum of squares of residuals (deviations) between the curve $\boldsymbol{y(x)=a + b / x}$ and the data poins (1/3, 2); (1/2, 0); (1, 1). Sketch the data points and the best-fit curve $y(x)$.