Lecture 7: Numerical solution of ODEs: Forward Euler Method

The Forward Euler method is just a linear approximation for the solution:

$$\begin{cases} \dfrac{dY}{dt} = f(t, Y) \\ Y(0) = Y_o \end{cases}$$

Denote $h \equiv \dfrac{t_{total}}{N}$, $t_n \equiv n \cdot h$, $Y_n \equiv Y(t_n) = Y(n \cdot h)$

Assuming $f(t, Y)$ is sufficiently smooth, so is the solution $Y(t)$, therefore we can use Taylor expansion:

$$Y(t_{n+1}) = \underbrace{Y(t_n + h) \approx Y(t_n) + \overbrace{\frac{dY}{dt}}^{f(t_n, Y_n)}(t_n) h}_{\text{Euler method: use Linearization}} + \underbrace{\frac{1}{2}\frac{d^2Y}{dt^2}(t_n) h^2}_{\text{Local error } \propto h^2} + \ldots$$

Therefore, we obtain $Y_{n+1} \approx Y_n + f(t_n, Y_n) \cdot h$

For an autonomous problem we get a simpler expression: $Y_{n+1} \approx Y_n + f(Y_n) \cdot h$

Example:

$$\begin{cases} \dfrac{dY}{dt} = -2Y + 5t \\ Y(0) = 1 \end{cases}$$

Let's solve numerically using $h = 0.2$

$t_0 = 0: \quad Y_0 = 1$

$t_1 = 0.2: \quad Y_1 = Y_0 + h\, f(t_0, Y_0) = 1 + 0.2(-2 \cdot 1 + 5 \cdot 0) = 1 - 0.4 = 0.6$

$t_2 = 0.4: \quad Y_2 = Y_1 + h\, f(t_1, Y_1) = 0.6 + 0.2(-2 \cdot 0.6 + 5 \cdot 0.2) = 0.56$

$t_3 = 0.6: \quad Y_3 = Y_2 + h\, f(t_2, Y_2) = 0.56 + 0.2(-2 \cdot 0.56 + 5 \cdot 0.4) = 0.736$

....