Math 613 * Fall 2019 * Homework 11 * Victor Matveev

Problem 1: The Cauchy Momentum Balance Equation (2nd Newton's law for flowing fluid):

Applying the conservation law $\partial_t X + \nabla\cdot\mathbf{J} = \sum \mathrm{Q}^{\mathrm{Source}}$ to momentum density $X_k = \rho u_k$ with flux $\mathbf{J}_{X_k} = X_k \mathbf{u}$, and forces as sources, we obtained: $\dfrac{\partial X_k}{\partial t} + \nabla\cdot J_{X_k} = \rho g_k + \nabla\cdot\vec{\sigma}_k \quad \left(\text{where } \vec{\sigma}_k \equiv \sigma_{jk}\right) \Rightarrow$

$$\Rightarrow \boxed{\frac{\partial(\rho u_k)}{\partial t} + \partial_j\left(\rho u_k u_j\right) = \rho g_k + \partial_j\left[\delta_{jk}\left(-p + \left(\lambda - \frac{2\mu}{3}\right)\partial_m u_m\right) + \mu\left(\partial_j u_k + \partial_k u_j\right)\right]}$$

Expand derivatives of all products in this expression, to find the equation of a ***compressible*** fluid flow

Problem 2: Consider the "Stokeslet" Green's function solution for the Stokes flow:

$$\mu\nabla^2\mathbf{u} - \nabla p = -\mathbf{F}\cdot\delta(\mathbf{r}) \quad \Rightarrow \quad \begin{cases} \mathbf{u}(\mathbf{r}) = \mathbf{F}\cdot\mathbf{J}(\mathbf{r}) & \Rightarrow \quad u_j = F_k J_{kj} \\ p(\mathbf{r}) = \dfrac{\mathbf{F}\cdot\mathbf{r}}{4\pi r^3} & \Rightarrow \quad p = \dfrac{F_m x_m}{4\pi r^3} \\ \mathbf{J}(\mathbf{r}) = \dfrac{1}{8\pi\mu}\left(\dfrac{\mathbf{I}}{r} + \dfrac{\mathbf{r}\,\mathbf{r}}{r^3}\right) & \Rightarrow \quad J_{kj} = \dfrac{1}{8\pi\mu}\left(\dfrac{\delta_{kj}}{r} + \dfrac{x_k x_j}{r^3}\right) \end{cases}$$

Complete the calculation below to show that $\mu\nabla^2\mathbf{u} - \nabla p = 0$ for $\mathbf{r}\neq\mathbf{0}$

Use the following differentiation rules: $\partial_k x_m = \delta_{km} \quad \left(\text{since } \dfrac{\partial x}{\partial x} = 1,\ \dfrac{\partial x}{\partial y} = 0,\ etc.\right)$

$$\partial_k f(r) = \frac{df}{dr}\partial_k r = \frac{df}{dr}\partial_k\sqrt{x_m x_m} = \frac{df}{dr}\frac{\delta_{mk}x_m}{r} = \boxed{\frac{df}{dr}\frac{x_k}{r}}$$

$$\mu\partial_m\partial_m u_j - \partial_j p = \mu\partial_m\partial_m\left(F_k J_{kj}\right) - \frac{1}{4\pi}\partial_j\left(\frac{F_m x_m}{r^3}\right)$$

$$= \frac{\mu}{8\pi\mu}F_k\partial_m\partial_m\left(\frac{\delta_{kj}}{r} + \frac{x_k x_j}{r^3}\right) - \frac{F_m}{4\pi}\partial_j\left(\frac{x_m}{r^3}\right)$$

$$= \frac{F_k}{8\pi}\left(\delta_{kj}\partial_m\partial_m\left(\frac{1}{r}\right) + x_k x_j\partial_m\partial_m\frac{1}{r^3} + \frac{1}{r^3}\partial_m\partial_m\left(x_k x_j\right) + 2\partial_m\left(x_k x_j\right)\partial_m\frac{1}{r^3}\right) - \frac{F_m}{4\pi}\left(\frac{\delta_{mj}}{r^3} - \frac{3x_m}{r^4}\frac{x_j}{r}\right)$$

$$= \frac{F_k}{8\pi}\left(\delta_{kj}\partial_m\left(-\frac{1}{r^2}\frac{x_m}{r}\right) + x_k x_j\partial_m\left(-\frac{3}{r^4}\frac{x_m}{r}\right) + \frac{1}{r^3}\partial_m\left(x_k\delta_{mj} + x_j\delta_{mk}\right) + 2\left(\delta_{mk}x_j + \delta_{mj}x_k\right)\left(-3\frac{1}{r^4}\frac{x_m}{r}\right)\right)$$

$$-\frac{1}{4\pi}\left(\frac{F_j}{r^3} - \frac{3F_m x_m x_j}{r^5}\right) = \ldots\ldots = \mathbf{0}$$