Math 613 * Fall 2019 * Victor Matveev * Homework 8

Problem 1 (30pts)

Without solving this problem, make a rough plot of characteristics $x(t)$, the initial condition $u_0(x)$, and $u(x,t_b)$ (the solution at *breaking time*). Find the breaking time, and try to show (at least roughly) the location of intersecting characteristics in your plot:

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial}{\partial x}\left(u^3\right)=\dfrac{\partial u}{\partial t}+\underbrace{3u^2}_{c(u)}\dfrac{\partial u}{\partial x}=0 & \begin{pmatrix} t>0 \\ -\infty<x<\infty \end{pmatrix} \\ u(x,0)=u_o(x)=\exp\left(-x^2\right) \end{cases}$$

Recall that $t_b=\min\limits_{x_0}\dfrac{1}{-c'\left(u_o(x_0)\right)u_o'(x_0)}$

Problem 2 (35pts)

Find the speed of the travelling wave solution by converting this PDE to an ODE using substitution $u(x,t)=U(\underbrace{x-ct}_{\xi})$. The speed of the wave *c* can be determined already after the first integration step, which gives you an equation for $\dfrac{dU}{d\xi}$, using the additional boundary condition $\dfrac{dU}{d\xi}\to 0 \text{ as } \xi\to\pm\infty$. You don't have to determine the solution itself by performing the 2nd integration step (although it's not too hard). How does the speed of the wave depend on parameter U_R?

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial}{\partial x}\left(u^2\right)=\dfrac{\partial^2 u}{\partial x^2} & \begin{pmatrix} t>0 \\ -\infty<x<\infty \end{pmatrix} \\ u(-\infty,0)=1, \quad u(+\infty,0)=U_R<1 \end{cases}$$

(1) Convert the PDE to an ODE. Note that $\dfrac{\partial U}{\partial t}=-c\dfrac{dU}{d\xi}; \quad \dfrac{\partial U}{\partial x}=\dfrac{dU}{d\xi}$

(2) Integrate the resulting equation once to obtain an ODE for $\dfrac{dU}{d\xi}$

(3) Consider the two limits of the resulting expression for $\xi\to\pm\infty$, using the two given boundary conditions and assuming that $\dfrac{dU}{d\xi}\to 0 \text{ as } \xi\to\pm\infty$

Problem 3 (35pts) Check your answer to problem 2 using MATLAB code for homework #6: add the advection term $\dfrac{\partial}{\partial x}\left(u^2\right)$ to your heat equation code (use a centered difference for this), set a certain number of initial values of *u* on the left part of the domain to 1, set the initial values on the rest of the domain (to the right) to U_R=1/2, and watch the wave form and then propagate. Determine the wave speed from your plots, and compare with the result of problem 2. Note that some time is needed for the initial transient to dissipate and for the solution to approach the travelling wave solution.