Math 676 * December 15, 2011 * Final Exam * Victor Matveev

1. **(20pts)** Consider the following system with a non-hyperbolic equilibrium at the origin:

$$\begin{cases} x' = x^2 y - y^2 x \\ y' = -x^3 - y^3 \end{cases}$$

   a) Analyze this system using polar coordinates, categorize the equilibrium, and carefully sketch the flow.

   b) Find a quadratic Lyapunov function (hint: part (a) reveals the form of this function). Does the LaSalle's Invariance Principle apply? Is the equilibrium asymptotically stable?

   c) Find the Poincare index of the origin using the flow sketch in part (a), and check that it agrees with the Bendixson formula $I_{x^*} = 1 + \frac{1}{2}(e-h)$

2. **(18pts)** Consider the following system with an equilibrium at the origin: $\begin{cases} x' = z^2 + yz - x \\ y' = z^2 + x^2 \\ z' = -y^2 \end{cases}$

   a) Use power series method to approximate all invariant manifolds of the equilibrium at the origin, up to second order in the variables.

   b) Write down the equations describing center manifold dynamics (recall the non-hyperbolic Hartman-Grobman Theorem), keeping all terms that you obtained. **You don't have to analyze the dynamics on $\mathbf{W}^{\mathbf{c}}_{\mathbf{loc}}$.**

3. **(30pts)** Consider the system $\begin{cases} x' = x + (x-y)^3 \\ y' = 2x - y + (x-y)^3 \end{cases}$

   a) Diagonalize this system; check that the diagonal system is Hamiltonian, and has the "skew-product" (i.e. partially decoupled) form $\{X' = f(X,Y); \;\; Y' = g(Y)\}$

   b) Solve the diagonalized system analytically (start with solving the de-coupled equation)

   c) Use your solution to obtain the equations for global invariant manifolds of the origin (explain all steps)

   d) Use the Hamiltonian of the diagonalized system to obtain the invariant manifolds, and compare with part (c)

   e) Sketch the flow for this system in the (x, y) phase plane.

   f) Compare the stable manifold you obtained with one iteration of the stable manifold contraction map (superscripts "s" and "u" denote projections onto stable and unstable linear subspaces):

$$T(X(t)) = e^{At} X^s(0) + \int_0^t e^{A(t-s)} g^s(X(s))\, ds - \int_t^{+\infty} e^{A(t-s)} g^u(X(s))\, ds$$

If you fail to successfully complete part (a) of Problem 3, please complete parts (b-f) of this problem using the following system instead: $\begin{cases} x' = 2x - 4y^3 \\ y' = -2y \end{cases}$

4. (20pts) Consider the following system: $\begin{cases} \dfrac{dx}{dt} = y + \mu x - 2xy^2 - x^3 \\ \dfrac{dy}{dt} = -x + y\mu - y^3 \end{cases}$

a) Use polar coordinates to show that this system has a circular limit cycle in some range of *μ*. Is the limit cycle stable? Use the same polar equations to examine the stability of the equilibrium at the origin, as a function of μ.

b) Describe the behavior of this system for different values of *μ*, categorize the bifurcation, and sketch the bifurcation diagram in the (μ, x, y) space

5. (12pts) Categorize the bifurcation(s) and sketch the bifurcation diagram for $\dfrac{dx}{dt} = \tan x + \mu x$