Fall 2015 * Math 430 * Math 635 * Prof. Victor Matveev

Lecture 11: Linear flows in 2D

Before solving non-linear 2D problems arising in modeling neuronal activity, it is important to review the solution of a linear system, since the behavior of a non-linear system near its equilibrium can sometimes be determined by its linear approximation near equilibrium (see below). Therefore, let's review linear systems:

Method 1: Matrix exponentiation (the only general method that works for any linear system!)

Consider a 2D flow field, where the velocity is a linear function of position (x, y): $\begin{cases} x'(t) = ax + by \\ y'(t) = cx + dy \\ x(0) = x_o; \quad y(0) = y_o \end{cases}$

Using vector-matrix product rule, this can be written as $\begin{cases} \dfrac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \\ \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}_{t=0} = \begin{pmatrix} x_o \\ y_o \end{pmatrix} \end{cases}$

Denoting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, this yields $\begin{cases} \dfrac{dY}{dt} = AY \\ Y(0) = Y_o \end{cases}$

It is quite easy to prove that the solution is the same as for a similar linear scalar equation, namely:

$$\boxed{Y(t) = e^{At}\, Y_o}$$

The order of terms in this product is important: initial condition must be multiplied from the left, since the exponential of a matrix is a matrix; it is defined in the familiar way as:

$$e^{At} = I + At + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \ldots = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

Here I is called the "identity matrix", $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, satisfying an important and easily verifiable matrix product property $A \cdot I = I \cdot A = A$. Thus, it is a matrix product analogue of a scalar unity, 1.

In the exponential series, any power of a square matrix, A^k, is a matrix of the same dimension as A, so all terms in this series are 2x2 square matrices (otherwise this expression wouldn't make sense).

In some cases matrix exponentiation is very simple and efficient, as in the important example below:

Example 1: solve the linear system $\begin{cases} x' = y \\ y' = 0 \\ x(0) = 1; \quad y(0) = 2 \end{cases}$

Step 1: Write in vector form: $\begin{cases} \dfrac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y(t) \\ Y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{cases}$

Step 2: Examine powers of system matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$: $A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \Rightarrow A^{k>1} = 0$

Matrices with this interesting property are called "nilpotent"

Step 3: Exponentiate the matrix: $e^{At} = I + At + \underbrace{\frac{t^2}{2}A^2 + \ldots}_{=0} = I + At = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

Step 4: Multiply by initial condition to obtain the solution:

$$Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At} Y_o = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_o \\ y_o \end{pmatrix} = \begin{pmatrix} x_o + t\, y_o \\ y_o \end{pmatrix} = \begin{pmatrix} 1 + 2t \\ 2 \end{pmatrix}$$ We are done!

As we saw from the flow sketch done in class, y(t) is constant, and x(t) is linearly increasing with velocity=y_o

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The matrix exponential method is also very useful when analyzing rotating flows. In general however, matrix exponentiation is a computationally expensive and indirect way to describe the solution. This brings us to the next method:

Method 2: Solution using eigenvectors and eigenvalues

Eigenvectors / eigenvalues (the "self"-vectors/"self"-values) of a given matrix *A* are solutions of equation

$$\boxed{A\mathbf{v}_{1,2} = \lambda_{1,2}\,\mathbf{v}_{1,2}} \quad \Leftrightarrow \quad A\mathbf{v}_{1,2} \text{ is parallel to } \mathbf{v}_{1,2} \text{ (or zero)}$$

- "Eigenvector" is a misnomer, since it is not a vector: it has a direction, but arbitrary length. Therefore, a more proper term for an eigenvector would be an **eigendirection** (or **linear vector space)**
- Since λ is a scalar, an eigendirection is a direction which is not changed by matrix multiplication: for our purposes, this means that eigendirections are directions where the velocity ($A\mathbf{v}_{1,2}$) is parallel to the position vector ($\mathbf{v}_{1,2}$), keeping the flow linear (an important exception is λ=0, in which case velocity is zero).
- Thus, eigenvectors are one-dimensional **linear vector spaces invariant under the flow**
- Non-zero N×N matrix *A* has at least one, and no more than N linearly independent eigenvectors

To find the eigenvalues, one has to solve the "characteristic equation" $\det(A - \lambda I) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$

Once eigenvalues are known, one can find eigenvectors directly, by solving $A\mathbf{v}_{1,2}=\lambda_{1,2}\,\mathbf{v}_{1,2}$. Since eigenvectors are really eigendirections with no fixed length, when looking for solutions one can set one of the vector components to 1: look for solutions of form $\begin{pmatrix}1\\c\end{pmatrix}$ or $\begin{pmatrix}c\\1\end{pmatrix}$: one of the two is bound to work (there are also some convenient shortcuts for 2x2 systems).

Here is why eigenvectors are important: if two linearly independent eigenvectors exist, then (and only then) the solution to the linear system Y' = AY equals

$$\boxed{Y(t)=e^{At}Y_o=c_1e^{\lambda_1 t}\mathbf{v}_1+c_2e^{\lambda_2 t}\mathbf{v}_2}\quad \text{where } c_{1,2} \text{ are found from initial condition } \boxed{Y_o=c_1\mathbf{v}_1+c_2\mathbf{v}_2}$$

Example 2: solve the linear system $\begin{cases}x'=2x-y\\ y'=-x+2y\\ x(0)=2;\quad y(0)=1\end{cases}$

Step 1: Find eigenvalues: $\det\begin{pmatrix}2-\lambda & -1\\ -1 & 2-\lambda\end{pmatrix}=(2-\lambda)^2-1=0 \;\Rightarrow\; 2-\lambda=\pm 1 \;\Rightarrow \boxed{\lambda_1=1,\ \lambda_2=3}$

Step 2: Find the two eigenvectors (if they exist) in the form $\mathbf{v}_{1,2}=\begin{pmatrix}1\\c\end{pmatrix}$:

$$\text{Find } \mathbf{v}_1:\ \begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}1\\c\end{pmatrix}=\lambda_1\begin{pmatrix}1\\c\end{pmatrix}=\begin{pmatrix}1\\c\end{pmatrix} \;\Rightarrow\; \begin{cases}2-c=1\\ -1+2c=c\end{cases} \;\Rightarrow\; c=1 \;\Rightarrow \boxed{\mathbf{v}_1=\begin{pmatrix}1\\1\end{pmatrix}}$$

$$\text{Find } \mathbf{v}_2:\ \begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}1\\c\end{pmatrix}=\lambda_2\begin{pmatrix}1\\c\end{pmatrix}=3\begin{pmatrix}1\\c\end{pmatrix} \;\Rightarrow\; \begin{cases}2-c=3\\ -1+2c=3c\end{cases} \;\Rightarrow\; c=-1 \;\Rightarrow \boxed{\mathbf{v}_2=\begin{pmatrix}1\\-1\end{pmatrix}}$$

[Note: in each case, the 2 equations for *c* are linearly dependent, so it is sufficient to consider one of them]

Step 3: Write down the vector solution:

$$Y(t)=c_1e^{\lambda_1 t}\mathbf{v}_1+c_2e^{\lambda_2 t}\mathbf{v}_2=c_1e^{t}\begin{pmatrix}1\\1\end{pmatrix}+c_2e^{3t}\begin{pmatrix}1\\-11\end{pmatrix}=\boxed{\begin{pmatrix}c_1e^t+c_2e^{3t}\\ c_1e^t-c_2e^{3t}\end{pmatrix}}$$

Step 4: Find the coefficients using the initial condition:

$$Y(0)=\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}c_1+c_2\\c_1-c_2\end{pmatrix} \;\Rightarrow\; \begin{cases}c_1=\dfrac{3}{2}\\[2mm] c_2=\dfrac{1}{2}\end{cases} \;\Rightarrow\; \boxed{\begin{cases}x(t)=\dfrac{3e^t+e^{3t}}{2}\\[2mm] y(t)=\dfrac{3e^t-e^{3t}}{2}\end{cases}}$$

Relationship between linear and non-linear systems: the Jacobian

The behavior of any non-linear system near its equilibrium cat be analyzed using its linear approximation near equilibrium, i.e. using the first terms in the Taylor series of the velocity field:

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases} \Rightarrow \text{ Express in terms of deviations from equilibrium } \Rightarrow \begin{cases} x(t) = x^* + \delta x(t) \\ x(t) = y^* + \delta y(t) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = \dfrac{d}{dt}\delta x = f(x^* + \delta x,\ y^* + \delta y) = \underbrace{f(x^*, y^*)}_{\text{0 at equilibrium}} + \dfrac{\partial f}{\partial x}(x^*, y^*)\delta x + \dfrac{\partial f}{\partial y}(x^*, y^*)\delta y \\ \dfrac{dy}{dt} = \dfrac{d}{dt}\delta y = g(x^* + \delta x,\ y^* + \delta y) = \underbrace{g(x^*, y^*)}_{\text{0 at equilibrium}} + \dfrac{\partial g}{\partial x}(x^*, y^*)\delta x + \dfrac{\partial g}{\partial y}(x^*, y^*)\delta y \end{cases}$$

We obtained a linear system with coefficients equal to the partial derivatives of the velocity field, which we can write more succinctly in vector-matrix product form:

$$\frac{d}{dt}\begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix} = \underbrace{\begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)}}_{\text{"Jacobian" } DF} \begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix}$$

$$\text{This can be written simply as } \frac{dY(t)}{dt} = DF \cdot Y(t) \text{ where } Y(t) = \begin{pmatrix} \delta x(t) \\ \delta y(t) \end{pmatrix} \text{ and } F = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

The matrix of partial derivatives of vector field $F(x,y)$ is denoted $DF(x,y)$ and called the Jacobian; note that it is a matrix of constants!

According to the method learned above, if the Jacobian has two distinct eigendirections, the behavior near equilibrium is approximated by

$$Y(t) \approx Y^* + \underbrace{c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2}_{\delta Y(t)}$$

If both eigenvalues of the Jacobian have positive real part, both terms diverge to infinity, and therefore the equilibrium is unstable; if the sign of the real part is negative, the equilibrium is stable, and if signs are distinct, the equilibrium is a saddle. For a full classification, see textbook sections 4.2.2-4.2.5