Fall 2015 * Math 430 * Math 635 * Prof. Victor Matveev
Homework 7 * Due date: October 22

The first two problems concern the FitzHugh-Nagumo model of excitable cell: $\begin{cases} V' = V - \dfrac{V^3}{3} - w + I \\ w' = \varepsilon\left(V - cw\right) \end{cases}$

1. **(30pts)** Consider parameter values $c = 1/2, \quad \varepsilon = 1/10$. Modify "PhasePlane2D.m" program to sketch the vector flow field (using "quiver"), the nullclines (using "plot"), the phase-plane trajectory (computed using the Euler method), and the V(*t*) vs *t* plot, for the following values of parameters:

   a) I = 0, initial condition: V(0) = –0.1; w(0)=0

   b) I = –2, initial condition V(0) = –0.8; w(0)= –2.57

   c) I = –2, initial condition V(0) = –0.2; w(0)= –2.57

   d) $I = I_{cr} = 2/3 - 1/c$, initial condition V(0) = –2; w(0)= –2 (you will see the so-called "subthreshold oscillation")

   e) Find numerically (using your program) the *V* threshold value for I = –2, assuming w(0) = –2.57

2. **(35pts)** Consider the FitzHugh-Nagumo model with parameters $c = 1/2, \quad \varepsilon = 1/2$.

   a) We found in class that for $\varepsilon \ll 1$ the bifurcation from stable equilibrium to a stable limit cycle occurs when the equilibrium lies on the left "knee" of the cubic, requiring $I = I_{cr} = 2/3 - 1/c$. Examine how accurate this result is when ε is *not* very small: use the program you've modified for problem 1 to find the *actual* critical value of current (correct to three decimal digits) corresponding to the transition from stable equilibrium to an oscillation, for the parameter values given above ($c = 1/2, \quad \varepsilon = 1/2$)

   b) Find also the equilibrium values V*, w* at the critical value of current you found in part "a" (you can use your plot for this: find the "zoom in" button on the plot), and calculate approximately the eigenvalues of the Jacobian at this equilibrium point.

   c) Find the eigenvalues of the Jacobian at the equilibrium for a value of current slightly above I_{cr}. Check the relationship between the oscillation period and the imaginary part of the eigenvalue close to the bifurcation: $\operatorname{Im}\lambda \approx 2\pi / T$

3. **(35pts) Midterm exam preparation:**

   In this problem you will use linear stability analysis to analyze the following model "by hand" (on paper); it's a version of the famous "Preditor-Prey" system (here c=const<1; you can pick c=1/2 if you like):

$$\begin{cases} x' = x\left(1-x\right) - c\,xy & (x\text{=relative population size of prey}) \\ y' = y\left(1-y\right) + c\,xy & (y\text{=relative population size of predator}) \end{cases}$$

   a) Sketch all nullclines (note: each nullcline consists of two lines)

   b) Find all equilibria (there are four of them)

   c) Find the Jacobian at each equilibrium, and categorize equilibrium (i.e. determine its type)

   d) Find the eigenvectors corresponding to the equilibrium with non-zero x and y (the "coexistence" equilibrium), and plot these eigenvectors as two linear directions emanating from this non-zero equilibrium.

   e) Complete your sketch of the flow in 2D plane using all information you found in parts "a"-"d"

4. Linear systems: extra credit problem (20pts)

Consider a linear system $\frac{dY(t)}{dt} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} Y(t)$, where $Y = \begin{pmatrix} x \\ y \end{pmatrix}$. Show that eigenvalues are a complex conjugate pair, and characterize the stability of the equilibrium at the origin (assume *a*, *b* are non-zero). Then, solve this system (for any initial condition) using matrix exponentiation, $Y(t) = e^{At} Y_o$, by following these steps:

a) Note that $A \equiv \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + b \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{J} = a\,I + b\,J$

b) By definition, the identity matrix satisfies $I^k = I$ for any integer power *k*. Now, find a simple expression for powers J^k (hint: the result only depends on whether *k* is even or odd).

c) Use the results of part "b" and the standard Taylor series expansion to find $e^{a t I}$ and $e^{b t J}$. Make sure to combine all like terms and simplify the result as much as possible, expressing each exponential as a single simple matrix.

d) Multiply the matrices you obtained in "c" to find the answer: $e^{At} = e^{(a I + b J)t} = e^{a t I} e^{b t J}$ (note: this product rule requires that $I\,J = J\,I$, which is easy to verify). Finally, multiply from the right by the initial condition vector (x_o, y_o) to find your general solution.