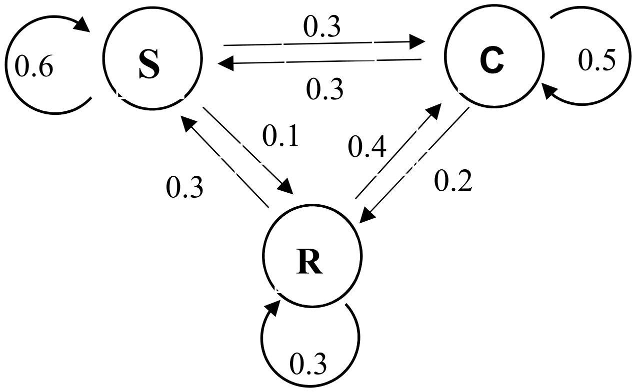
Math 613 * Fall 2018 * Victor Matveev * Homework #11

1. Consider the **discrete state, discrete time** Markov Chain describing a model of weather, with daily transitions between "S" (sunny), "C" (cloudy) and "R" (rainy) days (which are obtained using repeated observation):



a) Write down the Markov Matrix. What is the equilibrium weather probability distribution?

b) Find *all* eigenvectors and eigenvalues (you may use Wolfram Alpha if you like).

c) Write down the explicit solution of this discrete-time dynamical system, assuming that the weather was sunny on day zero.

2. Consider **the discrete state, continuous time** process describing the following chemical reaction, as considered in class:

$$A + A \xrightarrow{k_D} A$$
$$\varnothing \xrightarrow{k_B} A$$

Noting that the number of distinct pairs among n particles equal $n\,(n-1)\,/\,2$, we obtained the following Chemical Master Equation for this reaction:

$$\begin{cases} \dfrac{dp_0}{dt} = -k_B p_0 \\ \dfrac{dp_n}{dt} = k_B\left[p_{n-1} - p_n\right] + \dfrac{k_D}{2}\left[n(n+1)p_{n+1} - n(n-1)p_n\right] \quad (n > 0) \end{cases}$$

Multiplying the master equation by n and summing over all n, we obtained the following ODE describing the evolution of the average number of particles n:

$$\frac{d}{dt}\langle n\rangle = k_B\left(\langle n+1\rangle - \langle n\rangle\right) + \frac{k_D}{2}\left[\left\langle n(n-1)^2\right\rangle - \left\langle n^2(n-1)\right\rangle\right] = k_B + \frac{k_D}{2}\left[\left\langle n^3 - 2n^2 + n\right\rangle - \left\langle n^3 - n\right\rangle\right] = k_B - k_D\left\langle n^2 - n\right\rangle$$

As we noted, this disagrees with the mass-action ODE for the average number of particles: $\dfrac{d}{dt}N_A = k_B - k_D N_A^2$

a) Find a similar ODE describing the evolution of $\langle n^2\rangle$ (multiply the equation by n^2, and sum).

b) Find the relationships between probability values p_n^{EQ} recursively, by setting each Master Equation to zero, starting with n=1, up to n=5.

c) Find the PDE for the probability generating function, $F(z,t) = \sum_{n=0}^{\infty} p_n(t) z^n$. Hint: $\dfrac{\partial^2 F}{\partial z^2} = \sum_{n=0}^{\infty} n(n-1)p_n(t) z^{n-2}$. **Don't solve!**

3. Write down the Markov Chain and the Chemical Master Equations for the following system, without solving (we almost did this in class; note that there will be transitions between p_n and $p_{n\pm 2}$):

$$\left\{A + A \xrightarrow{k_D} \varnothing;\quad \varnothing \xrightarrow{k_B} A\right\}$$