Math 613 * Fall 2018 * Victor Matveev * Homework 8

1. In this problem you will determine the stability of the non-hyperbolic equilibrium of the following system:

$$\begin{cases} \dfrac{dx}{dt} = -x^3 - 2xy^2 \\ \dfrac{dy}{dt} = -2yx^2 - 4y^3 \end{cases}$$

a) Show that this vector flow field irrotational (has zero curl). To do this, re-write this equation in vector form $\dfrac{d\mathbf{r}}{dt} = \mathbf{U}(\mathbf{r}) = \begin{pmatrix} -x^3 - 2xy^2 \\ -2yx^2 - 4y^3 \\ 0 \end{pmatrix}$, where we introduced a z-variable with trivial dynamics. Now, show that $\nabla \times \mathbf{U} = 0$.

b) Therefore, **U** can be written as a gradient of potential of some scalar function φ. Find this potential by solving the system of equations $\mathbf{U} = -\nabla\phi \;\Rightarrow\; \left\langle -\dfrac{\partial\phi}{\partial x}, -\dfrac{\partial\phi}{\partial y} \right\rangle = \left\langle -x^3 - 2xy^2, -2yx^2 - 4y^3 \right\rangle$. Even if you don't recall how to do this systematically, you can still determine ϕ by trial and error.

c) Use the result of part "b", in particular the form of the potential function ϕ, to examine the stability of the non-hyperbolic zero equilibrium of the original system.

2. Consider the calcium-buffer system we considered in class:

$$\begin{cases} \text{Calcium:} & \dfrac{\partial C}{\partial t} = D_C \nabla^2 C - k^+ BC + k^- B^* + \displaystyle\sum_{k=1}^{\infty} \sigma_{Ca}^k \, \delta(\mathbf{r} - \mathbf{r}_k) \\ \text{Unbound buffer:} & \dfrac{\partial B}{\partial t} = D_B \nabla^2 B - k^+ BC + k^- B^* \\ \text{Bound buffer:} & \dfrac{\partial B^*}{\partial t} = D_B \nabla^2 B^* + k^+ BC - k^- B^* \end{cases}$$

a) Write down the conservation law for the total calcium concentration, $C_T = C + B^*$.

Note: as we learned previously, any conservation law can be written in the following form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{\mathbf{J}} = Q_{source}(\mathbf{x}, t)$$

b) Find the equilibrium value for the concentration combination $D_C C + D_B B^*$ (hint: we did a similar calculation several times in class)