Math 613 * Fall 2018 * Victor Matveev * Homework 2

1. Consider the damped pendulum equation derived in class: $\begin{cases} L\dfrac{d^2\theta}{dt^2}+k\dfrac{d\theta}{dt}+g\sin\theta=0 \\ \theta(0)=\theta_o \\ \dfrac{d\theta}{dt}(0)=\omega_o \end{cases}$

   Find ***all independent*** time scales t_c in this problem by considering $\Pi = t/t_c = t^a L^b k^c g^d \omega_0^e$ with a=1. The number of independent time scales t_c equals the number of linearly independent solutions of the resulting linear system satisfied by (b, c, d, e). First, you will need to find the physical units of k from the given equation (we did this in class). Since there is more than one time scale, the answer can be written in many ways that are all equivalent to each other.

2. We learned that generally the drag force F_D is determined by the flow speed v, fluid mass density ρ (units of M/L^3), object size R, and fluid viscosity μ (units of $M(LT)^{-1}$). If we do **not** non-dimensionalize, this means that F_D is a function of 4 variables (v, ρ, μ, R) – complicated! However, there are only 5-3=2 non-dimensional quantities, related to each other by $f(\Pi_1, \Pi_2)=0$, an implicit function that can be solved for $\Pi_1 = g(\Pi_2)$. Now, take Π_1 that depends on the drag force F_D (use any of the two we found in class) and find Π_2 that does **not** depend on the drag force (find it: $\Pi_2 = v^\alpha R^\beta \mu^\gamma \rho^\delta$, set α=1). Then, use $\Pi_1 = g(\Pi_2)$ to write down the most general dependence of F_D on (v, ρ, μ, R), in terms of a function g of **one** variable only. Note that there are many equivalent ways to write the answer, all trivially related to each other.

3. Explain in 1-2 sentences why a solution to the 1st order autonomous ordinary differential equation $dy/dt = f(y)$ **may never** have non-monotonic dependence on time. Argument by contradiction would be particularly simple here: consider a non-monotonic $y(t)$ and think of the corresponding "phase plot" ($dy/dt=f(y)$ vs y).

4. Consider the following three ODEs (note that (a) is the population growth model from homework #1):

$$(a)\begin{cases}\dfrac{dy}{dt}=(1+4y)^{3/2}-1\\ y(0)=-0.1\end{cases}\quad (b)\begin{cases}\dfrac{dy}{dt}=\left(1-e^{2y}\right)^3\\ y(0)=-0.1\end{cases}\quad (c)\begin{cases}\dfrac{dy}{dt}=-(y-1)\ln y\\ y(0)=1.1\end{cases}$$

   - Use the phase plot (the plot of dy/dt vs. y) to determine and sketch the qualitative behavior of solution (y vs. t), for the given initial condition.
   - Find any hyperbolic equilibria and analyze their stability. Indicate any non-hyperbolic equilibria.
   - For each non-hyperbolic equilibrium, use the first non-zero term in the Taylor expansion of the "velocity function" near the equilibrium to determine its stability (asymptotically stable, semi-stable or unstable).

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Taylor series review: it's important for problem 4 that you know well the Taylor series of the common analytic functions, from which the Taylor series of any composite analytic function can be derived without resorting to differentiation:

$$e^y = 1+y+\frac{y^2}{2!}+\ldots+\frac{y^n}{n!}+\ldots\ (\forall y\in\mathbb{C}) \qquad \frac{1}{1-y}=1+y+y^2+y^3+\ldots\ (|y|<1)$$

$$\cos y = 1-\frac{y^2}{2}+\frac{y^4}{4!}-\frac{y^6}{6!}+\ldots\ (\forall y\in\mathbb{C}) \qquad (1+y)^p = 1+py+\frac{p(p-1)}{2}y^3+\ldots\ (|y|<1)$$

$$\sin y = y-\frac{y^3}{3!}+\frac{y^4}{5!}-\frac{y^7}{7!}+\ldots\ (\forall y\in\mathbb{C}) \qquad \ln(1+y)=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\ldots\ (|y|<1)$$