Math 613 * Fall 2019 * Final Examination * Victor Matveev

You may drop one 12-point problem, but you have to solve *all* problems worth more than 12 points.

1. **(12pts)** Find all equilibria, and categorize their stability. Make two plots: (1) phase plot (*dy*/d*t* vs. *y*); (2) plot the solution for the given initial condition, *y*(*t*) vs *t.*

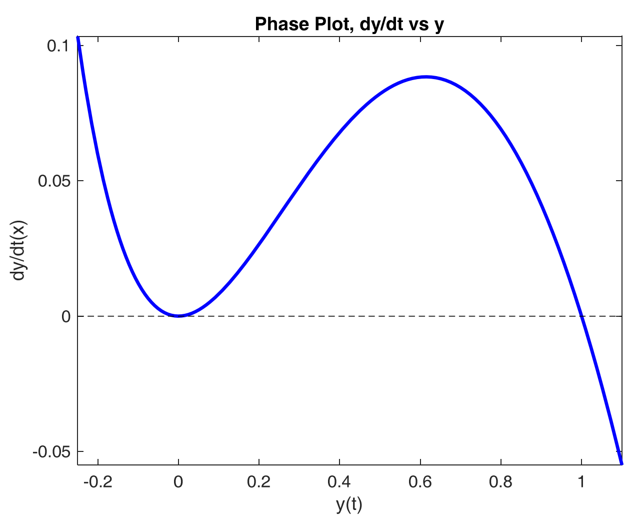
$$\begin{cases} \dfrac{dy}{dt} = (1-y)\left[\ln(1+y)\right]^2 \\ y(0) = 0.1 \end{cases}$$

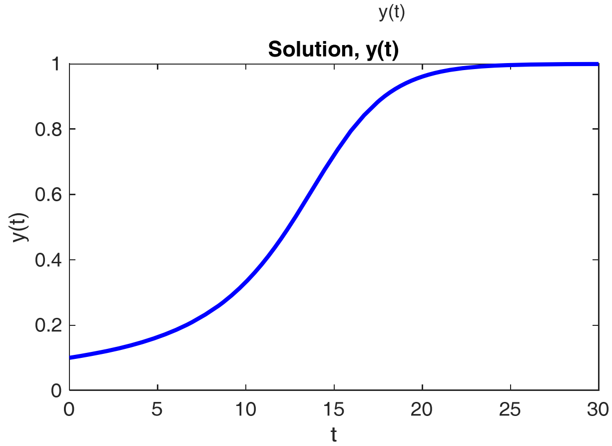
1. $\boxed{y_{eq}=0}$ $\ln(1+y) = y + O(y^2) \Rightarrow \dfrac{dy}{dt} \approx (1-y)y^2 \Rightarrow \dfrac{dy}{dt} \approx y^2 \Rightarrow y_{eq}=0$ is non-hyperbolic, semi-stable

2. $\boxed{y_{eq}=1}$ denote $Y = y-1 \Rightarrow \dfrac{dY}{dt} \approx \alpha Y$ where $\alpha = -\ln^2 2 < 0 \Rightarrow$ linearly (therefore, asymptotically) stable

Note: you could use direct differentiation to prove that $f'(1) = -\ln^2 2$, but that's unnecessary, and causes mistakes

Instead, note that near $y=1$ we have $\ln(1+y) = \ln 2 + O(y-1)$





2. (12pts) Consider the so-called RLC electric circuit equation (you don't have to know what it means):

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q(t)}{C}=\Phi(t)$$

Fundamental units are: **[t] = T** (time), **[q] = Q** (charge), **[Φ] = V** (electric potential)

a) Determine the fundamental units of constants R, L and C (resistance, inductance and capacitance).

b) Find any two distinct time scales t_c, in terms of model parameters. You don't have to use linear algebra.

c) Explain why you can only eliminate two parameters by non-dimensionalization in this case, not three.

d) Non-dimensionalize this equation, using electron charge e as an extra scale: $q_c = e$, $\bar{q}(t)=\frac{q(t)}{e}$.

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q(t)}{C}=\Phi(t) \Rightarrow [L]\frac{Q}{T^2}+[R]\frac{Q}{T}+\frac{Q}{[C]}=V \Rightarrow \begin{cases}[L]=\frac{V}{Q}T^2\\ [R]=\frac{V}{Q}T\\ [C]=\frac{Q}{V}\end{cases} \Rightarrow \begin{cases}t_c=RC\\ t_c=\sqrt{LC}\\ t_c=\frac{L}{R}\end{cases} \text{ (2 out of 3 is enough)}$$

- Can eliminate 2 parameters, since only N_U=2 units are independent, $\frac{V}{Q}$ and T: $N_D-N_U=6-2=N_\Pi=4$
- Of course, this is only true in the context of this problem: Q and V are, in general, independent units

- Start by non-dimensionalizing the charge, the potential, and the entire equation:

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q(t)}{C}=\Phi(t)\ \Big\|\times\frac{C}{e} \Rightarrow LC\frac{d^2\bar{q}}{dt^2}+RC\frac{d\bar{q}}{dt}+\bar{q}=\frac{C\Phi}{e}=\bar{\Phi}$$

- Three simplest non-dimensniolalization options, depending on time scale choices (either of these is fine):

1. $t_c=RC \Rightarrow t=RC\,\bar{t} \quad \Rightarrow \frac{LC}{(RC)^2}\frac{d^2\bar{q}}{d\bar{t}^2}+\frac{d\bar{q}}{d\bar{t}}+\bar{q}=\bar{\Phi} \Rightarrow \boxed{p\frac{d^2\bar{q}}{d\bar{t}^2}+\frac{d\bar{q}}{d\bar{t}}+\bar{q}=\bar{\Phi}}$ where $p=\frac{L}{R^2C}$

2. $t_c=\sqrt{LC} \Rightarrow t=\sqrt{LC}\,\bar{t} \quad \Rightarrow \frac{d^2\bar{q}}{d\bar{t}^2}+\frac{RC}{\sqrt{LC}}\frac{d\bar{q}}{d\bar{t}}+\bar{q}=\bar{\Phi} \Rightarrow \boxed{\frac{d^2\bar{q}}{d\bar{t}^2}+p\frac{d\bar{q}}{d\bar{t}}+\bar{q}=\bar{\Phi}}$ where $p=R\sqrt{\frac{C}{L}}$

3. $t_c=\frac{L}{R} \Rightarrow t=\frac{L}{R}\bar{t} \quad \Rightarrow LC\frac{R^2}{L^2}\frac{d^2\bar{q}}{d\bar{t}^2}+RC\frac{R}{L}\frac{d\bar{q}}{d\bar{t}}+\bar{q}=\bar{\Phi}\ \Big\|\times\frac{L}{CR^2}$

$$\Rightarrow \boxed{\frac{d^2\bar{q}}{d\bar{t}^2}+\frac{d\bar{q}}{d\bar{t}}+p\bar{q}=\bar{\Phi}} \quad \text{where } p=\frac{L}{R^2C};\ \bar{\Phi}=\frac{C\Phi}{e}\frac{L}{CR^2}=\frac{L\Phi}{R^2e}$$

3. (12pts) Consider a charged ball of radius R with spherically-symmetric charge density equal to

$$\rho(\mathbf{r}) = \rho(r) = \begin{cases} \dfrac{\gamma}{r}, & r \le R \quad (\gamma = const) \\ 0, & r > R \end{cases}$$

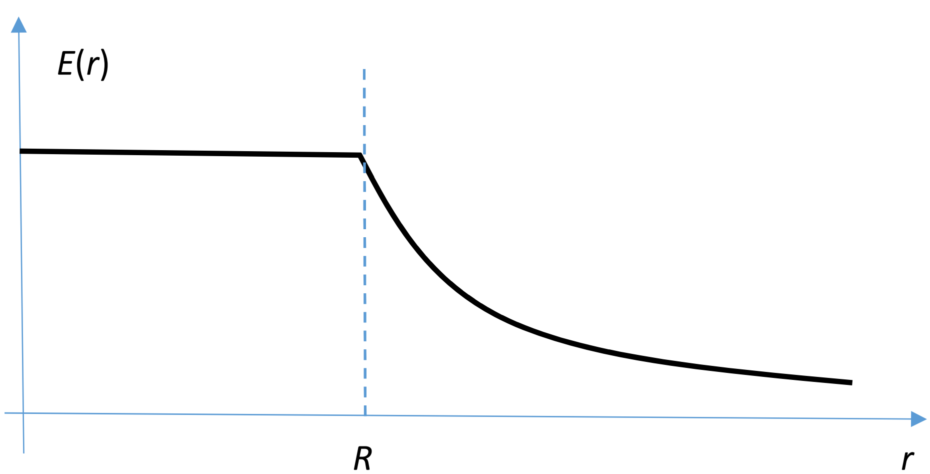
Apply the divergence theorem to the Gauss law $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ to find the electric field **E**(r) both inside and outside of this ball. Plot $E(r) = |\mathbf{E}|$ as a function of r. Make sure to explain all steps clearly

$$\iiint\limits_{\text{BALL}} \nabla \cdot \mathbf{E}(\mathbf{r})\, dV = \iint\limits_{\text{SPHERE}} \mathbf{E}(\mathbf{r}) \cdot \mathbf{n}\, dS = \iint\limits_{\text{SPHERE}} E(r)(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})\, dS = E(r) \cdot 4\pi r^2$$

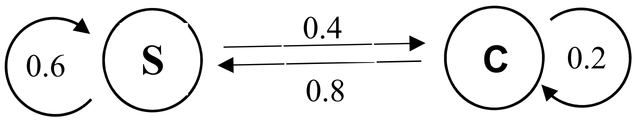
$$= \frac{1}{\varepsilon_0} \iiint\limits_{\text{BALL}} \rho(\mathbf{r})\, dV = \begin{cases} r \le R: \ = \dfrac{1}{\varepsilon_0} 4\pi \displaystyle\int_0^r \frac{\gamma}{\widehat{r}} \widehat{r}^2 d\widehat{r} \ = \ = \dfrac{4\pi\gamma}{\varepsilon_0} \displaystyle\int_0^r \widehat{r}\, d\widehat{r} \ = \ \dfrac{4\pi\gamma r^2}{2\varepsilon_0} \\ r > R: \ \dfrac{1}{\varepsilon_0} 4\pi \displaystyle\int_0^R \frac{\gamma}{\widehat{r}} \widehat{r}^2 d\widehat{r} \ = \ = \dfrac{4\pi\gamma}{\varepsilon_0} \displaystyle\int_0^R \widehat{r}\, d\widehat{r} \ = \ \dfrac{4\pi\gamma R^2}{2\varepsilon_0} \end{cases}$$

$$\Rightarrow \boxed{E(r) = \begin{cases} r \le R: \ \dfrac{4\pi\gamma r^2}{2\varepsilon_0} \dfrac{1}{4\pi r^2} \ = \dfrac{\gamma}{2\varepsilon_0} = const \\ r > R: \ \dfrac{4\pi\gamma R^2}{2\varepsilon_0} \dfrac{1}{4\pi r^2} \ = \dfrac{\gamma}{2\varepsilon_0} \dfrac{R^2}{r^2} \end{cases}}$$

Solution is finite, despite unnounded density



4. (12pts) Consider the **discrete state, discrete time** Markov Chain describing a toy weather model, with daily transitions between "S" (sunny) and "C" (cloudy) days (supposedly obtained using repeated observation):



a) Write down the explicit solution of this discrete-time dynamical system, assuming that the weather was cloudy on day zero

b) What is the probability that it is sunny on day 4, given that it is cloudy on day zero? One decimal digit of precision is enough in your answer.

$$M = \begin{pmatrix} 0.6 & 0.8 \\ 0.4 & 0.2 \end{pmatrix} \begin{matrix} S \\ C \end{matrix} \qquad trace(M) = 0.8 = \lambda_1 + \lambda_2 = 1 + \lambda_2 \Rightarrow \lambda_2 = 0.8 - 1 = -0.2$$

$$\boxed{\lambda_1 = 1} \quad \Rightarrow (M - \lambda_1 I)\,\mathbf{v}_1 = \begin{pmatrix} -0.4 & 0.8 \\ 0.4 & -0.8 \end{pmatrix}\mathbf{v}_1 = 0 \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \mathbf{p}^{EQ} = c_1\mathbf{v}_1 = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$\boxed{\lambda_2 = -0.2} \Rightarrow (M - \lambda_2 I)\,\mathbf{v}_1 = \begin{pmatrix} 0.8 & 0.8 \\ 0.4 & 0.4 \end{pmatrix}\mathbf{v}_2 = 0 \Rightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Use initial condition: $\mathbf{p}^0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{p}^{EQ} + c_2\mathbf{v}_2 = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \boxed{c_2 = -\frac{2}{3}}$

Therefore, the solution is $\mathbf{p}^m = c_1\lambda_1^m\mathbf{v}_1 + c_2\lambda_2^m\mathbf{v}_2 = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} - \frac{2}{3}(-0.2)^m\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{2}{3}\left[1-(-0.2)^m\right] \\ \frac{1}{3}\left[1-2(-0.2)^m\right] \end{pmatrix}}$

Probability it is sunny on day 4: $S^4 = \frac{2}{3}\left[1-(-0.2)^4\right] = \frac{2}{3}[1-0.0016] \approx \frac{2}{3}$

(information about initial condition is lost very quickly)

5. **(12pts)** Consider diffusion in a thin cylindrical tube of constant cross-section radius *R*, with molecule loss through the side surface of the tube satisfying the following property: the loss per unit time per surface *side area* of the tube is proportional to concentration at a particular location, $u(x, t)$, with the constant of proportionality denoted γ. Derive the diffusion equation for the concentration $u(x, t)$ in this case. Start by re-deriving the conservation law.

$$\frac{dN(t)}{dt} = A\Delta x \frac{\partial u\left(x^*,t\right)}{\partial t}$$

$$= \begin{bmatrix}\text{Inflow rate} \\ \text{from the left}\end{bmatrix} - \begin{bmatrix}\text{Outflow rate} \\ \text{from the right}\end{bmatrix} - \begin{bmatrix}\text{loss} \\ \text{from the sides}\end{bmatrix} = A\,J(x,t) - A\,J(x+\Delta x,t) - \gamma u\left(x^*,t\right) A_{SIDE}$$

where $A_{SIDE} = 2\pi R\Delta x$ is the side area of a cylindrical "slice" $\left(\text{different compared to cross-section area } A = \pi R^2\right)$

Now, divide by Δx times A: $\quad \dfrac{\partial u\left(x^*,t\right)}{\partial t} = -\dfrac{J(x+\Delta x,t) - J(x,t)}{\Delta x} - \gamma \dfrac{A_{SIDE}}{A\,\Delta x} u\left(x,t\right)$

Limit $\Delta x \to 0 \;\Rightarrow\; \dfrac{\partial u(x,t)}{\partial t} = -\dfrac{\partial J}{\partial x} - \gamma u \dfrac{A_{SIDE}}{A\,\Delta x} \;\Rightarrow\; \dfrac{\partial u}{\partial t} = -\dfrac{\partial J}{\partial x} - \gamma u \dfrac{2\pi R\,\Delta x}{\pi R^2 \Delta x} \equiv -\dfrac{\partial J}{\partial x} - \dfrac{2\gamma}{R} u$

Now plug into this equation the Fick's law of diffusion: $\quad J = -D\dfrac{\partial u}{\partial x} \;\Rightarrow\; \boxed{\dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} - \alpha u}$ where $\alpha = \dfrac{2\gamma}{R}$

6. (17pts) Consider the following ODE in $\mathbb{R}^2$: $\frac{d\mathbf{r}}{dt} = \mathbf{u}(\mathbf{r}) = \begin{pmatrix} -y \\ x - y^3 \end{pmatrix}$ $\left(\text{i.e. } \frac{dx}{dt} = -y;\ \frac{dy}{dt} = x - y^3 \right)$

a) Make a rough plot of the flow field in the (x, y) phase plane.

b) Perform linear stability analysis of the equilibrium. It linear stability analysis sufficient? Categorize the stability of the equilibrium.

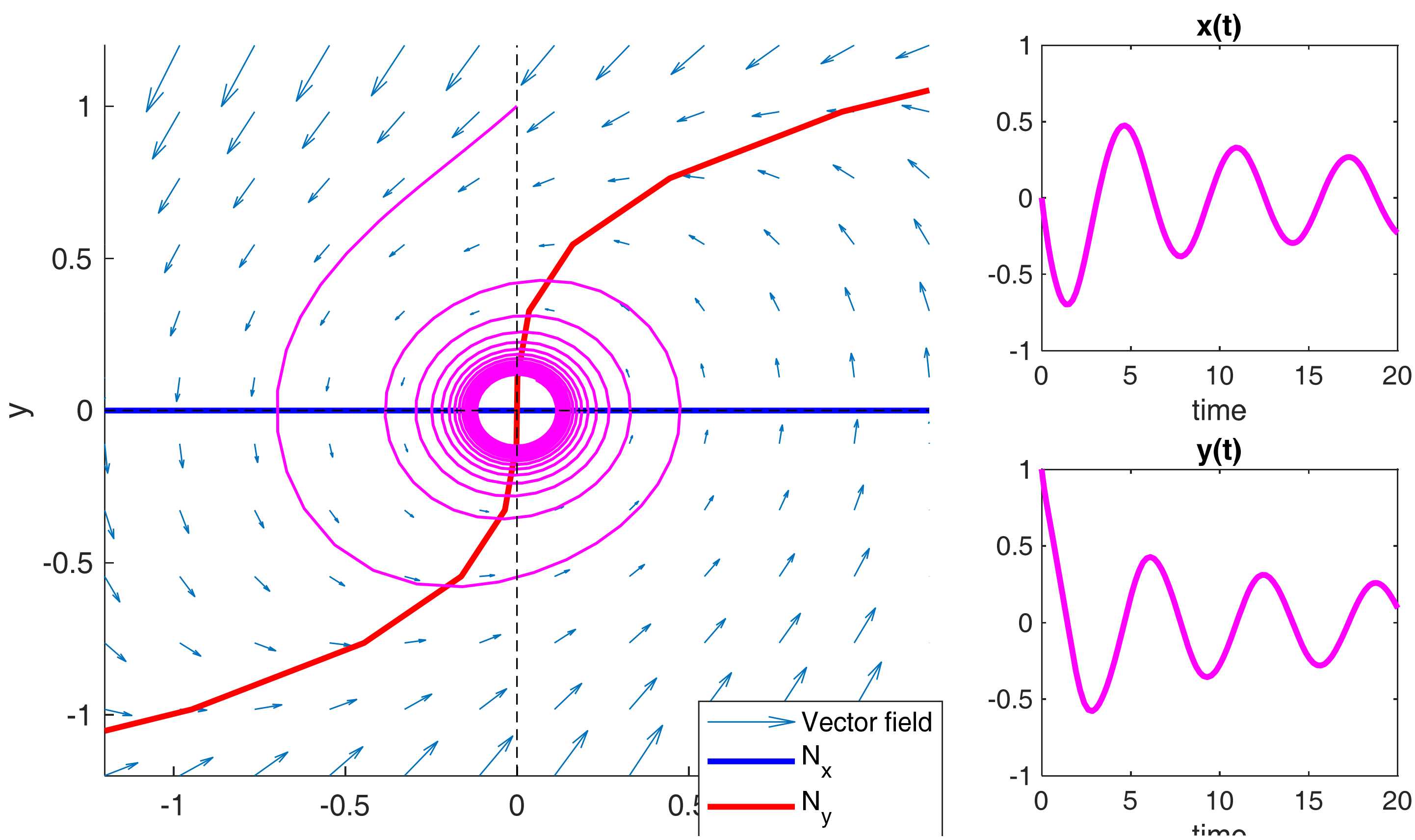
c) For the initial condition at (0, 1), plot the trajectory in the (*x*, *y*) phase-plane, and plot *x*(*t*) and *y*(*t*) vs *t*. Be as accurate as possible.

$$J(\mathbf{r}_{eq}) = \begin{pmatrix} 0 & -1 \\ 1 & -y^2 \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{cases} T = 0 \\ D = 1 \end{cases} \Rightarrow \lambda = \pm i$$

$\Rightarrow$ Non-hyperbolic equiliibrium $\Rightarrow$ linear analysis is **insufficient**

Examine distance from the origin: $\frac{d}{dt} r^2 = 2\left(x\frac{dx}{dt} + y\frac{dy}{dt} \right) = 2\left(-xy + xy - y^4\right) = -2y^4 \le 0$

The origin is **asymptotically** stable (non-hyperbolic stable spiral)



7. (17pts) Consider the traffic flow equation, with physical traffic velocity depending linearly on traffic density:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}\left[(1-\rho)\rho\right] = 0 & (t>0,\ x \in \mathbb{R}) \\ \rho(x,0) \equiv \rho_0(x) = \begin{cases} -x, & x<0 \\ 0, & x \ge 0 \end{cases} \end{cases}$$

Don't forget to use the chain rule to convert this equation to standard advection form!

a) Start by plotting the initial condition. Be careful with the minus signs.

b) Plot the characteristics corresponding to $x_0 = -2$, $x_0 = -1$, and $x_0 = 0$. Is there a shock wave / break-up?

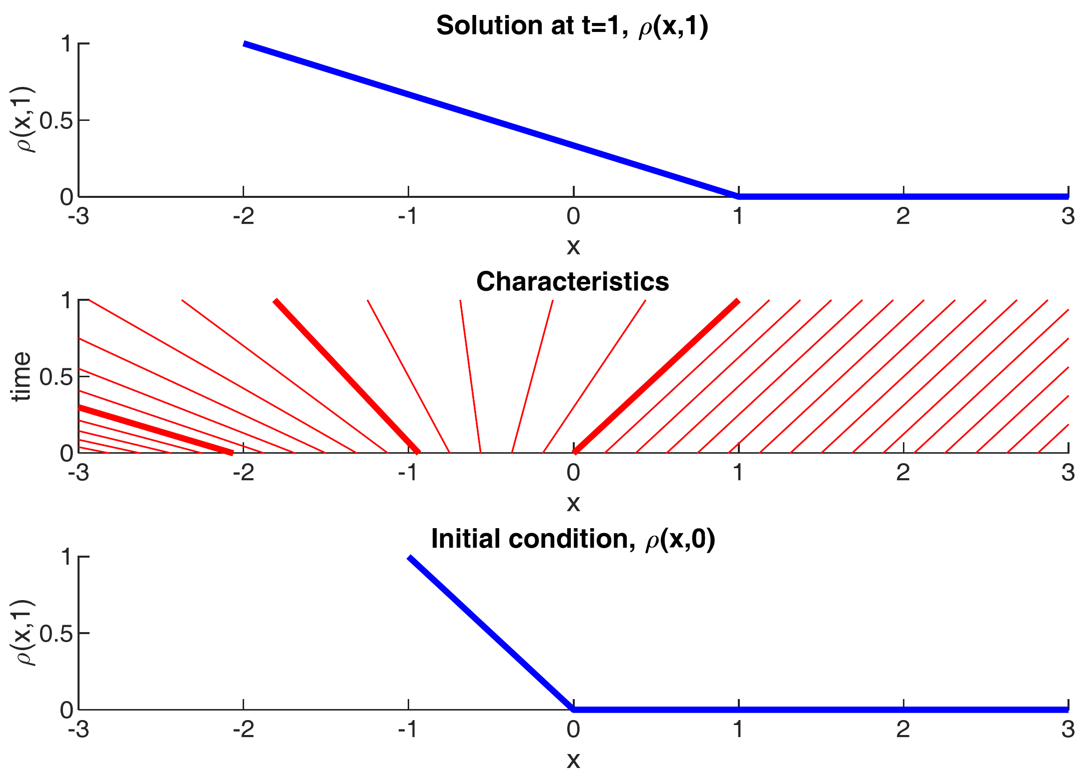
c) Make a rough plot of traffic density $\rho(x, t)$ at t=1.

d) Write down the explicit solution to this problem. It may help to separate the (x, t) domain into two regions.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho - \rho^2\right) = \frac{\partial \rho}{\partial t} + \underbrace{(1-2\rho)}_{\left.\frac{dx}{dt}\right|_{cc}}\frac{\partial \rho}{\partial x} = 0 \quad \Rightarrow \quad \left.\frac{dx}{dt}\right|_{cc} = 1-2\rho$$

$$x(t;x_0)\Big|_{cc} = x_0 + (1-2\rho_0)\,t = \begin{cases} x_0 + (1+2x_0)t = (1+2t)x_0 + t, & x_0 < 0 \\ x_0 + t, & x_0 \ge 0 \end{cases} \Rightarrow \begin{cases} x_0 = -2: & x(t) = -2-3t \\ x_0 = -1: & x(t) = -1-t \\ x_0 = 0: & x(t) = t \end{cases}$$

$$\text{Invert:}\quad x_0 = \begin{cases} \dfrac{x-t}{1+2t}, & x<t \\ x-t, & x \ge t \end{cases} \Rightarrow \rho(x,t) = \rho_0(x_0) = -x_0 = \boxed{\begin{cases} \dfrac{t-x}{1+2t}, & x<t \\ 0, & x \ge t \end{cases}} \Rightarrow \rho(x,1) = \begin{cases} \dfrac{1-x}{3}, & x<1 \\ 0, & x \ge 1 \end{cases}$$



8. (18pts) Convert to index notation, then use index notation to expand or simplify, and finally convert the result back to vector notation. Here $\mathbf{u}(\mathbf{r})$ is a smooth vector field, $\mathbf{r}$ is the position vector, and $r = |\mathbf{r}|$:

a) $\nabla \times [\mathbf{r} \times \mathbf{u}(\mathbf{r})]$ Solved in last year's exam:

$$\nabla \times (\mathbf{r} \times \mathbf{u}) = \varepsilon_{ijk} \partial_j (\mathbf{r} \times \mathbf{u})_k = \varepsilon_{ijk} \partial_j (\varepsilon_{knm} x_n u_m) = (\delta_{in} \delta_{jm} - \delta_{im} \delta_{jn}) \partial_j (x_n u_m)$$

$$= \partial_m (x_i u_m) - \partial_n (x_n u_i) = x_i \partial_m u_m + u_m \underbrace{\partial_m x_i}_{\delta_{mi}} - x_n \partial_n u_i - u_i \underbrace{\partial_n x_n}_{\delta_{nn}=3}$$

$$= x_i \partial_m u_m + u_i - x_n \partial_n u_i - 3u_i = \boxed{\mathbf{r}\, \nabla \cdot \mathbf{u} - (\mathbf{r} \cdot \nabla) \mathbf{u} - 2\mathbf{u}}$$

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b) $\nabla^2 \left(\frac{1}{r^p} \right)$, where p = const. For which p does this equal zero?

This was part of the calculation In problem 2 of homework 11, where you showed that the Laplacian of 1/r is zero. This is also why the potential of a point charge equals const / r (in other words, the Green's function of the Laplacian in free space = const / r). This gives the significance of this exercise

Use the chain rule: $\partial_m f(r) = f'(r) \frac{x_m}{r}$

$$\partial_m \partial_m \left(\frac{1}{r^p} \right) = \partial_m \left(-\frac{p}{r^{p+1}} \frac{x_m}{r} \right) = -p \left[x_m \partial_m \left(\frac{1}{r^{p+2}} \right) + \frac{1}{r^{p+2}} \partial_m x_m \right] = -p \left[-(p+2) \frac{\overbrace{x_m x_m}^{r^2}}{r^{p+4}} + \frac{\overbrace{\delta_{mm}}^{N}}{r^{p+2}} \right]$$

Now, assume $\mathbf{r} \in \mathbb{R}^N$: $= p \frac{p+2-N}{r^{p+2}} = 0$ if $p = 0$ or $p = N-2$

If N=3: $p = 0$ or $p = 1$ as expected!