Math 613 * Fall 2018 * Victor Matveev * Homework 2

1. Consider the following Schrödinger equation for the angle-symmetric orbitals of the hydrogen atom (the two terms on the left-hand side are the kinetic and the electrostatic potential energy of the electron, whose sum equals the total energy, $E\psi$):

$$-\frac{\hbar^2}{2m_e}\nabla^2\psi(r)-\frac{e^2}{4\pi\varepsilon_o r}\psi(r)=E\psi \quad \text{where} \quad \nabla^2\psi(r)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi(r)}{\partial r}\right)$$

Here $\psi(r)$ is the **dimensionless** wave function, r is the distance of the electron from the nucleus, m_e is the electron mass, e is the elementary electron charge, and E is the energy of the electron.

a) Determine the units of the Planck constant $\hbar$ and permittivity ε_o, noting that all terms in this equation have units of energy (recall that units of energy are mass · velocity^2 = ML^2/T^2)

b) How many parameters can we eliminate from this problem?

c) Non-dimensionalize this equation (make sure all terms are dimensionless). Hint: you only need to introduce one scale, [r]. Note that you start with the following dimensional variables and parameters: $\{r,\hbar,m_e,e,\varepsilon_o,E\}$

2. Use the phase plot to categorize the stability of the zero equilibrium of the ODE $y'(t)=y^n$ for any positive integer value of n (i.e. for which n is $y_{eq}=0$ stable, unstable, semistable?)

3. Consider the following three ODEs (note that (a) is the population growth model from homework #1):

$$(a)\ \begin{cases}\dfrac{dy}{dt}=y(1-y)\\ n(0)=1.2\end{cases} \qquad (b)\ \begin{cases}\dfrac{dy}{dt}=y\sin y\\ y(0)=-0.1\end{cases} \qquad (c)\ \begin{cases}\dfrac{dy}{dt}=-(\ln y)^3\\ y(0)=1.1\end{cases}$$

- Use the phase plot (the plot of dy/dt vs. y) to determine and sketch the qualitative behavior of solution (y vs. t), for the given initial condition.
- Find all equilibria and analyze their stability using linear stability analysis. Indicate any non-hyperbolic equilibria.
- For each non-hyperbolic equilibrium, use the first non-zero term in the Taylor expansion of the "velocity function" near the equilibrium to determine its stability (asymptotically stable, semi-stable or unstable).

4. Explain in 1-2 sentences why a solution to the 1st order autonomous ordinary differential equation cannot have non-monotonic dependence on time. Argument by contradiction would be particularly simple here; perhaps a phase plot and/or the trajectory plot may help, as well.

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Taylor series review: it's important for this homework and all future homework that you recall the Taylor series of the following fundamental functions, from which the Taylor series of any composite analytic function can be derived by composition, without any differentiation (this applies to quiz #0 problem and problems 3(b) and 3(c) above):

$$e^y=1+y+\frac{y^2}{2!}+\ldots+\frac{y^n}{n!}+\ldots\ (\forall y\in\mathbb{C}) \qquad \frac{1}{1-y}=1+y+y^2+y^3+\ldots\ (|y|<1)$$

$$\cos y=1-\frac{y^2}{2}+\frac{y^4}{4!}-\frac{y^6}{6!}+\ldots\ (\forall y\in\mathbb{C}) \qquad (1+y)^p=1+py+\frac{p(p-1)}{2}y^3+\ldots\ (|y|<1)$$

$$\sin y=y-\frac{y^3}{3!}+\frac{y^4}{5!}-\frac{y^7}{7!}+\ldots\ (\forall y\in\mathbb{C}) \qquad \ln(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\ldots\ (|y|<1)$$