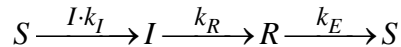


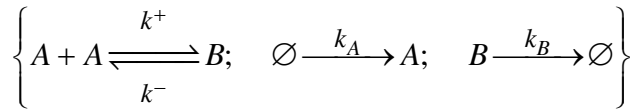
Math 613 * Fall 2018 * Victor Matveev * Homework 4

1. Consider the SIR (susceptible-infected-recovered) epidemiology model with immunity extinction rate k_E (note that in this case the transition chain forms a loop):



- Using the conservation law $(S(t) + I(t) + R(t) = \text{const} = N)$, eliminate the variable $R(t)$
 - Non-dimensionalize the system of 2 ODEs, using time scale $[t] = 1 / k_I$. Denote $\rho_R = k_R / k_I$; $\rho_E = k_E / k_I$
 - Sketch the nullclines and the flow field in the phase plane $(s = S^*, i = I^*)$. Assume $\rho_E \ll 1$; $\rho_R = O(1) \ll N$.
 - Find all equilibria and analyze their linear stability
2. Finish the problem we started in class concerning the bimolecular reaction $C + B \xrightleftharpoons[k^-]{k^+} A$
- Write down the single differential equation for dA/dt that you obtain by eliminating $C(t)$ and $B(t)$ using conservation laws, for initial conditions $C(0)=C_0$, $B(0)=B_0$ and $A(0)=0$
 - Non-dimensionalize this equation using time scale $[t] = 1 / k^-$ and concentration scale $[A] = K_D = k^- / k^+$
 - Find the equilibrium and analyze its stability, both graphically and using linear stability analysis

3. Examine the chemical reactor for B production considered in class:



- Non-dimensionalize this system using time scale $[t] = 1 / k^-$ and concentration scale $[A] = [B] = k^- / k^+$.
- Find the equilibrium, and analyze the linear stability of the equilibrium. You **don't** have to examine whether the equilibrium is a spiral or a node.
- For the special case $k_B = k^-$, sketch the nullclines in the phase plane ($a=A^*$, $b=B^*$), and show the trajectory for the initial condition $A(0)=B(0)=0$

4. Consider the function $f(x) = \int_0^{\infty} \frac{e^{-t} dt}{1 - xt}$

- Obtain the asymptotic expansion of this function for $x \rightarrow 0$ using the familiar geometric series / Taylor series identity $\frac{1}{1-z} = \sum_{k=0}^N z^k + \frac{z^{N+1}}{1-z}$, and then taking the integral term-by-term (the last term will give you an integral representation of the remainder). Use the familiar gamma function identity $\int_0^{\infty} t^k e^{-t} dt = \Gamma(k+1) = k!$
- Take a value of $x=0.002$, and plot the dependence of the partial sum of the asymptotic series, $S_N(x)$, on N . What do you think would be the best estimate for $f(0.002)$?