Math 430 * Math 635 * Fall 2015 * Victor Matveev

Lecture 5: solving differential equations

Part 1: Method of constant variation (an example):

Consider a passive cell which at time zero starts receiving a **linearly increasing** current $I(t) \propto t$:

$$\begin{cases} \tau_m V' = -(V - V_R) + R\,I(t) \\ V(0) = V_R \end{cases} \Rightarrow \begin{cases} \tau_m V' = -(V - V_R) + \beta t \\ V(0) = V_R \end{cases} \quad \text{where } \beta = const$$

Let's calculate the potential of the cell as a function of time, using the **method of *constant variation***.

Before we do that, let's shift the potential by introducing variable $v(t) = V(t) - V_R$

This simplifies our equation, transforming it to $\tau_m v' = -v + \beta t, \quad v(0) = 0$

1. Solve the linear part, which is a *homogeneous* equation $\tau_m v' = -v \quad \Rightarrow v(t) = C e^{-t/\tau_m}$

2. Replace "C" with time-dependent C(*t*) ("vary the constant"), then plug it into full equation $\tau_m v' = -v + \beta t$ to find and solve the equation for C(*t*):

$$v(t) = C e^{-t/\tau_m} \Rightarrow v(t) = C(t) e^{-t/\tau_m} \quad (\text{where } C(0) = 0 \text{ since } v(0) = C(0) = 0)$$

Plug this into our full equation $\tau_m v' = -v + \beta t$:

$$\tau_m \frac{dv}{dt} = \tau_m \frac{d}{dt}\left[C(t) e^{-t/\tau_m}\right] = \tau_m \left(\frac{dC}{dt} - \frac{C}{\tau_m}\right) e^{-t/\tau_m}$$

$$= \tau_m \frac{dC}{dt} e^{-t/\tau_m} - v = -v + \beta t \quad \Rightarrow \tau_m \frac{dC}{dt} e^{-t/\tau_m} = \beta t \Rightarrow \boxed{\frac{dC}{dt} = \frac{\beta}{\tau_m} t e^{t/\tau_m}}$$

Now integrate between $t = 0$ and t, in order to find $C(t)$:

$$C(t) - \underbrace{C(0)}_{0} = \frac{\beta}{\tau_m} \int_0^t \tau e^{\tau/\tau_m} d\tau = \underbrace{\beta \int_0^t \tau\, d\left(e^{\tau/\tau_m}\right) = \beta \left[\tau e^{\tau/\tau_m}\Big|_0^t - \int_0^t e^{\tau/\tau_m} d\tau \right]}_{\text{Integration by parts}}$$

$$= \beta \left[t e^{t/\tau_m} - \tau_m e^{\tau/\tau_m}\Big|_0^t \right] = \boxed{\beta \left[t e^{t/\tau_m} - \tau_m \left(e^{t/\tau_m} - 1\right)\right]}$$

3. Write down the solution V(t), and make a rough plot

$$v(t) = V(t) - V_R \quad \Rightarrow \quad V(t) = V_R + v(t)$$

$$= V_R + C(t)e^{-t/\tau_m}$$

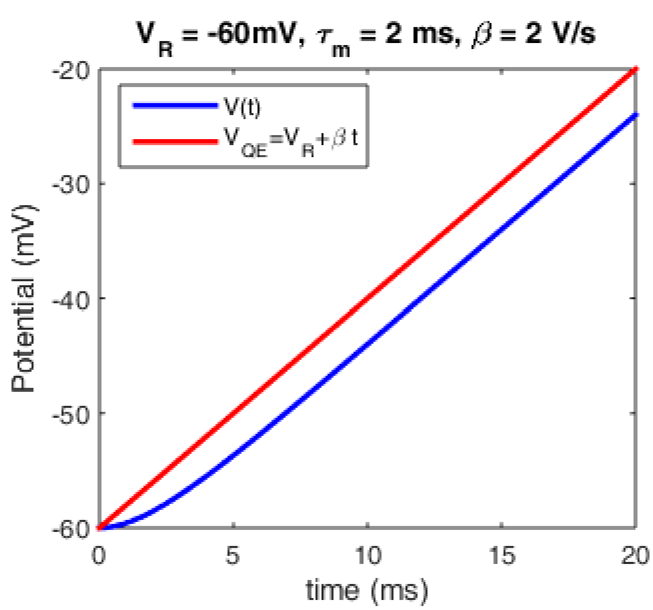
$$= V_R + \beta\left[t\,e^{\,t/\tau_m} - \tau_m\left(e^{t/\tau_m} - 1\right)\right]e^{-t/\tau_m}$$

$$= \boxed{V_R + \beta\left[t - \tau_m\left(1 - e^{-t/\tau_m}\right)\right]}$$

At large time $t >> 1/\tau_m$, we have: $V(t) \approx V_R + \beta[t - \tau_m]$

- **Voltage "chases" the "quazi-equilibrium"** $V_{QE} = V_R + \beta t$ **with a time lag that approaches** τ_m

See Figure for an example:



Part 2: Linear Stability Analysis

Consider an *autonomous first-order* differential equation equation $\begin{cases} \frac{dY}{dt} = f(Y) \\ Y(0) = Y_o \end{cases}$

Where ***f*(*Y*) is a differentiable function** (this condition can be relaxed, but is good enough for us)

Note that if *Y*=position, the right-hand side *f*(*Y*) describes how velocity depends on position, so it gives us a lot of intuition about the solution, without even solving this equation!

Note the following:

- Oscillatory behavior is not possible, since *f*(*Y*) is single-valued (we can't both increase *and* decrease for same *Y*, can we?)
- Therefore, the solution is either monotonically increasing or monotonically decreasing, unless it is simply constant.
- Still, the following different outcomes are possible: increasing to infinity in infinite time, increasing to infinity in finite time ("blow up"), or increasing to some steady state
- At a steady-state *Y**, there is no change, so $\frac{dY}{dt} = f(Y^*) = 0$
- If we approach this equilibrium, the equilibrium is called *stable*
- If we go away from this equilibrium, it is called *unstable*
- Stability can be determined using linearization: examine the perturbation from equilibrium, *y*(*t*)

  $$Y(t) = Y^* + y(t)$$

  Plug this into our equation, and use linearization

$$\frac{d}{dt}\left(Y^* + y(t)\right) = f(Y^* + y(t)) \approx \underbrace{\overbrace{f(Y^*)}^{0} + \frac{df}{dY}\left(Y^*\right)\overbrace{\left(Y - Y^*\right)}^{y(t)} + \frac{1}{2}\frac{d^2 f}{dY^2}\left(Y^*\right)\overbrace{\left(Y - Y^*\right)^2}^{y(t)^2} + \ldots}_{\text{Taylor series (assume function is smooth)}}$$

Note that $\frac{d}{dt}\left(Y^* + y(t)\right) = \underbrace{\frac{dY^*}{dt}}_{0} + \frac{dy(t)}{dt}$ and that $f(Y^*) = 0$ by definition of an equilibrium point

Therefore, our equation becomes $\frac{dy}{dt} \approx \underbrace{\frac{df}{dY}\left(Y^*\right)y + \frac{1}{2}\frac{d^2 f}{dY^2}\left(Y^*\right)y^2 + \ldots}_{\text{Taylor series (assume function is smooth)}}$

This may look complicated, but remember that the factors in the Taylor series are simple constants, so

$$\frac{dy}{dt} \approx \alpha y + \beta y^2 + \ldots$$

If we keep only the first term, then we obtain a simplified, linear equation for the deviation from equilibrium of form $\frac{dy}{dt} \approx \alpha y$ where $\alpha = \frac{df}{dY}\left(Y^*\right) = const$

We know that the solution is $y(t) \approx y_o e^{\alpha t}$

Thus, the following outcomes are possible:

- $\alpha = \frac{df}{dY}\left(Y^*\right) > 0 \;\Rightarrow\;$ deviation grows $\;\Rightarrow\; Y^*$ is unstable equilibrium (unstable fixed point)
- $\alpha = \frac{df}{dY}\left(Y^*\right) < 0 \;\Rightarrow\;$ deviation decreases $\;\Rightarrow\; Y^*$ is a stable equilibrium (stable fixed point)
- $\alpha = \frac{df}{dY}\left(Y^*\right) = 0 \;\Rightarrow\;$ inconclusive $\;\Rightarrow\;$ examine $\frac{dy}{dt} \approx \beta y^2$, where $\beta = \frac{d^2 f}{dY^2}\left(Y^*\right) = const$

Example: qualitatively describe the solution to $\begin{cases} Y' = \cos Y \\ Y(0) = 0.2 \end{cases}$

Equilibria: $Y' = \cos Y^* = 0 \;\Rightarrow\; Y^* = \pi\left(k + \frac{1}{2}\right)$

Examine their stability: $\alpha = \frac{df}{dY}\left(Y^*\right) = -\sin Y^* = -\sin\left[\pi\left(k + \frac{1}{2}\right)\right] = \begin{cases} -1, \text{ even } k \\ +1, \text{ odd } k \end{cases}$

Thus, equilibria corresponding to even *k* are stable (green circles below): $Y^*_{SE} = \frac{\pi}{2}, \; \frac{5\pi}{2}, \ldots$

Equilibria corresponding to odd *k* are unstable (yellow circles below): $Y^*_{UE} = -\frac{\pi}{2}, \; \frac{3\pi}{2}, \ldots$

Now, our initial condition is between $Y^*_{UE} = -\frac{\pi}{2}$ and $Y^*_{SE} = \frac{\pi}{2}$, so the solution keeps increasing, going away from $Y^*_{UE} = -\frac{\pi}{2}$, and asymptotically approaches $Y^*_{SE} = \frac{\pi}{2}$ (see phase plot below)

Part 3: Geometric analysis: "Phase plot" / "velocity plot":

A powerful method to figure out qualitative behavior of an autonomous ODE is to plot the "velocity" $f(Y)$ as a function of "position" Y.

For instance, for the above example, at the initial condition cos Y(0) = cos 0.2 > 0, therefore $Y(t)$ grows with t. However, as $Y(t)$ increases and approaches $\pi/2$, the rate of that monotonic growth decreases (since cos(Y) is a decreasing function on that interval of Y from 0 to $\pi/2$), and the solution can never actually reach that equilibrium point in finite time. Look at the plot of cos Y to understand this.