Victor Matveev, March 26, 2017 * Modified Nov 1, 2018 * Lecture Notes

I. DIVERGENCE: DEFINITION

Consider a vector field **F**(**r**) defined within some region in $\mathbf{R}^3$ (everything generalizes to $\mathbf{R}^n$ however).

Divergence of a vector field at a given point $\mathbf{r_o}$ is a scalar quantifying the local *expansion* of a vector field in the neighborhood of that point. It is therefore defined as the flux of this field out of any closed surface surrounding any point $\mathbf{r_o}$, per unit volume, as the volume enclosed by this closed surface approaches zero:

$$\boxed{\text{Definition}} \quad \text{div } \mathbf{F}(\mathbf{r}_o) \equiv \nabla \cdot \mathbf{F}(\mathbf{r}_o) \equiv \lim_{V \to 0} \frac{1}{V}\left(\oiint_{\partial V} \mathbf{F} \cdot \mathbf{dS} \right)$$

where $\mathbf{r}_0$ remains an interior point of V as the volume measure approaches zero. Here $\mathbf{dS} = \mathbf{n}\, dS$, where **n** is the unit normal field (facing outward) on the surface ∂V , and dS is the differential measure on this surface area.

Theorem 1

If a vector field **F** is continuously differentiable at point $\mathbf{r_o}$, by applying the definition to the particular case of a cube volume, and taking the limit of the sides of the cube approaching zero, in $\mathbf{R}^3$ one can easily prove that

$$\text{div } \mathbf{F}(\mathbf{r}_o) \equiv \nabla \cdot \mathbf{F}(\mathbf{r}_o) = \frac{\partial F_1}{\partial x}(\mathbf{r}_o) + \frac{\partial F_2}{\partial y}(\mathbf{r}_o) + \frac{\partial F_3}{\partial z}(\mathbf{r}_o)$$

Note that this is **not** a definition: definition of divergence does **not** require the field to be differentiable.

II. DIVERGENCE THEOREM

Theorem 2 (Divergence theorem)

Consider a connected bounded (i.e. compact) volume V with a piece-wise smooth boundary surface ∂V , and let $\mathbf{dS} = \mathbf{n}\, dS$ denote the outward normal vector field multiplied by surface area differential. Further, suppose vector field **F** is continuously differentiable (C^1) within this volume V. Then the following result holds:

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_{\partial V} \mathbf{F} \cdot \mathbf{dS}$$

Example 1: If we apply the divergence theorem to vector field **F** which is equal to position vector field **r**, we obtain:

$$\nabla \cdot \mathbf{r} = 3 \;\Rightarrow\; \oiint_{\partial V} \mathbf{r} \cdot \mathbf{dS} = \oiint_{\partial V} \mathbf{r} \cdot \mathbf{dS} = 3 \iiint_V dV = 3V \;\Rightarrow\; V = \frac{1}{3} \oiint_{\partial V} \mathbf{r} \cdot \mathbf{dS}$$

This means that the volume enclosed by the surface equals the flux of position vector field out of this surface, divided by the number of dimensions (here, 3). This is a highly non-trivial topological result.

III. DIVERGENCE THEOREM AND CONTINUITY / CONSERVATION

The divergence theorem is intricately connected to the continuity/conservation condition arising in a wide variety of physical problems. Consider a very general problem of flow of any quantity such as mass, charge, molecule number (#), or any other substance:

- $\rho(\mathbf{r}, t)$ = density of charge (or mass or # of molecules) per unit volume
- $Q(t)$ = total charge (or mass or # of molecules) within a connected bounded volume V
- $\mathbf{J}(\mathbf{r}, t)$ = flux of charge (or mass or #) per unit time per unit area perpendicular to the flow direction. Flux is also known as current density (current per unit area), in the case of electric current.

By these definitions, the following two relationships between these three quantities have to hold:

1. $Q = \iiint_V \rho(\mathbf{r},t)\, dV$
2. $\dfrac{dQ}{dt} = -\oint\!\!\oint_{\partial V} \mathbf{J}\cdot\mathbf{dS}$ (total flux out of any volume V = rate of decrease per unit time within V)

Note that relationship 1 leads to the following:

3. $\dfrac{dQ}{dt} = \iiint_V \dfrac{\partial \rho(\mathbf{r},t)}{\partial t}\, dV$

Further, we can use the divergence theorem in relationship 2, leading to

4. $\dfrac{dQ}{dt} = -\oint\!\!\oint_{S=\partial V} \mathbf{J}\cdot\mathbf{dS} = -\iiint_V \nabla\cdot\mathbf{J}\, dV$

Combining results 3 and 4, we obtain

$$\frac{dQ}{dt} = -\iiint_V \nabla\cdot\mathbf{J}\, dV = \iiint_V \frac{\partial \rho(\mathbf{r},t)}{\partial t}\, dV$$

Subtracting the two sides of this equation yields

$$\iiint_V \left(\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla\cdot\mathbf{J} \right) dV = 0$$

Since this integral is zero for *any* integration domain (volume) V, the integrand has to be zero as well, leading to the well-known and very general continuity condition:

Continuity / conservation equation: $\boxed{\dfrac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla\cdot\mathbf{J} = 0}$

Note that if we have information about the velocity field $\mathbf{v}(\mathbf{r},t)$, then we have the following additional relationship between flux (current density) and the density:

$$\mathbf{J}(\mathbf{r}, t) = \rho(\mathbf{r}, t)\, \mathbf{v}(\mathbf{r}, t)$$

Check the physical units to understand this better: $\left[\dfrac{\#}{m^2\,\text{sec}}\right]=\left[\dfrac{\#}{m^3}\right]\cdot\left[\dfrac{m}{\text{sec}}\right]$

This allows us to write the conservation condition as

$$\boxed{\frac{\partial\rho(\mathbf{r},t)}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0}$$

Further, by the chain rule this can be written as

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t}+\nabla\rho\cdot v+\rho\nabla\cdot v=0$$

If density is constant $(\rho_t=0,\ \nabla\rho=0)$, i.e. the flow is incompressible, then we obtain a well-known result that

$$\nabla\cdot v=0$$

IV. Application 1: Diffusion around a point source

Consider diffusion of some substance with volume concentration of $C(\mathbf{r},t)$, with a single source at the origin characterized by current of σ_C in units of molecule number (say, expressed in moles) per unit time:

$$\begin{cases}\dfrac{\partial C}{\partial t}=\nabla\cdot(D_C\nabla C)+\sigma_C\delta(\mathbf{r})\\ \lim\limits_{|r|\to\infty}C(r)=0\end{cases}$$

Integrate both sides of the PDE over a sphere of radius R centered at the source:

$$\iiint_V\frac{\partial C}{\partial t}dV=\iiint_V\nabla\cdot(D_C\nabla C)\,dV+\sigma_C\iiint_V\delta(r)dV$$

Recalling that Q(t) = $\iiint_V C(\mathbf{r},t)dV$ and using the divergence theorem, we obtain

$$\frac{dQ}{dt}=D_C\oiint_{\partial V}\nabla C\cdot\mathbf{dS}+\sigma_C$$

In case of radial symmetry we have $\nabla C(\mathbf{r},t)=\dfrac{\partial C}{\partial r}(r,t)\,\mathbf{e_r}$ and $\mathbf{dS}=\mathbf{e_r}\,dS$, which leads to

$$\frac{dQ}{dt}=D_C\oiint_{\partial V}\frac{\partial C}{\partial r}\mathbf{e_r}\cdot\mathbf{e_r}\,dS+\sigma_C=D_C\oiint_{\partial V}\frac{\partial C}{\partial r}dS+\sigma_C=D_C\frac{\partial C}{\partial r}(R,t)\oiint_{\partial V}dS+\sigma_C=4\pi R^2D_C\frac{\partial C}{\partial r}(R,t)+\sigma_C$$

This time-dependent equation is not simple, but it allows to obtain the equilibrium solution quite readily:

$$4\pi R^2D_C\frac{\partial C_{eq}}{\partial r}(R)+\sigma_C=0\ \Rightarrow\ \frac{\partial C_{eq}}{\partial r}(R)=-\frac{\sigma_C}{4\pi D_C}\frac{1}{R^2}\ \Rightarrow\ \boxed{C_{eq}(R)=\frac{\sigma_C}{4\pi D_C}\frac{1}{R}+C_\infty}$$

V. APPLICATION II: POTENTIAL OF POINT CHARGE

The Gauss law is the first in the system of Maxwell's equations, connecting electric field and charge density. In SI units, it reads:

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = \frac{\rho(\mathbf{r},t)}{\varepsilon_o}$$

If we consider a stationary point charge at the origin, this becomes

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{Q}{\varepsilon_o}\delta(\mathbf{r})$$

Integrate both sides of the PDE over a sphere of radius R centered at the source:

$$\iiint_V \nabla \cdot \mathbf{E}(\mathbf{r})\, dV = \frac{Q}{\varepsilon_o}\iiint_V \delta(\mathbf{r})\, dV = \frac{Q}{\varepsilon_o}$$

Using the divergence theorem, we obtain

$$\oint\!\!\oint_{\partial V} \mathbf{E} \cdot \mathbf{dS} = \frac{Q}{\varepsilon_o}$$

Now, note that both **dS** and **E** have the same orientation, pointing away from the origin; and that the magnitude of **E** depends only on the distance from the point charge:

$$\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}} \quad \text{where } E = |\mathbf{E}| \text{ and } \hat{\mathbf{r}} = \mathbf{e}_r = \frac{\mathbf{r}}{|\mathbf{r}|}$$ (unit normal pointing away from the origin)

$$\mathbf{dS} = \hat{\mathbf{r}}\, dS$$

Therefore, the dot product in the flux integral simplifies:

$$\oint\!\!\oint_{\partial V} \mathbf{E}(\mathbf{r}) \cdot \mathbf{dS} = \oint\!\!\oint_{\partial V} E(r)\underbrace{\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}}_{=1} dS = \oint\!\!\oint_{\partial V} E(r) dS = E(R)\oint\!\!\oint_{\partial V} dS = 4\pi R^2 E(R) = \frac{Q}{\varepsilon_o}$$

Thus, we obtain: $$\boxed{E(R) = \frac{Q}{4\pi\varepsilon_o}\frac{1}{R^2}}$$

NOTE: the same derivation holds for a sphere of radius < R with any angle-independent charge distribution (the entire derivation is unchanged).