Math 430 * Math 635 * Fall 2015 * Victor Matveev

Derivation of the Nernst Equation for ion "X":

Let's think of a pore across the membrane as a one-dimensional space, and direct the "x"-axis across the pore (into or out of the cell). Consider an ion "X" of charge *q* moving within this pore along axis *x*.

At reversal potential, the diffusive current of ions precisely balances their field-driven current:

$J_D + J_E = 0$ ← These are current densities in units of # / (s m^2)

Diffusive current is given by the Fick's law of diffusion (ions move against concentration gradient):

$$J_D = -D\frac{d[\mathrm{X}]}{dx} = -u\; k_B T\,\frac{d[\mathrm{X}]}{dx}$$

Here we used Einstein's relationship between diffusion coefficient and temperature: $D = u\,k_B T$

Parameter u is the mobility, which is the proportionality constant between equilibrium ("terminal") velocity and force in viscous flow: v = u Force

Extra curious fact (Stokes-Einstein-Sutherland equation): $u = 1/\left[6\pi(\text{radius})(\text{viscosity})\right]$

Now, recall that electrostatic force (in 1D) equals Force = $q\mathbf{E} = -q\nabla V = -q\dfrac{dV}{dx}$

Note that the product between velocity and density gives current density (flux): $\dfrac{m}{s}\dfrac{\#}{m^3} = \dfrac{\#}{m^2 s}$

Therefore, for the electric field-driven flux, we have to multiply velocity and density, yielding

$$J_E = \mathrm{v}[X] = [X]\cdot u\underbrace{\;\text{Force}\;}_{-q\,dV/dx} = -u\,q\,[X]\frac{dV}{dx}$$

We can finally examine the equality between the two current densities:

$$J_E = -J_D \quad\Rightarrow\quad -u\,q\,[X]\frac{dV}{dx} = u\,k_B T\,\frac{d[\mathrm{X}]}{dx}$$

$$\Rightarrow\quad \frac{dV}{dx} = -\frac{k_B T}{q}\frac{1}{[X]}\frac{d[\mathrm{X}]}{dx} = -\frac{k_B T}{q}\frac{d\ln[\mathrm{X}]}{dx}$$

Now integrate from the outside to the inside of the membrane:

$$V = V_{in} - V_{out} = -\frac{k_B T}{q}\left(\ln[X]_{in} - \ln[X]_{out}\right) = -\frac{k_B T}{q}\ln\frac{[X]_{in}}{[X]_{out}} = \boxed{\frac{k_B T}{q}\ln\frac{[X]_{out}}{[X]_{in}}}$$

Note that the ion charge *q* is the product of its valence *z* and the elementary charge *e*