Math 613 * Fall 2018 * Victor Matveev * Homework 5

1. Consider the function $f(x)=\int_0^\infty \frac{e^{-t}dt}{1-xt}$

a) Obtain the asymptotic expansion of this function for x → 0 using the familiar geometric series / Taylor series identity $\frac{1}{1-z}=\sum_{k=0}^{N} z^k+\frac{z^{N+1}}{1-z}$, and then taking the integral term-by-term (the last term will give you an integral representation of the remainder). Use the gamma function identity $\int_0^\infty t^k e^{-t}dt=\Gamma(k+1)=k!$

b) Take a value of x=**0.1**, and plot the dependence of the partial sum of the asymptotic series, S_N(x), on *N*, for *N* from **1 to 22**. What do you think would be the best estimate for $f(\mathbf{0.1})$?

2. Obtain asymptotic series representation of the solution to the following ODE, up to second order in ε:

$$\begin{cases} \dfrac{d^2y}{dt^2}=\dfrac{1}{1+\varepsilon y} \\ y(0)=0;\ \dfrac{dy}{dt}(0)=1 \end{cases}$$

Hint: expand the right-hand side in the Taylor series up to second order in ε, and consider solutions in the form $y(t)=y_o(t)+\varepsilon y_1(t)+\varepsilon^2 y_2(t)+O(\varepsilon^3)$, with the following initial conditions:

$$\begin{cases} y_0(0)=0 \\ \dfrac{dy_o}{dt}(0)=1 \end{cases} \quad \begin{cases} y_1(0)=0 \\ \dfrac{dy_1}{dt}(0)=0 \end{cases} \quad \begin{cases} y_2(0)=0 \\ \dfrac{dy_2}{dt}(0)=0 \end{cases}$$

3. Show by direct differentiation that $\rho(x,t)=\frac{1}{\sqrt{4\pi kt}}\exp\left(-\frac{x^2}{4kt}\right)$ satisfies the partial differential equation of diffusion on an infinite domain: $\rho_t(x,t)=k\rho_{xx}(x,t)$

Fourth problem on the other side of the page

4. Consider a substance of linear density $\rho(x,t)$ diffusing in a tube of length ***L*,** with a sealed left end, and with extra molecules injected according to the source function $Q(x) = \alpha \sin\left(\frac{\pi x}{L}\right)$ ($\alpha = const$):

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = k\dfrac{\partial^2 \rho}{\partial x^2} + \alpha \sin\left(\dfrac{\pi x}{L}\right) & (0 < x < L,\ \ t > 0) \\ \dfrac{\partial \rho}{\partial x}(0,t) = 0 \\ \dfrac{\partial \rho}{\partial x}(L,\, t) = \beta = const \\ \rho(x,0) = \gamma = const \end{cases}$$

You can non-dimensionalize this equation if you wish.

a) Explain in one sentence what the boundary condition $\frac{\partial \rho}{\partial x}(L,\, t) = \beta$ describes, physically, for different signs of constant β. Hint: recall the relationship between the flux and the density: $\phi(x,t) = -k\rho_x(x,t)$

b) Integrate both sides of this equation from 0 to *L* to obtain an ordinary differential equation (conservation law) describing the time evolution of $N(t) = \int_0^L \rho(x,t)\,dx$, which is the total number of molecules (or total mass, charge, energy, probability, etc.). Solve this equation to find $N(t)$

c) Use your results to part "b" to answer the following question: does the equilibrium exist for **any** value of constant β? Why or why not?

d) Find the density function at equilibrium