Math 613 * Fall 2018 * Victor Matveev * Homework 6

1. Derive the diffusion equation in a one-dimensional tube/cable for the case where the tube cross-section varies along the length, *A*(*x*) (you don't have to consider the source term). Bonus question: does the resulting equation remind you of anything from Calculus III, in the special case *A*(*x*) = const·*x* or *A*(*x*) = const·x^2 ?

Hint: the Fick's law of diffusion $q = -k\frac{\partial \rho}{\partial x}$ still has the same form in this case; you only have to modify the conservation law derivation, and then combine the two equations. As a reminder, below is the integral derivation of the conservation law, in the case of constant cross-section and zero sources:

$$N_{ab}(t) \equiv \iiint_V \rho(\mathbf{r},t)\underbrace{dV}_{A\,dx} = A\int_a^b \rho(x,t)\,dx$$

$$\Rightarrow \frac{dN_{ab}(t)}{dt} = A\frac{d}{dt}\int_a^b \rho(x,t)\,dx = A\int_a^b \frac{\partial \rho(x,t)}{\partial t}dx = (\text{inflow rate}) - (\text{outflow rate})$$

$$= Aq(a) - Aq(b) = -A\int_a^b \frac{\partial q}{\partial x}dx$$

$$\Rightarrow A\int_a^b \left(\frac{\partial q}{\partial t} + \frac{\partial q}{\partial x}\right)dx = 0 \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0} \text{ (since the interval of integration is arbitrary)}$$

2. Solve the following wave / advection equation using the method of characteristics. Make two plots: plot the characteristics, and plot the solution at *t*=10 (note: units are non-dimensional)

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\sqrt{x}}{1+t}\frac{\partial \rho}{\partial x} = 0 & (t > 0) \\ \rho(x,0) = \rho_o(x) = 4x & (0 < x < +\infty) \end{cases}$$

3. Consider the traffic flow problem discussed in class, but with no speed limit, and a different dependence of velocity on density: $u(\rho) = \alpha \ln \frac{\rho_{\max}}{\rho}$ (assume constants $\alpha, \rho_{\max}, \beta, \gamma$ are positive and real):

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\big(\rho u(\rho)\big) = 0 & (-\infty < x < +\infty,\ \ t > 0) \\ \rho(x,0) \equiv \rho_o(x) = \beta \exp(-\gamma x) \end{cases}$$

a) Non-dimensionalize this problem

b) Use the chain rule to convert the problem to the standard advection form $\left.\frac{d\rho}{dt}\right|_{\Phi} = \frac{\partial \rho}{\partial t} + \left.\frac{dx}{dt}\right|_{\Phi}\frac{\partial \rho}{\partial x} = 0$

c) Find the characteristic curves and plot them (consider non-dimensional position in the range $-1 < x^* < 1$)

d) Find and plot the solution at *t* = 0 and at some future time *t* > 0

e) Repeat steps (c-d) for a different initial condition: $\beta \exp(+\gamma x)$ $(\beta, \gamma = const > 0)$. Choose initial non-dimensional position on the interval $-1 < x^*(0) < 1$. Do you notice anything funny about the characteristic curves at large time values? Explain.