Math 613 * Fall 2018 * Final Examination * Victor Matveev

Note: bold quantities are vectors or vector fields; *italics* **denote scalars or scalar fields.**

1. **(10pts)** Consider the vector field $\mathbf{u}(\mathbf{r}) = \langle e^{x+y},\ y^2,\ 0 \rangle$. Compute the following derivatives (all of which appear in the generalized compressible Navier-Stokes equation):

$$a)\quad (\mathbf{u}\cdot\nabla)\mathbf{u} \qquad b)\quad \nabla(\nabla\cdot\mathbf{u}) \qquad c)\quad \Delta\mathbf{u} \equiv \nabla^2\mathbf{u}$$

2. **(12pts)** Non-dimensionalize the following one-dimensional advection-diffusion-absorption equation for volume mass density function $\rho(\mathbf{r}, t)$, reducing the number of parameters as much as possible:

$$\begin{cases} \dfrac{\partial\rho}{\partial t} = D\dfrac{\partial^2\rho}{\partial x^2} + u_0\dfrac{\partial\rho}{\partial x} - \gamma\rho & (-\infty < x < \infty,\ t > 0) \\ \rho(x \to \pm\infty, t) = \rho_0 = const \end{cases}$$

Here D is the diffusion coefficient, $\gamma = const$ is the absorption rate, and $u_0 = const$ is the externally imposed flow velocity.

3. **(16pts)** Consider the following 2D flow:

$$\begin{cases} \dfrac{dx}{dt} = y + x^2 \\ \dfrac{dy}{dt} = x + y^2 \end{cases}$$

   a) Find all equilibria of this system, and analyze their stability using linear stability analysis.
   b) Sketch the nullclines.
   c) Make a rough plot of the flow field. Hint: start by showing the flow along the coordinate axes and the nullclines

4. **(18pts)** Consider the continuous-time stochastic process describing the following chemical reaction:

$$\left\{ A \xrightarrow{k_D} \varnothing;\quad \varnothing \xrightarrow{k_B} A \right\}$$

   a) Write down the Chemical Master Equations (CME).
   b) Find the equation for the evolution of the second moment, $\dfrac{d\langle n^2\rangle}{dt}$.
   c) Find the partial differential equation (PDE) for the probability-generating function, $F(z,t) = \sum_{n=0}^{\infty} p_n(t) z^n$
   d) Find the equilibrium probability distribution. Make sure that completeness is satisfied: $\sum_{n=0}^{\infty} p_n = 1$

5. **(16pts)** Convert to index notation, then use index notation to expand or simplify, and finally convert the result back to vector notation (here **U** is a vector field, ϕ is a scalar field, and **r** is the position vector: $\mathbf{r} \equiv x_j,\ j = 1,2,3$):

   a) $\nabla\times(\phi\mathbf{U})$ b) $\nabla\times(\mathbf{r}\times\mathbf{U})$ (hints: $\partial_k x_j = \delta_{kj}$; $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$)

6. (16pts) Consider the following advection equation (assume that the equation is already non-dimensionalized):

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + 3xt^2 \dfrac{\partial \rho}{\partial x} = 0 & (t > 0) \\ \rho(x_0, 0) = \rho_o(x_0) = x_0^2 & (-\infty < x_0 < +\infty) \end{cases}$$

a) Find and plot the characteristics corresponding to 3 values of x_0: $x_0 = -1$, $x_0 = 0$, $x_0 = 1$.

b) Is there a shock-wave / break-up?

c) Find the solution, and make a rough plot of $\rho(x, t)$ at t=1 and at t=2.

7. (12pts) Consider a charged spherical shell, with charge distributed within $r_0 < r < r_1$ according to

$$\rho(\mathbf{r}) = \rho(r) = \begin{cases} \gamma r, & r_0 < r < r_1 \quad (\gamma = const) \\ 0, & r < r_0 \text{ or } r > r_1 \end{cases}$$

Apply the divergence theorem to the Gauss law $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ to find the electric field $\mathbf{E}(r)$ in three regions:

(a) $r < r_0$ (b) $r_0 < r < r_1$ (c) $r > r_1$