Math 613 * Fall 2018 * Victor Matveev * Homework 4

1. Consider the SIR (susceptible-infected-recovered) epidemiology model with immunity extinction rate k_E (note that in this case the transition chain forms a loop):

$$S \xrightarrow{I \cdot k_I} I \xrightarrow{k_R} R \xrightarrow{k_E} S$$

   a) Using the conservation law $\left(S(t)+I(t)+R(t)=const=N\right)$, eliminate the variable R(t)

   b) Non-dimensionalize the system of 2 ODEs, using time scale $[t]=1/k_I$. Denote $\rho_R = k_R / k_I$; $\rho_E = k_E / k_I$

   c) Sketch the nullclines and the flow field in the phase plane $\left(s=S^*,\ i=I^*\right)$. Assume $\rho_E << 1$; $\rho_R = O(1) << \mathrm{N}$.

   d) Find all equilibria and analyze their linear stability

2. Finish the problem we started in class concerning the bimolecular reaction $C + B \underset{k^-}{\overset{k^+}{\rightleftharpoons}} A$

   a) Write down the single differential equation for dA/dt that you obtain by eliminating C(t) and B(t) using conservation laws, for initial conditions C(0)=C_o, B(0)=B_o and A(0)=0

   b) Non-dimensionalize this equation using time scale $[t]=1/k^{-}$ and concentration scale $[\mathrm{A}]=K_D=k^-/k^+$

   c) Find the equilibrium and analyze its stability, both graphically and using linear stability analysis

3. Examine the chemical reactor for B production considered in class:

$$\left\{ A + A \underset{k^-}{\overset{k^+}{\rightleftharpoons}} B; \quad \varnothing \xrightarrow{k_A} A; \quad B \xrightarrow{k_B} \varnothing \right\}$$

   a) Non-dimensionalize this system using time scale $[t]=1/k^-$ and concentration scale $[A]=[B]=k^-/k^+$.

   b) Find the equilibrium, and analyze the linear stability of the equilibrium. You **don't** have to examine whether the equilibrium is a spiral or a node.

   c) For the special case $k_B = k^-$, sketch the nullclines in the phase plane (a=A*, b=B*), and show the trajectory for the initial condition A(0)=B(0)=0

4. Consider the function $f(x)=\int_0^\infty \frac{e^{-t}dt}{1-xt}$

   a) Obtain the asymptotic expansion of this function for x → 0 using the familiar geometric series / Taylor series identity $\frac{1}{1-z}=\sum_{k=0}^{N} z^k + \frac{z^{N+1}}{1-z}$, and then taking the integral term-by-term (the last term will give you an integral representation of the remainder). Use the familiar gamma function identity $\int_0^\infty t^k e^{-t}dt = \Gamma(k+1)=k!$

   b) Take a value of x=0.002, and plot the dependence of the partial sum of the asymptotic series, S_N(x), on N. What do you think would be the best estimate for $f(0.002)$?