Math 613 * Fall 2018 * Victor Matveev * Homework 10

1. **Divergence Theorem**

   Explain why question 2(b) of homework 9 **cannot** be solved using the divergence theorem, without using linearity to break up the problem first. Namely, consider the same problem with only two sources, positioned at locations $\mathbf{r}_{1,2}$:

   $$\nabla \cdot \vec{\mathbf{J}} = \sigma_1\, \delta(\mathbf{r} - \mathbf{r}_1) + \sigma_2\, \delta(\mathbf{r} - \mathbf{r}_2)$$

   Note that in homework 9 the flux field (the current) is $\vec{\mathbf{J}} = \nabla\left(D_C C + D_B B^*\right)$, but that's unimportant here. Now, integrate both sides of this equation over some volume (describe the volume that you choose):

   $$\iiint_V \nabla \cdot \vec{\mathbf{J}}\, dV = \iiint_V \left[\sigma_1\, \delta(\mathbf{r} - \mathbf{r}_1) + \sigma_2\, \delta(\mathbf{r} - \mathbf{r}_2)\right]\, dV$$

   Explain carefully why this doesn't lead to any useful result (i.e. explain **exactly** what part of the calculation does not lead to a simple answer when we have two sources instead of one source).

2. **Navier-Stokes Equation**

   Use suffix notation to repeat the last steps in the derivation of the Navier-Stokes equation, starting with the equation in the red box on the 2nd page of the notes, for the case of **compressible** fluid, and **expanding derivatives of all products**. Make sure the final expression is in vector form.

   Notes: https://web.njit.edu/~matveev/Courses/M613_F18/M613-Derivation-Navier-Stokes-Equation.pdf

3. **Two-dimensional flows, streamlines and "convective acceleration"**

   Consider a stationary two-dimensional flow field $\mathbf{u}(\mathbf{r}) = \left(-\dfrac{\alpha y}{x^2+y^2}, \dfrac{\alpha x}{x^2+y^2}, 0\right)$

   a) Show that this flow is incompressible. Therefore, there exists a "stream function" ψ such that $\mathbf{u}(\mathbf{r}) = \left(-\dfrac{\partial \psi}{\partial y}, \dfrac{\partial \psi}{\partial x}, 0\right)$. Find this function ψ by integration. Side note: the relationship between **u** and ψ can also be written as $\mathbf{u}(\mathbf{r}) = -\nabla \times \langle 0, 0, \psi(\mathbf{r})\rangle$ (some books have the opposite signs)

   b) The curves of constant ψ represent the streamlines of the flow: sketch some of these curves ψ=const in the plane.

   c) Find the "convective acceleration" for this flow, $(\mathbf{u} \cdot \nabla)\mathbf{u}$. Are there points where the convective acceleration is zero? Make sure your answer agrees with your picture in part "b"