**Math 613 * Fall 2018 * Victor Matveev**

**Homework 1: units, nondimensionalization, and scaling**

**1.** Non-dimensionalize the bacterial population growth model, examined in class, using the following scales: $[n]=K$, $[t]=1/(\alpha K)$

$$\begin{cases} \dfrac{dn}{dt} = \alpha n(K-n) & (t>0) \\ n(0) = n_o \end{cases}$$

**2.** Consider the drag force $F_D$ on a sphere of radius $R$ moving with speed $v$ in a fluid with viscosity $\mu$ and mass density $\rho$.

a) Figure out the units of $\mu$ by examining the units in the Navier-Stokes equation for incompressible fluid (this does not require any understanding of this equation beyond partial differentiation; recall that units of *all* terms have to equal to each other in any equation:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \mu\nabla^2\mathbf{v}$$

b) Find the unique dimensionless quantity relating ($R$, $v$, $\mu$, $\rho$) and proportional to v, using the Buckingham's theorem (i.e. find the unique solution to $\Pi = v^\alpha R^\beta \mu^\gamma \rho^\delta$ with $\alpha=1$). This quantity is called the Reynolds number.

c) Out of given 5 quantities ($F_D$, $R$, $v$, $\mu$, $\rho$), we can only form 5 – 3 = 2 dimensionless quantities, given fundamental units of (M,L,T) in this equation. One of these we derived in class, $\Pi_1=F/(v^2\rho R^2)$, to obtain the high velocity (high Reynolds number) limit for $F_D(v)=C\,v^2\rho R^2$. Apply the Buckingham's Π Theorem (i.e. consider the linear homogeneous system for unit powers) to find the second dimensionless quantity $\Pi_2$ that also contains $F_D$ and $v$, in order to find the low Reynolds number (low-velocity) drag force dependence on velocity (i.e. find the unique solution to $\Pi_2 = F_D^\alpha R^\beta v^\gamma \mu^\delta \rho^\varepsilon$ with $\alpha=1$, and $\gamma\neq 0$ since we are interested in $F_D$ as function of v).

**3.** Consider the following model of damped pendulum (easily found from Newton's law and the result of problem 2c)

$$\begin{cases} mL\dfrac{d^2\theta}{dt^2} + \gamma\dfrac{d\theta}{dt} + mg\sin\theta = 0 \\ \theta(0) = \theta_o \\ \dfrac{d\theta}{dt}(0) = \omega_o \end{cases}$$

Here $m$ is the mass at the end of the pendulum of length $L$, $g$ is the acceleration of free fall, $\gamma$ is the drag parameter, and $\theta$ is the angle with respect to the vertical.

a) What are the units of the drag parameter, $\gamma$? Examine units of other terms in this equation to answer.

b) How many model parameters can we eliminate by non-dimensionalization? Hint: although mathematically angles expressed in radians are non-dimensional, you can formally consider radians as a unit, which allows to rescale the angle by its initial value, $\theta_o$.

c) How many different choices for time scale [t] can you construct?

Hint: consider all independent solutions to $\Pi = t/[t] = t\, m^\alpha L^\beta \gamma^\delta g^\varepsilon \omega_o^\lambda$

d) Perform a full non-dimensionalization of this system using any time scale **except for** $[t]=1/\omega_o$ that is considered in the textbook. Try to choose the non-dimensionalization that gives the simplest form of the differential equation.