Math 613 * Fall 2018 * Victor Matveev

Derivation of the Navier-Stokes Equation

1. Relationship between force (stress), stress tensor, and strain:

- Consider any sub-volume inside the fluid, with variable unit normal **n** to the surface of this sub-volume.
- Definition: **Force per area** at each point along the surface of this sub-volume is called the **stress vector T**.

When fluid is not in motion, **T** is pointing parallel to the outward normal **n**, and its magnitude equals pressure *p*: **T** = –*p* **n**. However, if there is shear flow, the two are not parallel to each other, so we need a marix (a tensor), called the **stress-tensor** σ, to express the force direction relative to the normal direction, defined as follows:

$$\mathbf{T} = \mathbf{n}^T \sigma \quad \text{or} \quad T_k = n_j \sigma_{jk}$$

As we will see below, σ is a symmetric matrix, so we can also write

$$\mathbf{T} = \sigma \mathbf{n} \quad \text{or} \quad T_k = \sigma_{kj} n_j$$

The difference in directions of **T** and **n** is due to the non-diagonal “deviatoric” part of the stress tensor, τ_{jk}, which makes the force deviate from the normal:

$$\sigma_{jk} = -p\,\delta_{jk} + \tau_{jk} \quad \text{where } p \text{ is the usual (scalar) pressure}$$

From general considerations, it is clear that the only source of such “skew” / ”deviatoric” force in fluid is the shear component of the flow, described by the shear (non-diagonal) part of the “strain rate” tensor e_{kj}:

$$\tau_{jk} = 2\mu e_{jk} + \left(\lambda - \frac{2}{3}\mu\right) e_{mm}\,\delta_{jk} \quad \text{where } e_{jk} = \frac{1}{2}\left(\partial_j u_k + \partial_k u_j\right) \quad \text{(strain rate tensor)}$$

Note: the funny construct $\lambda - 2\mu/3$ guarantees that the part of τ proportional to μ has a zero trace.

The two terms above represent the most general (and the only possible) mathematical expression that depends on first-order velocity derivatives and is invariant under coordinate transformations like rotations.

Thus, we have:

$$\begin{aligned}
\sigma_{jk} &= -p\,\delta_{jk} + \tau_{jk} \\
&= -p\,\delta_{jk} + 2\mu e_{jk} + \left(\lambda - \frac{2\mu}{3}\right) e_{mm}\,\delta_{jk} \\
&= \delta_{jk}\left[-p + \left(\lambda - \frac{2\mu}{3}\right)\partial_m u_m\right] + \mu\left(\partial_j u_k + \partial_k u_j\right)
\end{aligned}$$

Proportionality constant ***μ*** between shear stress and the shear strain rate is called ***dynamic viscosity***.

Proportionality constant *λ* between compression stress and compression strain rate is ***volume viscosity***.

2. Second Newton's law: The Cauchy Momentum Balance Equation

The Navier-Stokes equation represents the 2nd Newton's law: the rate of change of the integral of momentum volume density $X_k = \rho u_k$ equals sum of forces. The derivation is also analogous to the derivation of the continuity / mass conservation law (see older hand-out / lecture notes): $\rho_t + \nabla \cdot (\rho \mathbf{u}) = \sum \sigma_{\text{SOURCE}}$.
https://web.njit.edu/~matveev/Courses/M613_F18/DivergenceTheorem_DiffusionEquation_2018.pdf

According to the 2nd Newton's law, the source of the momentum (mass times velocity) change is the sum of forces. Thus, our starting point is the conservation of each component of momentum, $X_k = \rho u_k$:

$$\frac{d}{dt}\iiint_V X_k dV + \begin{pmatrix} \text{X}_k \text{ loss} \\ \text{through } \partial\text{V} \end{pmatrix} = \sum F_k$$

$$\Rightarrow \frac{d}{dt}\iiint_V X_k dV + \oiint_{\partial V} (X_k \mathbf{u}) \cdot \mathbf{n}\, dS = \iiint_V f_k^{BODY}\, dV + \oiint_{\partial V} f_k^{SURFACE} dS$$

Here we used the fact that X_k leaves the volume when being carried out by the velocity field **u**. The forces are:

- f_k^{BODY} is the k^{th} component of the body force **per unit volume**; for the case of gravity we have $f_k^{BODY} = \rho g_k$
- $f_k^{SURFACE}$ is the k^{th} component of the surface force **per unit area**, the stress, so $f_k^{SURFACE} = T_k = n_j\, \sigma_{jk}$

Therefore, we obtain:

$$\iiint_V \frac{\partial X_k}{\partial t} dV + \oiint_{\partial V} (X_k \mathbf{u}) \cdot \mathbf{n}\, dS = \iiint_V \rho g_k\, dV + \oiint_{\partial V} T_k dS$$
$$= \iiint_V \rho g_k\, dV + \oiint_{\partial V} \underbrace{n_j\, \sigma_{jk}}_{\mathbf{n}\cdot\boldsymbol{\sigma}_k} dS$$

Here we introduced the notation $T_k = n_j\, \sigma_{jk} = \mathbf{n} \cdot \boldsymbol{\sigma}_k$, to make the application of the Divergence Theorem more obvious. Let's now apply the divergence theorem to the two surface integrals in the above momentum equation:

$$\iiint_V \left[\frac{\partial X_k}{\partial t} + \nabla \cdot (X_k \mathbf{u}) \right] dV = \iiint_V (\rho g_k + \nabla \cdot \boldsymbol{\sigma}_k)\, dV \quad \text{where} \quad \boldsymbol{\sigma}_k = \sigma_{jk}$$

Since the volume we have chosen is arbitrary, we can equate the integrands:

$$\frac{\partial X_k}{\partial t} + \nabla \cdot (X_k \mathbf{u}) = \rho g_k + \nabla \cdot \boldsymbol{\sigma}_k = \rho g_k + \partial_j \sigma_{jk}$$

Now, all that is left is to plug in the definition $X_k = \rho u_k$, and the stress-strain relationship:

$$\boxed{\frac{\partial(\rho u_k)}{\partial t} + \partial_j (\rho u_k u_j) = \rho g_k + \partial_j \left[\delta_{jk} \left(-p + \left(\lambda - \frac{2\mu}{3} \right) \partial_m u_m \right) + \mu (\partial_j u_k + \partial_k u_j) \right]}$$

Taking into account incompressive fluid case, ρ=const, $\nabla \cdot \mathbf{u} = 0$, performing some simplifications and converting back to vector notation, we obtain the Navier-Stokes equation of incompressible fluid flow (with gravity):

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \rho\, \mathbf{g} + \mu \nabla^2 \mathbf{u}$$

HOMEWORK: starting with the equation in the red box, expand derivatives of all products, and write down the Navier Stokes equation for the case of *compressible* fluid (non-constant ρ). Make sure to convert the final result to vector notation. The result should contain gradients, divergences, and Laplacian(s).