Math 613 * Fall 2018 * Final Examination * Victor Matveev

Note: bold quantities are vectors or vector fields; *italics* **denote scalars or scalar fields.**

1. **(10pts)** Consider the vector field $\mathbf{u}(\mathbf{r}) = \langle e^{x+y},\ y^2,\ 0 \rangle$. Compute the following derivatives (all of which appear in the generalized compressible Navier-Stokes equation):

a) $$(\mathbf{u}\cdot\nabla)\mathbf{u} = \left(e^{x+y}\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right)\langle e^{x+y},\ y^2,\ 0\rangle$$

$$= \left\langle \left(e^{x+y}\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right)e^{x+y},\ \left(e^{x+y}\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right)y^2,\ \left(e^{x+y}\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right)0 \right\rangle$$

$$= \boxed{\left\langle e^{2(x+y)} + y^2 e^{x+y},\ 2y^3,\ 0 \right\rangle}$$

b) $$\nabla(\nabla\cdot\mathbf{u}) = \text{grad div } \mathbf{u} = \text{grad}\left(\frac{\partial e^{x+y}}{\partial x} + \frac{\partial y^2}{\partial y}\right) = \text{grad}\left(e^{x+y} + 2y\right) = \boxed{\langle e^{x+y},\ 2,\ 0\rangle}$$

c) $$\Delta\mathbf{u} \equiv \nabla^2\mathbf{u} = \langle \nabla^2 e^{x+y},\ \nabla^2 y^2,\ \nabla^2 0\rangle = \langle e^{x+y} + \mathrm{e}^{x+y},\ 2,\ 0\rangle = \boxed{\langle 2e^{x+y},\ 2,\ 0\rangle}$$

2. **(12pts)** Non-dimensionalize the following one-dimensional advection-diffusion-absorption equation for volume mass density function $\rho(\mathbf{r}, t)$, reducing the number of parameters as much as possible:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = D\dfrac{\partial^2 \rho}{\partial x^2} + u_0 \dfrac{\partial \rho}{\partial x} - \gamma\rho \\ \rho(x \to \pm\infty, t) = \rho_0 = const \end{cases} \Rightarrow \boxed{[\rho] = \rho_0;\ \rho^* = \frac{\rho}{\rho_0}} \Rightarrow \boxed{\rho^*(x^* \to \pm\infty, t^*) = 1}$$

$$\frac{[\rho]\partial\rho^*}{[t]\,\partial t^*} = D\frac{[\rho]}{[x]^2}\frac{\partial^2\rho^*}{\partial x^{*2}} + u_0\frac{[\rho]}{[x]}\frac{\partial\rho^*}{\partial x^*} - \gamma[\rho]\rho^* \,\Bigg\|\, \times\frac{[t]}{[\rho]} \Rightarrow \boxed{\frac{\partial\rho^*}{\partial t^*} = D\frac{[t]}{[x]^2}\frac{\partial^2\rho^*}{\partial x^{*2}} + u_0\frac{[t]}{[x]}\frac{\partial\rho^*}{\partial x^*} - \gamma[t]\rho^*}$$

We already got rid of one parameter in the boundary condition, so we can eliminate two more (out of three) parameters in the equation above, by setting them to equal 1, therefore we have three choices:

1. $D\dfrac{[t]}{[x]^2} = u_0\dfrac{[t]}{[x]} = 1 \Rightarrow \dfrac{D}{[x]} = u_0 \Rightarrow \begin{cases} [x] = \dfrac{D}{u_0} \\ [t] = \dfrac{[x]}{u_0} = \dfrac{D}{u_0^2} \end{cases} \Rightarrow \boxed{\dfrac{\partial\rho^*}{\partial t^*} = \dfrac{\partial^2\rho^*}{\partial x^{*2}} + \dfrac{\partial\rho^*}{\partial x^*} - p\rho^*}$ where $p = \gamma[t] = \dfrac{\gamma D}{u_0^2}$

2. $D\dfrac{[t]}{[x]^2} = \gamma[t] = 1 \Rightarrow \dfrac{D}{[x]^2} = \gamma \Rightarrow \begin{cases} [x] = \sqrt{\dfrac{D}{\gamma}} \\ [t] = \dfrac{1}{\gamma} \end{cases} \Rightarrow \boxed{\dfrac{\partial\rho^*}{\partial t^*} = \dfrac{\partial^2\rho^*}{\partial x^{*2}} + p\dfrac{\partial\rho^*}{\partial x^*} - \rho^*}$ where $p = u_0\dfrac{[t]}{[x]} = \dfrac{u_0}{\sqrt{\gamma D}}$

3. $u_0\dfrac{[t]}{[x]} = \gamma[t] = 1 \Rightarrow \begin{cases} [x] = \dfrac{u_0}{\gamma} \\ [t] = \dfrac{1}{\gamma} \end{cases} \Rightarrow \boxed{\dfrac{\partial\rho^*}{\partial t^*} = p\dfrac{\partial^2\rho^*}{\partial x^{*2}} + \dfrac{\partial\rho^*}{\partial x^*} - \rho^*}$ where $p = D\dfrac{[t]}{[x]^2} = D\dfrac{1/\gamma}{u_0^2/\gamma^2} = \dfrac{\gamma D}{u_0^2}$

3. (16pts) Consider the following 2D flow:

$$\begin{cases} \dfrac{dx}{dt} = y + x^2 \\ \dfrac{dy}{dt} = x + y^2 \end{cases}$$

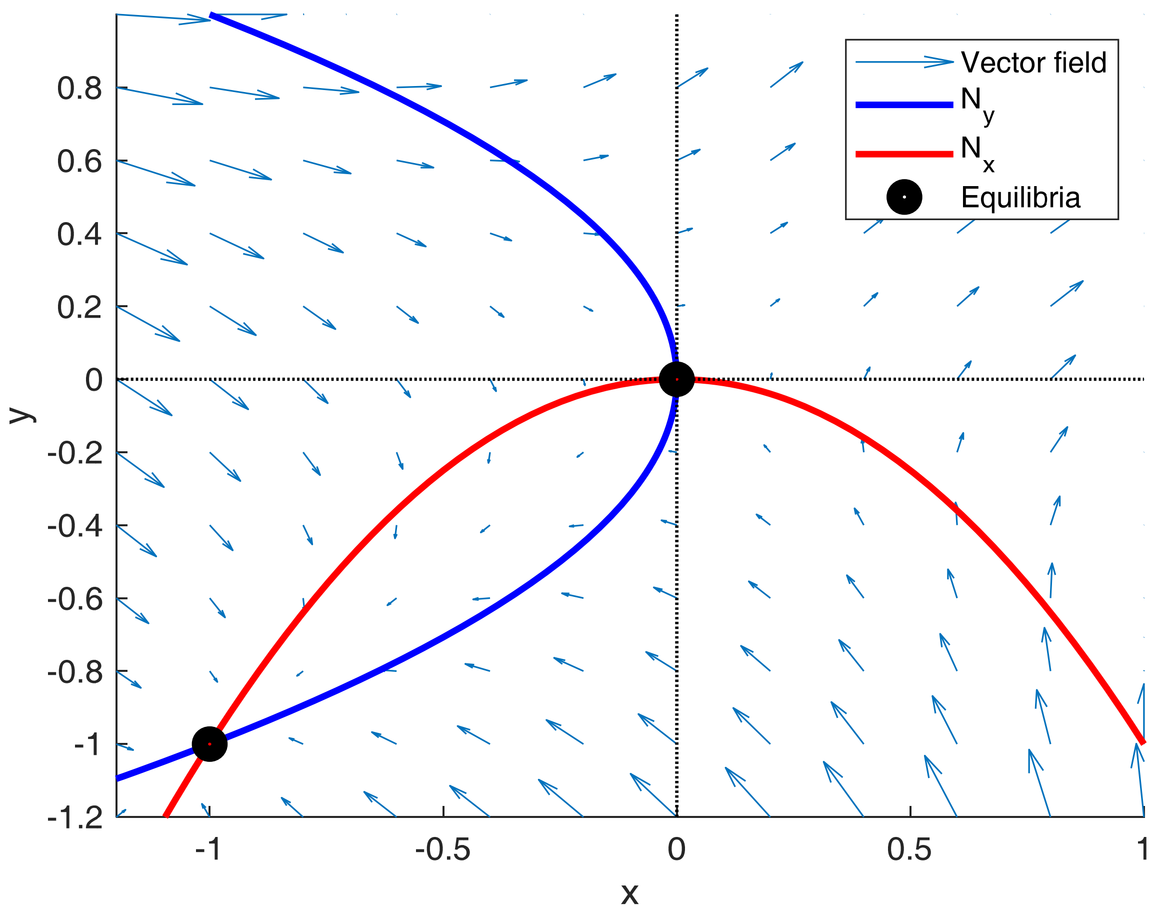
a) Find all equilibria of this system, and analyze their stability using linear stability analysis.

b) Sketch the nullclines.

c) Make a rough plot of the flow field. Hint: start by showing the flow along the coordinate axes and the nullclines

$$\begin{cases} \dfrac{dx}{dt} = y + x^2 = 0 \Rightarrow y_{eq} + x_{eq}^2 = 0 \Rightarrow x_{eq}^2 = -y_{eq} \\ \dfrac{dy}{dt} = x + y^2 = 0 \Rightarrow x_{eq} + y_{eq}^2 = 0 \Rightarrow y_{eq}^2 = -x_{eq} \end{cases} \Rightarrow y_{eq} = -y_{eq}^4 \Rightarrow y_{eq} \in \{-1, 0\} \Rightarrow \boxed{(x_{eq}, y_{eq}) \in \{(-1,-1), (0,0)\}}$$

$$J = \begin{bmatrix} 2x_{eq} & 1 \\ 1 & 2y_{eq} \end{bmatrix} = \begin{cases} x_{eq} = y_{eq} = 0: \ J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ Det } J = -1 < 0 \Rightarrow \boxed{\textit{Saddle} \text{ point (unstable)}} \\ x_{eq} = y_{eq} = -1: J = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \text{ Det } J = 2 > 0; \text{ Trace } J = -4 < 0; \ T^2 - 4D > 0 \Rightarrow \boxed{\textit{Stable node}} \end{cases}$$



4. (18pts) Consider the continuous-time stochastic process describing the following chemical reaction:

$$\left\{ A \xrightarrow{k_D} \varnothing; \quad \varnothing \xrightarrow{k_B} A \right\}$$

a) Write down the Chemical Master Equations (CME).

$$\begin{cases} \dfrac{dp_0}{dt} = -k_B p_0 + k_D p_1 \\ \dfrac{dp_n}{dt} = k_B \left[p_{n-1} - p_n \right] + k_D \left[(n+1) p_{n+1} - n p_n \right] \quad (n > 0) \end{cases}$$

b) Find the equation for the evolution of the second moment, $\dfrac{d\langle n^2 \rangle}{dt}$.

$$\frac{d}{dt}\sum_{n=0}^{\infty} n^2 p_n = k_B \left[\sum_{n=1}^{\infty} n^2 p_{n-1} - \sum_{n=0}^{\infty} n^2 p_n \right] + k_D \left[\sum_{n=0}^{\infty} n^2 (n+1) p_{n+1} - \sum_{n=0}^{\infty} n^3 p_n \right]$$

$$\Rightarrow \frac{d}{dt}\sum_{n=0}^{\infty} n^2 p_n = k_B \left[\sum_{m=0}^{\infty} (m+1)^2 p_m - \sum_{n=0}^{\infty} n^2 p_n \right] + k_D \left[\sum_{m=0}^{\infty} (m-1)^2 m p_m - \sum_{n=0}^{\infty} n^3 p_n \right]$$

$$\Rightarrow \frac{d}{dt}\langle n^2 \rangle = k_B \left[\left\langle (n+1)^2 \right\rangle - \left\langle n^2 \right\rangle \right] + k_D \left[\left\langle n(n-1)^2 \right\rangle - \left\langle n^3 \right\rangle \right]$$

$$= k_B \left[2\langle n \rangle + 1 \right] + k_D \left[\left\langle n^3 - 2n^2 + n \right\rangle - \left\langle n^3 \right\rangle \right] = \boxed{k_B \left[2\langle n \rangle + 1 \right] + k_D \left[-2\langle n^2 \rangle + \langle n \rangle \right]}$$

c) Find the partial differential equation (PDE) for the probability-generating function, $F(z,t) = \sum_{n=0}^{\infty} p_n(t) z^n$

$$\frac{d}{dt}\sum_{n=0}^{\infty} z^n p_n = k_B \left[\sum_{n=1}^{\infty} z^n p_{n-1} - \sum_{n=0}^{\infty} z^n p_n \right] + k_D \left[\sum_{n=0}^{\infty} (n+1) z^n p_{n+1} - \sum_{n=0}^{\infty} n z^n p_n \right]$$

$$\Rightarrow \frac{\partial F}{\partial t} = k_B \left[\sum_{m=0}^{\infty} z^{m+1} p_m - F(z) \right] + \frac{k_D}{2} \left[\sum_{m=1}^{\infty} m z^{m-1} p_m - z \frac{\partial F}{\partial z} \right]$$

$$\Rightarrow \frac{\partial F}{\partial t} = k_B \left[zF - F \right] + k_D \left[\frac{\partial F}{\partial z} - z \frac{\partial F}{\partial z} \right] \Rightarrow \boxed{\frac{\partial F}{\partial t} = (z-1) \left[k_B F - k_D \frac{\partial F}{\partial z} \right]}$$

d) Find the equilibrium probability distribution. Make sure that completeness is satisfied: $\sum_{n=0}^{\infty} p_n = 1$

$$\frac{\partial F_{eq}}{\partial t} = 0 \Rightarrow k_B F_{eq} - k_D \frac{dF_{eq}}{dz} = 0 \Rightarrow \frac{dF_{eq}(z)}{dz} = \frac{k_B}{k_D} F(z) \Rightarrow \boxed{F_{eq}(z) = Ce^{\lambda z}} \quad \text{where } \lambda = \frac{k_B}{k_D}$$

$$\sum_{n=0}^{\infty} p_n = 1 \Rightarrow F_{eq}(1) = Ce^{\lambda} = 1 \Rightarrow C = e^{-\lambda} \Rightarrow F_{eq}(z) = e^{-\lambda} e^{\lambda z} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} z^n \Rightarrow \boxed{p_n^{eq} = \frac{\lambda^n}{n!} e^{-\lambda}}$$

Of course, you would find the same result if you used the CME instead of $F(z,t)$

5. (16pts) Convert to index notation, then use index notation to expand or simplify, and finally convert the result back to vector notation (here $\mathbf{U}$ is a vector field, ϕ is a scalar field, and $\mathbf{r}$ is the position vector: $\mathbf{r} \equiv x_j, \;\; j = 1,2,3$):

a) $\nabla \times (\phi \mathbf{U}) = \varepsilon_{klm} \partial_l (\phi U_m) = \phi \varepsilon_{klm} (\partial_l U_m) + \varepsilon_{klm} (\partial_l \phi) U_m = \boxed{\phi \nabla \times \mathbf{U} + \nabla \phi \times \mathbf{U}}$

b) $\nabla \times (\mathbf{r} \times \mathbf{U}) = \varepsilon_{ijk} \partial_j (\mathbf{r} \times \mathbf{U})_k = \varepsilon_{ijk} \partial_j (\varepsilon_{knm} x_n U_m) = (\delta_{in} \delta_{jm} - \delta_{im} \delta_{jn}) \partial_j (x_n U_m)$

$$= \partial_m (x_i U_m) - \partial_n (x_n U_i) = x_i \partial_m U_m + U_m \underbrace{\partial_m x_i}_{\delta_{mi}} - x_n \partial_n U_i - U_i \underbrace{\partial_n x_n}_{\delta_{nn}=3}$$

$$= x_i \partial_m U_m + U_i - x_n \partial_n U_i - 3U_i = \boxed{\mathbf{r} \; \nabla \cdot \mathbf{U} - (\mathbf{r} \cdot \nabla) \mathbf{U} - 2\mathbf{U}}$$

6. (16pts) Consider the following advection equation (assume that the equation is already non-dimensionalized):

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + 3xt^2 \dfrac{\partial \rho}{\partial x} = 0 & (t > 0) \\ \rho(x_0, 0) = \rho_o(x_0) = x_0^2 & (-\infty < x_0 < +\infty) \end{cases}$$

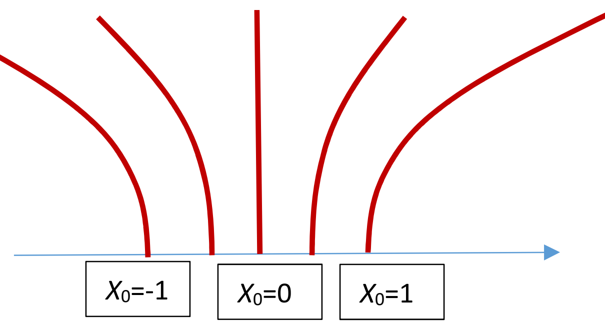
a) Find and plot the characteristics corresponding to 3 values of x_0: $x_0 = -1$, $x_0 = 0$, $x_0 = 1$.

b) Is there a shock-wave / break-up? **No shock wave (see plot of characteristics)**

c) Find the solution, and make a rough plot of $\rho(x, t)$ at t=1 and at t=2.

$$\frac{\partial \rho}{\partial t} + \underbrace{3xt^2}_{\left.\frac{dx}{dt}\right|_\Phi} \frac{\partial \rho}{\partial x} = \left.\frac{d\rho(x_\Phi(t), t)}{dt}\right|_\Phi = 0 \Rightarrow \left.\frac{dx}{dt}\right|_\Phi = 3xt^2 \Rightarrow \frac{dx}{x} = 3t^2\, dt$$

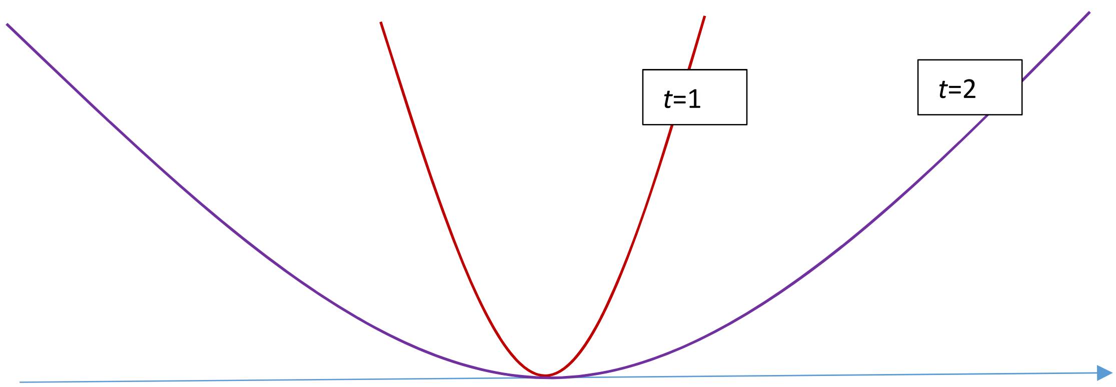
$$\Rightarrow \int_0^t \ldots dt \Rightarrow \ln x - \ln x_0 = \ln \frac{x}{x_0} = t^3 \Rightarrow \boxed{x(t)|_\Phi = x_0 \exp(t^3)}$$



$$\Rightarrow x_0 = x \exp(-t^3) \Rightarrow \rho(x,t) = F(x_0(x,t)) = F(x \exp(-t^3))$$

Now, find the unknown function using the initial condition
(advection simply propagates the initial condition forward along the characteristic curves):

$$\rho(x_0, 0) = F\left(x_0 \underbrace{\exp 0}_{=1}\right) = \rho_o(x_0) = x_0^2 \Rightarrow F(x_0) = x_0^2 \Rightarrow \boxed{\rho(x,t) = x_0^2 \exp(-2t^3)}$$



7. (12pts) Consider a charged spherical shell, with charge distributed within $r_0 < r < r_1$ according to

$$\rho(\mathbf{r}) = \rho(r) = \begin{cases} \gamma r, & r_0 < r < r_1 \quad (\gamma = const) \\ 0, & r < r_0 \text{ or } r > r_1 \end{cases}$$

Apply the divergence theorem to the Gauss law $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ to find the electric field $\mathbf{E}(r)$ in three regions:

(a) $r < r_0 \Rightarrow$ **Integrate over a sphere of radius** $r < r_0$

$$\rho(r < r_0) = 0 \Rightarrow \iiint_{V(r<r_0)} \nabla \cdot \mathbf{E}(\mathbf{r})\, dV = \oint\!\!\oint_{\partial Vr} \mathbf{E} \cdot d\mathbf{S} = 0 \Rightarrow \boxed{\mathbf{E}(\mathbf{r}) = 0} \text{ when } r < r_0$$

(b) $r_0 < r < r_1 \Rightarrow$ **Integrate over a sphere of radius** $r_0 < r < r_1$

$$\iiint_{V(r)} \nabla \cdot \mathbf{E}(\mathbf{r})\, dV = \oint\!\!\oint_{\partial Vr} \mathbf{E} \cdot d\mathbf{S} = \oint\!\!\oint_{\partial V(r)} E\, \underbrace{\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}}_{1}\, dS = \oint\!\!\oint_{\partial V(r)} E(r)\, dS = E(r) \oint\!\!\oint_{\partial V(R)} dS = 4\pi r^2 \cdot E(\mathrm{r})$$

$$= \frac{1}{\varepsilon_0} \iiint_{V(r)} \rho(r)\, dV = \frac{1}{\varepsilon_0} \int_{r_0}^{r} \underbrace{\gamma r}_{\rho}\, \underbrace{4\pi r^2 dr}_{dV} = \frac{4\gamma\pi}{\varepsilon_0} \int_{r_0}^{r} r^3 dr = \frac{\gamma\pi}{\varepsilon_0}\left(r^4 - r_0^4\right) \Rightarrow \boxed{E(r) = \frac{\gamma}{4\varepsilon_0} \frac{r^4 - r_0^4}{r^2}}$$

$$\boxed{\mathbf{E}(r_0 < r < r_1) = \frac{\gamma}{4\varepsilon_0} \frac{r^4 - r_0^4}{r^2} \hat{\mathbf{r}}} = \frac{\gamma}{4\varepsilon_0} \frac{r^4 - r_0^4}{r^3} \mathbf{r}$$

(c) $r > r_1 \Rightarrow$ **Integrate over a sphere of radius** $r > r_1$:

$$\iiint_{V(r)} \nabla \cdot \mathbf{E}(\mathbf{r})\, dV = 4\pi r^2 \cdot E(\mathrm{r})$$

$$= \frac{1}{\varepsilon_0} \iiint_{V(r)} \rho(r)\, dV = \frac{1}{\varepsilon_0} \int_{r_0}^{r_1} \underbrace{\gamma r}_{\rho}\, \underbrace{4\pi r^2 dr}_{dV} = \frac{4\gamma\pi}{\varepsilon_0} \int_{r_0}^{r_1} r^3 dr = \frac{\gamma\pi}{\varepsilon_0}\left(r_1^4 - r_0^4\right) \Rightarrow \boxed{E(r) = \frac{\gamma}{4\varepsilon_0} \frac{r_1^4 - r_0^4}{r^2}}$$

$$\boxed{\mathbf{E}(r_0 < r < r_1) = \frac{\gamma}{4\varepsilon_0} \frac{r_1^4 - r_0^4}{r^2} \hat{\mathbf{r}}} = \frac{\gamma}{4\varepsilon_0} \frac{r_1^4 - r_0^4}{r^3} \mathbf{r}$$