Math 613 * Fall 2019 * Victor Matveev * Homework 7

1. **(25pts)** Consider the ***equilibrium*** solution $u_{eq}(x)$ to the heat / diffusion equation with Neumann boundary conditions on both sides (here γ=const, β=const, length $\boldsymbol{L}=\pi$):

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} + \gamma e^{-x} \quad (0 < x < \pi;\ t > 0) \\ \dfrac{\partial u}{\partial x}(0, t) = \beta;\ \ \dfrac{\partial u}{\partial x}(\pi, t) = 0 \\ u(x,0) = \sin(x) \end{cases}$$

   a) Find the equilibrium solution $u_{eq}(x)$. For which values of constant β does the equilibrium exist?
   b) To determine the last constant of integration in $u_{eq}(x)$, consider the integral of the entire PDE over the length of the tube/cable, as we did in class.
   c) Make a rough plot by hand of $u_{eq}(x)$ and explain the heat balance: where does the heat enter, and where does it exit at equilibrium?
   d) Do you think this equilibrium is asymptotically stable or just neutrally stable?

2. **(25pts each)** Solve the following wave / advection equations using the method of characteristics. For each problem, make 3 ***rough*** plots (by hand is fine): characteristics $x(t)$, the initial condition $u_0(x)$, and $u(x,2)$ (the solution at t=2). Note that all three plots can be done without solving the problem!

$$\textbf{(a)} \begin{cases} \dfrac{\partial u}{\partial t} + x^{1/3}\dfrac{\partial u}{\partial x} = 0 \quad \begin{pmatrix} t>0 \\ -\infty < x < \infty \end{pmatrix} \\ u(x,0) = \sin x \end{cases} \quad \textbf{(b)} \begin{cases} \dfrac{\partial u}{\partial t} + e^x t\dfrac{\partial u}{\partial x} = 0 \quad \begin{pmatrix} t>0 \\ -\infty < x < \infty \end{pmatrix} \\ u(x,0) = u_o(x) = \sin x \end{cases} \quad \textbf{(c)} \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(u^3\right) = 0 \quad \begin{pmatrix} t>0 \\ -\infty < x < \infty \end{pmatrix} \\ u(x,0) = u_o(x) = \begin{cases} x, & x \ge 0 \\ 0, & x < 0 \end{cases} \end{cases}$$

In (**c**), before you start, expand the second term using product rule for differentiation. We will practice with problems more similar to problem (c) on Monday

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The steps are always the same for the method of characteristics:

1) Compute the characteristics $x(t;\, x_0)$ Easy (if you can solve simple ODEs)
2) Invert step 1 to look-up x_0 corresponding to any space-time coordinate (x, t): $x_0 = x_0(x, t)$ Not always easy!
3) Look-up initial value u_0 corresponding to this $x_0(x, t)$: $u(x, t) = u_0(x_0(x, t))$ No work required!

Note that the only tricky step is usually step 2, since this involves inverting a function $x(t;\, x_0)$. As we know from Calculus I, not all functions have an inverse: when the function is not invertible, we get something interesting, as we will see Monday.