

Note: bold quantities are vectors or vector fields; *italics* denote scalars or scalar fields.

1. (10pts) Consider the vector field  $\mathbf{u}(\mathbf{r}) = \langle e^{x+y}, y^2, 0 \rangle$ . Compute the following derivatives (all of which appear in the generalized compressible Navier-Stokes equation):

a)  $(\mathbf{u} \cdot \nabla)\mathbf{u}$     b)  $\nabla(\nabla \cdot \mathbf{u})$     c)  $\Delta \mathbf{u} \equiv \nabla^2 \mathbf{u}$

2. (12pts) Non-dimensionalize the following one-dimensional advection-diffusion-absorption equation for volume mass density function  $\rho(\mathbf{r}, t)$ , reducing the number of parameters as much as possible:

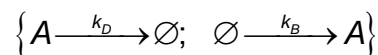
$$\begin{cases} \frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2} + u_0 \frac{\partial \rho}{\partial x} - \gamma \rho & (-\infty < x < \infty, t > 0) \\ \rho(x \rightarrow \pm \infty, t) = \rho_0 = \text{const} \end{cases}$$

Here  $D$  is the diffusion coefficient,  $\gamma = \text{const}$  is the absorption rate, and  $u_0 = \text{const}$  is the externally imposed flow velocity.

3. (16pts) Consider the following 2D flow:

$$\begin{cases} \frac{dx}{dt} = y + x^2 \\ \frac{dy}{dt} = x + y^2 \end{cases}$$

- a) Find all equilibria of this system, and analyze their stability using linear stability analysis.  
 b) Sketch the nullclines.  
 c) Make a rough plot of the flow field. Hint: start by showing the flow along the coordinate axes and the nullclines
4. (18pts) Consider the continuous-time stochastic process describing the following chemical reaction:



- a) Write down the Chemical Master Equations (CME).

- b) Find the equation for the evolution of the second moment,  $\frac{d\langle n^2 \rangle}{dt}$ .

- c) Find the partial differential equation (PDE) for the probability-generating function,  $F(z, t) = \sum_{n=0}^{\infty} p_n(t) z^n$

- d) Find the equilibrium probability distribution. Make sure that completeness is satisfied:  $\sum_{n=0}^{\infty} p_n = 1$

5. (16pts) Convert to index notation, then use index notation to expand or simplify, and finally convert the result back to vector notation (here  $\mathbf{U}$  is a vector field,  $\phi$  is a scalar field, and  $\mathbf{r}$  is the position vector:  $\mathbf{r} \equiv x_j$ ,  $j = 1, 2, 3$ ):

a)  $\nabla \times (\phi \mathbf{U})$     b)  $\nabla \times (\mathbf{r} \times \mathbf{U})$  (hints:  $\partial_k x_j = \delta_{kj}$ ;  $\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$ )

6. (16pts) Consider the following advection equation (assume that the equation is already non-dimensionalized):

$$\begin{cases} \frac{\partial \rho}{\partial t} + 3xt^2 \frac{\partial \rho}{\partial x} = 0 & (t > 0) \\ \rho(x_0, 0) = \rho_0(x_0) = x_0^2 & (-\infty < x_0 < +\infty) \end{cases}$$

- Find and plot the characteristics corresponding to 3 values of  $x_0$ :  $x_0 = -1$ ,  $x_0 = 0$ ,  $x_0 = 1$ .
- Is there a shock-wave / break-up?
- Find the solution, and make a rough plot of  $\rho(x, t)$  at  $t=1$  and at  $t=2$ .

7. (12pts) Consider a charged spherical shell, with charge distributed within  $r_0 < r < r_1$  according to

$$\rho(\mathbf{r}) = \rho(r) = \begin{cases} \gamma r, & r_0 < r < r_1 \quad (\gamma = \text{const}) \\ 0, & r < r_0 \text{ or } r > r_1 \end{cases}$$

Apply the divergence theorem to the Gauss law  $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$  to find the electric field  $\mathbf{E}(r)$  in three regions:

- (a)  $r < r_0$       (b)  $r_0 < r < r_1$       (c)  $r > r_1$