Math 613 * Fall 2018 * Victor Matveev * Homework 8

1. Re-write the following expressions using index (Einstein) notation (do not simplify):

$$a)\ \ \mathrm{trace}(AB) \qquad b)\ \ \det(A)$$

2. Use index notation to derive the product rules for the following expressions, converting the results back to vector form:

a) $\nabla\cdot\left(\dfrac{\mathbf{U}}{\phi}\right)$ b) $\nabla(\mathbf{U}\cdot\mathbf{V})$ c) $\nabla\cdot(\mathbf{U}\times\mathbf{V})$

3. Use the identity $\varepsilon_{ijk}\varepsilon_{ilm}=\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}$ to expand and simplify the expression $\nabla\times(\nabla\times\mathbf{U})$. Make sure to convert the final expression back to vector form.

4. Find the electric potential φ both inside and outside a uniformly charged sphere of radius r_0 and total charge of Q, by directly solving the corresponding Poisson-Laplace equations; plot the electric potential as a function of distance from the origin:

$$\begin{cases}\nabla^2\phi_{in}=\dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2\dfrac{d\phi_{in}}{dr}\right)=\dfrac{\rho}{\varepsilon_0} & (r<r_0)\\ \nabla^2\phi_{out}=\dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2\dfrac{d\phi_{out}}{dr}\right)=0 & (r>r_0)\\ \phi_{in}(0)\ \text{bounded};\ \ \phi_{out}(r\to\infty)\to 0 & \\ \phi_{in}(r_0)=\phi_{out}(r_0);\ \ \phi_{in}'(r_0)=\phi_{out}'(r_0) & \end{cases}$$

Finally, to compare your results to the results of homework 7, find and plot the electric field strength both inside and outside of the sphere using the definition of electric potential: $E=|\mathbf{E}|=\left|\dfrac{d\phi}{dr}\right|$

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Some index/suffix/Einstein/tensor notation basics:

$\mathbf{u}=A\mathbf{v}:\quad u_i=A_{ij}v_j$

$\mathbf{u}=\mathrm{I}\,\mathbf{u}:\quad u_i=\delta_{ij}u_j$

$\mathbf{a}=\mathbf{b}\times\mathbf{c}:\quad a_i=\varepsilon_{ijk}b_jc_k$

$A=B\,C:\quad A_{ij}=B_{ik}C_{kj}$

$grad\ \phi=\nabla\phi:\quad \partial_i\phi$

$div\ \mathbf{U}=\nabla\cdot\mathbf{U}:\quad \partial_k U_k$

$curl\,\mathbf{U}=\nabla\times\mathbf{U}:\quad \varepsilon_{ijk}\partial_j U_k$

$\varepsilon_{ijk}=\varepsilon_{jki}=\varepsilon_{kij}=-\varepsilon_{jik}=-\varepsilon_{ikj}=-\varepsilon_{kji}$

$\delta_{ij}=\delta_{ji}$