Fall 2015 * Math 430 / 635 * Final Examination * December 17, 2015 * Victor Matveev

1. **(10pts)** Make a *rough* sketch of the *equilibrium* $I_{Na}(V)$ and $I_K(V)$ curves for the Hodgkin-Huxley model. Recall that at equilibrium $I_{Na}(V) = g_{Na}\, m_\infty^3(V)\, h_\infty(V)\,(V - V_{Na})$ and $I_K(V) = g_K n_\infty^4(V)(V - V_K)$. Start by making a sketch of the equilibrium activation functions $m_\infty(V),\ h_\infty(V)$ and $n_\infty(V)$. *V*-scale doesn't have to be accurate.

2. **(10pts)** Recall that in the Morris-Lecar model of excitable cell, the activation of I_{Ca} is instantaneous, while the activation of I_K is much slower. Consider reversing the time scales of the two activation gates, making I_K activation instantaneous, while making I_{Ca} activation time-dependent, but without changing other parameters, and keeping the same gate activation functions $m_\infty(V)$ and $n_\infty(V)$. The modified model would look like this:

$$\begin{cases} CV' = -g_L(V - V_L) - g_{Ca} m(t)(V - V_{Ca}) - g_K n_\infty(V)(V - V_K) \\ m' = (m_\infty(V) - m)/\tau_m(V) \end{cases}$$

   Answer the following questions:

   a) Would this modified system have the same equilibria as the original Morris-Lecar model? Briefly explain.

   b) Could such a model generate an action potential? Why or why not? Explain in no more than 1-2 sentences (hint: what happens if the potential is increased above the equilibrium?)

3. **(16pts)** Consider a passive cell receiving an "up-ramp" applied current $I_{app}(t) = \gamma t$, where γ = const > 0. Calculate the solution V(t) as a function of parameters $\gamma,\ C,\ R$ and V_R, assuming $V(0) = V_R$

4. **(16pts)** Consider the Fitzhugh-Nagumo model with the applied current parameter I :

$$\begin{cases} V' = V - \dfrac{V^3}{3} - w + I \\ w' = V - \dfrac{3w}{4} \end{cases}$$

   a) For the case $I = 0$, make a rough sketch of the nullclines and the flow field in the (V, w) phase-plane

   b) Express the trace and determinant of the Jacobian *DF* as a function of equilibrium value V^* (don't compute V^*)

   c) Recall that at the Hopf bifurcation the eigenvalues of *DF* satisfy $\lambda_{1,2} = \pm i\omega$, and that any matrix satisfies $\mathrm{trace}(DF) = \lambda_1 + \lambda_2$ and $\det(DF) = \lambda_1\lambda_2$. Use this knowledge to find value(s) of V^* at the Hopf bifurcations.

5. **(16pts: MATLAB)** Consider a quadratic integrate-and-fire model given by:

$$\begin{cases} \dfrac{dV}{dt} = V^2 + I_{app}(t) \\ V(t_k^-) = V_{spike} \Rightarrow V(t_k^+) = V_{reset} \end{cases}$$

   where $I_{app}(t)$ is a given applied current function. Write down a program that uses the Euler's method to produce an array "Tspike" of spike-times (denoted t_k above), as a MATLAB function with the following header:

   function Tspike = QuadIandF(Iapp, Vreset, Vspike, V0, Ttotal, h)

   Assume that the first argument **Iapp** is a function, not a constant (use it as a function of time in your Euler's method). Other arguments include the model parameters **Vreset** and **Vspike**, the initial condition **V0**, the total integration time **Ttotal**, and the Euler time step **h**. The output **Tspike** is an array of spike times produced within the [0, Ttotal] time interval (the size of this output array is not known *a priori*).

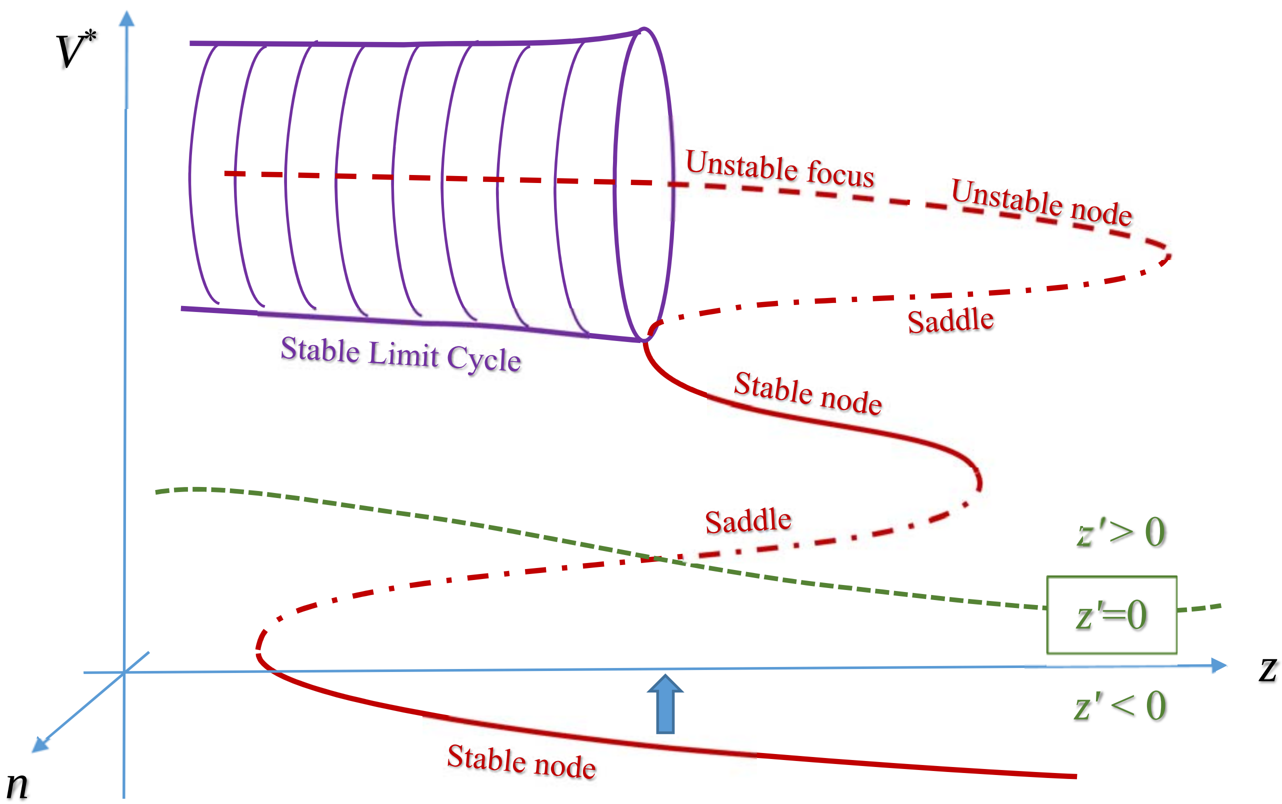
6. **(16pts)** Consider a parameter-dependent 1D differential equation $\frac{dy}{dt} = f\left(y;\, I\right) = \sin y - I\, y$. Make a rough sketch of the bifurcation diagram with respect to parameter I, and categorize all bifurcations. Hint: analyze the equation $f\left(y;\, I\right) = 0$ graphically, by plotting separately the two terms, and start by considering large values of I. There is an infinite set of bifurcations in this problem.

7. **(16pts)** Consider a 3D (V, n, z) model of excitable cell with strong time-scale separation, so that z(*t*) is slow and the 2D subsystem (V, n) is fast. The dynamics of the fast subsystem at different fixed values of *z* is summarized by a bifurcation diagram in the figure below.

a) Classify all bifurcations of the fast subsystem

b) Make a rough sketch of the 2D flow in the fast subsystem (V, n) for a value of *z* corresponding to one of the bifurcations, indicated with an arrow along the z-axis

c) Sketch the trace V(t) for the bursting solution in the full 3D system



If you encounter difficulties with any of the problems above, you can solve the following problem instead, but note that it is worth 10 points only:

8. **(10pts)** Consider the non-diagonalizable linear system
$$\begin{cases} x' = -x + y \\ y' = -y \\ x(0) = -1; \;\; y(0) = 1 \end{cases}$$

a) Find and categorize the equilibrium, plot the nullclines, and sketch the flow in the 2D phase-plane.

b) Use your vector field sketch in part "a" to make a rough sketch of x(t) and y(t), for the given initial condition
Note: if you'd like to check your result, you can find exact solution for y(t), and then solve for x(t)