Math 756 Complex Variables II

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Contents

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Book: M.J. Ablowitz & A.S. Fokas. Complex variables: Introduction and Applications (2nd edition). *Cambridge University Press* (2003).

1 Review of Complex I, and extensions

1.1 Elementary definitions

A **neighborhood** of a point $z_0 \in \mathbb{C}$ is the set of points z such that $|z - z_0| < \epsilon$. This may sometimes be referred to as an ϵ-neighborhood of z_0.

$z_0 \in S$ is an **interior point** of the set $S \subset \mathbb{C}$ if S contains a neighborhood of z_0.

Example: $z_0 = 1/2$ is an interior point of the set $S_1 = \{z \in \mathbb{C} : |z| < 1\} \subset \mathbb{C}$, but not of the set $S_2 = \{z \in \mathbb{C} : -1 < \Re(z) < 1, \Im(z) = 0\} \subset \mathbb{C}$.

The set S is **open** if all its points are interior. (S_1 in the example above is open, but S_2 is not.)

A **region** R is an open subset of $\mathbb{C}$, plus some, all or none of the boundary points.

Example: S_1 (the open unit disc) is a region, as is $R_1 = S_1 \cup \{z \in \mathbb{C} : |z| = 1, 0 < \arg z < \pi/2\}$, and $R_2 = S_1 \cup \{z \in \mathbb{C} : |z| = 1\}$ (R_2 is the closed unit disc – the open unit disc plus its boundary points).

A region R is **closed** if it contains all its boundary points.

Example: R_2 in the example above is a closed region, but R_1 and S_1 are non-closed regions (S_1 is an open region).

A region R is **bounded** if $\exists M > 0$ such that $|z| \leq M \forall z \in R$.

Example: S_1, R_1 and R_2 above are all bounded by $M = 1$, but $R_3 = \{z \in \mathbb{C} : |z| > 1\}$ is unbounded.

A region that is both closed and bounded is **compact.**

Example: The closed unit disc $R_2 = \{z \in \mathbb{C} : |z| \leq 1\}$ is compact.

(Path) connectedness (simplest definition): Given points $z_1, z_2, \ldots, z_n$ in $\mathbb{C}$, the $(n-1)$ line segments $[z_1 z_2]$, $[z_2 z_3], \ldots, [z_{n-1} z_n]$ form a piecewise linear curve in $\mathbb{C}$. A region R is said to be **connected** if any two of its points can be joined by such a curve that is contained wholly within R.

Example: All of the regions S_1, R_1, R_2, R_3 given above are connected. The region $R_4 = S_1 \cup R_3$ is not connected, since no point in S_1 can be connected to a point in R_3 by a piecewise linear curve lying entirely within R_4.

A **domain** is a connected open region.

Simply-connected: A domain D is said to be **simply-connected** if it is path-connected, and any path joining 2 points in D can be continuously transformed into any other.

Example: All of the regions S_1, R_1, R_2 given above are simply-connected. The region R_3 is not, since any two distinct points can be joined by topologically-distinct paths within R_3, passing on either side of the "hole".

1.2 Analytic functions: The basics

Definition 1.1 *(Differentiable function) Let D be an open set in $\mathbb{C}$. $f : D \to \mathbb{C}$ is* **differentiable** *at $a \in D$ if*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists (independently of how the limit $h \to 0$, $h \in \mathbb{C}$ is taken).

Definition 1.2 *(Analytic function) A function $f : D \to \mathbb{C}$ is* **analytic** *at $a \in D$ if f is differentiable in a neighborhood of a. f is analytic on D if it is analytic at every point in D. Analytic functions are also sometimes referred to as* **holomorphic** *functions.*

It can be shown (from Cauchy's integral formula; see (5)) that if a function f is analytic then its derivatives of all orders exist in the region of analyticity, and all these derivatives are themselves analytic. So, analytic $\Rightarrow$ infinitely differentiable.

Remark At first sight it appears quite hard to write down non-differentiable complex functions f, and one might think that most functions one could write down are differentiable, and even analytic. This seems at odds with the statement often made in textbooks that analytic functions are very special and rare. However, if one writes $z = x + iy$, and thinks of a general complex function f as being composed of real and imaginary parts that are functions of x and y, $f(z) = u(x, y) + iv(x, y)$, then the rarity becomes more apparent. To extract the representation in terms of the complex variable z we have to substitute for $x = (z + \bar{z})/2$ and $y = -i(z - \bar{z})/2$, so that in fact f is, in general, a function of **both** z **and its complex conjugate** $\bar{z}$. Only if f turns out to be **purely a function of** z can it be analytic.

Example 1.3 *The function* $f(z) = z^2$ *is differentiable for all* $z \in \mathbb{C}$ *because*

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{(z+h)^2 - z^2}{h} = \lim_{h\to 0} \frac{2zh + h^2}{h} = 2z$$

independently of how $h \in \mathbb{C}$ *approaches zero.*

To show non-differentiability at a point, it suffices to show that different choices of $h \to 0$ give different results.

Example 1.4 *The function* $f(z) = |z|^2$ *is not differentiable anywhere, since*

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{(z+h)(\overline{z+h}) - z\bar{z}}{h} = \lim_{h\to 0} \frac{z\bar{h} + h\bar{z}}{h}.$$

Choosing $h = \epsilon e^{i\alpha}$ *for* $0 < \epsilon \ll 1$ *we find*

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{\epsilon\to 0} \frac{\epsilon z e^{-i\alpha} + \epsilon\bar{z}e^{i\alpha}}{\epsilon e^{i\alpha}} = ze^{-2i\alpha} + \bar{z},$$

a result which depends on the way we take the limit $h \to 0$.

This example is related to the remark above, since in fact f is a function of both z and $\bar{z}$: $f(z, \bar{z}) = z\bar{z}$, and cannot be expected to be differentiable.

Non-differentiability may also be established by appealing to the **Cauchy-Riemann theorem.**

Theorem 1.5 *(Cauchy-Riemann) The function $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$ of $D \subset \mathbb{C}$ if and only if the partial derivatives u_x, u_y, v_x, v_y are continuous and satisfy the* **Cauchy-Riemann equations**

$$u_x = v_y, \qquad u_y = -v_x \tag{1}$$

in a neighborhood of z.

Proof Bookwork. E.g., Ablowitz & Fokas, pages 32-34.

It follows that, for an analytic function $f(z) = u(x, y) + iv(x, y)$, we have $f'(z) = u_x + iv_x = v_y - iu_y$. The Cauchy-Riemann equations also confirm the statement made in the remark above, that only if f turns out to be purely a function of z can it be analytic. To see this we note that

$$x = \frac{1}{2}(z + \bar{z}), \quad y = -\frac{i}{2}(z - \bar{z}) \quad \Rightarrow \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

(using the chain rule for partial differentiation). Thus, if u and v satisfy the Cauchy-Riemann equations then

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(u_x + iu_y + i(v_x + iv_y)) = \frac{1}{2}(u_x - v_y + i(u_y + v_x)) = 0,$$

so that f is independent of $\bar{z}$.

Returning to example 1.4 above, with $f(z) = |z|^2$ we have $u(x, y) = x^2 + y^2$, $v(x, y) = 0$. The Cauchy-Riemann equations are not satisfied in the neighborhood of any $z \in \mathbb{C}$, confirming the above finding that this function is nowhere analytic.

Definition 1.6 *(Harmonic function) Any function $u(x, y)$ with continuous 2nd derivatives satisfying*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2}$$

is **harmonic.** *Equation (2) is* **Laplace's equation.**

Lemma 1.7 *The real and imaginary parts of an analytic function $f(z) = u(x, y) + iv(x, y)$ are harmonic.*

Proof Since f is analytic we know that its derivatives of all orders exist (and are analytic), and thus the partial derivatives of u and v exist and are continuous at all orders in the domain of analyticity. By the Cauchy-Riemann equations we then have

$$u_{xx} + u_{yy} = \frac{\partial}{\partial x}(u_x) + \frac{\partial}{\partial y}(u_y) = \frac{\partial}{\partial x}(v_y) + \frac{\partial}{\partial y}(-v_x) = v_{xy} - v_{xy} = 0.$$

Similarly we can show that $v_{xx} + v_{yy} = 0$, so that both u and v are harmonic.

Lemma 1.8 *If $u(x,y)$ is a harmonic function on a simply-connected domain D then a* **harmonic conjugate** *v exists such that u and v satisfy the Cauchy-Riemann equations (1), and $f = u + iv$ is an analytic function on D.*

Proof We find a harmonic conjugate v by construction. Let $(x_0, y_0) \in D$ and set $v(x_0, y_0) = 0$. Define the value of v at other points $(x, y) \in D$ by the following integral, taken along some path within D joining (x_0, y_0) to (x, y):

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -u_y\, dx + u_x\, dy.$$

This is a good definition provided the value of the integral is independent of the path taken within D from (x_0, y_0) to (x, y). Suppose C_1 and C_2 are two such different paths within D, then the integrals along these two paths are the same provided that the integral around the closed curve C, made by joining C_1 and C_2, is zero. But for any closed curve C we have

$$\oint_C -u_y\, dx + u_x\, dy = \int_S \left(\frac{\partial}{\partial x}(u_x) - \frac{\partial}{\partial y}(-u_y) \right) dxdy, \tag{3}$$

where S is a spanning surface within D for the closed curve C (this is where the simple-connectedness is required), by Green's theorem in the plane; and the right-hand side of (3) vanishes because u is harmonic on D. Thus a conjugate function v can always be defined in this way; and by construction u and v satisfy the Cauchy-Riemann equations

$$v_y = u_x, \qquad v_x = -u_y,$$

so that $u + iv$ defines an analytic function on D by theorem 1.5.

Example 1.9 *We can show that* $u(x,y) = \ln(x^2+y^2)^{1/2}$ *is harmonic away from the origin without having to do any differentiating, by noting that*

$$u(x,y) = \Re(\ln z),$$

and $\ln z$ *is analytic in regions* D *that do not contain the origin* $z=0$*. (Note: We have to define a single-valued* **branch** *of the function, but that is easy to do. See for example [1].)*

We now recall an important result about complex analytic functions, which has implications for real harmonic functions.

Theorem 1.10 *(Maximum modulus principle for analytic functions) (i) If* f *is a non-constant analytic function on the domain* $D \subset \mathbb{C}$ *then* $|f(z)|$ *has no maximum in* D*.*
(ii) If g *is analytic on the bounded domain* $D \subset \mathbb{C}$ *and continuous on* $\overline{D} = D \cup \partial D$ *then* $\exists z_{max} \in \partial D$ *such that* $|g(z)| \leq |g(z_{max})| \forall z \in D$*.*

Proof (i) The proof relies on the fact that analytic functions map open sets to open sets. Let $a \in D$. Then $f(a) \in f(D)$, and $f(D)$, being open, contains a neighborhood of $f(a)$, and therefore a point of larger modulus.
(ii) Since the real function $|g|$ is continuous on the closed bounded set $\overline{D}$, $|g|$ has a maximum at some point $z_{max} \in \overline{D}$,

$$|g(z_{max})| = M = \sup\{|g(z)| : z \in \overline{D}\}.$$

If $z_{max} \in D$ then g is constant by (i) (so it attains its maximum modulus trivially on ∂D). Otherwise $z_{max} \in \partial D$.

Corollary 1.11 *(Minimum modulus principle) If* $f(z)$ *does not vanish on the domain* $D \subset \mathbb{C}$ *then* $|f|$ *attains its minimum value on the boundary* ∂D*.*

Proof If f does not vanish on D then $g(z) = 1/f(z)$ is analytic on D and so, by theorem 1.10, attains its maximum modulus on the boundary ∂D (or is constant, in which case the result follows trivially).

Corollary 1.12 *(Maximum principle for the Laplace equation) A function* $u(x,y)$ *harmonic on a domain* D *attains both its maximum and minimum values on the boundary* ∂D*.*

Proof Since u is harmonic on D it is the real part of a function $f(z)$ analytic on D. The function $g(z) = \exp(f(z))$ is also analytic on D, and so attains its maximum modulus and its minimum modulus on the boundary ∂D. Since

$$|g(z)| = |\exp(f(z))| = |\exp(u + iv)| = \exp(u),$$

it follows that $u(x, y)$ must attain its maximum and minimum values on the boundary ∂D. (The result can also be deduced from the maximum modulus principle, theorem 1.10, alone, by consideration of the analytic function $\exp(-f(z))$ to show that $u(x, y)$ attains its minimum value on the boundary.)

Homework: (If you feel you need practice on these topics.)
Ablowitz & Fokas, Problems for Section 2.1, questions 1,2,4,5. You can also attempt Q7 if you wish.

1.3 Complex Integration

Complex functions may be integrated along contours in the complex plane in much the same way as real functions of 2 real variables can be integrated along a given curve in (x, y)-space. An integration may be performed directly, by introducing a convenient parametrization of the contour, or it can often be done by appealing to one of several powerful theorems of complex analysis, which we recall below.

Example 1.13 *(Direct integration by parametrization) Evaluate*

$$I = \oint \left(\frac{a}{z} + bz\right)\, dz$$

where C is the unit circle $|z| = 1$ in $\mathbb{C}$.

This contour is conveniently parametrized by the real variable $\theta \in [0, 2\pi]$ by $C = \{z \in \mathbb{C} : z = e^{i\theta}\}$, so that $dz = ie^{i\theta} d\theta = iz\, d\theta$. Then

$$I = \int_0^{2\pi} i(a + be^{2i\theta})\, d\theta = 2\pi ia + \frac{b}{2}[e^{2i\theta}]_0^{2\pi} = 2\pi ia.$$

The nonzero result here is a reflection of the fact that the function f has a singularity within the contour of integration at $z = 0$. A function analytic on the domain within the contour would give a zero result, as the following theorem shows:

Theorem 1.14 *(Cauchy) If a function f is analytic on a simply-connected domain D then along any simple closed contour $C \in D$*

$$\oint_C f(z)\, dz = 0. \tag{4}$$

Theorem 1.15 *(Morera's theorem – a Cauchy converse) If $f(z)$ is continuous in a domain D and if*

$$\oint_C f(z)\, dz = 0$$

for every simple closed contour C lying in D, then $f(z)$ is analytic in D.

Theorem 1.16 *(Cauchy's Integral Formula) Let $f(z)$ be analytic inside and on a simple closed contour C. Then for any z inside C,*

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{5}$$

Corollary 1.17 *If f is analytic inside and on a simple closed contour C then all its derivatives exist in the domain D interior to C, and*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta. \tag{6}$$

Corollary 1.18 *(Mean value representations for f) If f is analytic inside and on a circular contour of radius R centered at z then*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{i\theta}) d\theta \tag{7}$$

and

$$f(z) = \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} f(z + re^{i\theta}) r dr d\theta. \tag{8}$$

These representations (7) and (8) state that the value of f at the point z is equal to its mean value integrated around the circle of radius R, and is also equal to its mean value integrated over the area of the same circle, respectively. Proofs of (7) and (8) are easy – for (7) just take $C = \{z \in \mathbb{C} : z = Re^{i\theta}, 0 \leq \theta < 2\pi\}$. Then (8) follows by multiplying both sides of (7) by $r\, dr$ and integrating from $r = 0$ to $r = R$.

Proofs of the results 1.14–1.17 may be found in Ablowitz & Fokas or other textbooks.

Homework: Use the result (8) to prove the maximum modulus theorem 1.10 (restated below) directly, in the case that D is a disc centered on the origin, $D = B(0, R)$.

Theorem 1.19 *(Maximum principles) (i) If f is analytic in domain D then $|f|$ cannot have a maximum in D unless f is constant. (ii) If f is analytic in a bounded region D and $|f|$ is continuous on the closed region $\overline{D}$, then $|f|$ assumes its maximum on the boundary of the region.*

Definition 1.20 *(Entire function) A function that is analytic in the whole complex plane $\mathbb{C}$ is called an entire function.*

Theorem 1.21 *(Liouville) If $f(z)$ is entire and bounded in $\mathbb{C}$ (including at infinity) then $f(z)$ is constant.*

Proof The proof uses the expression (6) for $f'(z)$. Since f is entire we may choose any contour C we wish in this expression, and we choose the circle of arbitrary radius R centered on z: $|\zeta - z| = R$, or $\zeta = z + Re^{i\theta}$, $0 \leq \theta < 2\pi$. Then, since we know $|f| \leq M$ for some $M > 0$, and $d\zeta = iRe^{i\theta}d\theta$, we have

$$|f'(z)| = \frac{1}{2\pi}\left|\oint_C \frac{f(\zeta)}{(\zeta - z)^2}d\zeta\right| \leq \frac{1}{2\pi}\oint_C \frac{|f(\zeta)|}{|\zeta - z|^2}|d\zeta| \leq \frac{M}{2\pi R}\int_0^{2\pi} d\theta = \frac{M}{R}.$$

Since the contour radius R is arbitrary, we may take it to be as large as we wish, and we conclude that $f'(z) = 0$, so that f is constant as claimed.

We note without proof the following much stronger result due to Picard:

Theorem 1.22 *(Picard's little theorem) If f is entire and non-constant, then the range of f is either the entire complex plane, or the plane minus a single point.*

Homework: Ablowitz & Fokas, problems for section 2.6. Questions 1 (a),(b),(c),(d) (integrate by direct parametrization), 3,4,5,7. You could also try 9.

1.4 Poisson's integral formulae for solutions of Laplace's equation

We return now to the concept of a harmonic function (definition 1.6) and see how Cauchy's theorem enables us to write down explicit solutions for the Laplace equation in certain simple geometries with appropriate boundary conditions. Laplace's equation is ubiquitious throughout physics and applied mathematics, governing countless real-world phenomena, so it is very important to be able to solve it. Moreover, we will see later that the ability to solve the equation on even a simple geometry, such as a half-space or a circular disc, enables us to generate solutions on very complicated domains, via the technique of **conformal mapping.**

1.4.1 Poisson integral formula for a circle

We first consider solving Laplace's equation (2) for u on the unit disc, $D = B(0,1) = \{z : |z| \leq 1\}$, subject to the value of u being specified on the boundary (this boundary condition is known as a **Dirichlet** boundary condition, and the associated boundary value problem is the **Dirichlet problem**). When translating between real and complex variables we will use the real polar coordinates (r, θ), so that the complex number $z = re^{i\theta}$. The function u satisfies

$$\left.\begin{aligned} \nabla^2 u = 0 \quad & \text{on } |z| = r \leq 1, \\ u = u_0(\theta) \quad & \text{on } |z| = r = 1. \end{aligned}\right\} \tag{9}$$

Since u is harmonic it is the real part of a function $f(z)$ that is complex analytic on $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Then, for points $z \in D$ Cauchy's integral formula (5) gives

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - z} d\phi, \quad \text{for } |z| < 1, \tag{10}$$

parametrizing the curve ∂D by $\zeta = e^{i\phi}$, so that $d\zeta = i\zeta\, d\phi$. We cannot take the real part of this equation directly to find u, because we don't know $f_{\partial D}$, only its real part. If we wish to find an expression for u in terms of its boundary data, we need to work towards some expression in which f in the integrand is multiplied by some quantity that is purely real. It turns out that we can do this by writing down a similar integral where z is replaced by its *image* in the unit circle, $1/\bar{z}$. Since $|z| < 1$ in the formula above, $|1/\overline{z}| > 1$

and $f(\zeta)/(\zeta - 1/\overline{z})$ is analytic on $|\zeta| \leq 1$. Theorem 1.14 (Cauchy's theorem) thus gives

$$0 = \frac{1}{2\pi i}\oint_{\partial D} \frac{f(\zeta)}{\zeta - 1/\overline{z}} d\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - 1/\overline{z}} d\phi = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\overline{z}}{\overline{z} - \overline{\zeta}} d\phi, \tag{11}$$

using the fact that $\zeta = 1/\overline{\zeta}$ on the boundary in the last equality. Adding and subtracting the above results (10) and (11),

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left(\frac{\zeta}{\zeta - z} \pm \frac{\overline{z}}{\overline{\zeta} - \overline{z}}\right) d\phi. \tag{12}$$

Taking the positive sign in (12) gives

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left(\frac{1 - |z|^2}{|\zeta - z|^2}\right) d\phi,$$

and then taking the real part here gives u:

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} u_0(\phi)\left(\frac{1 - r^2}{|e^{i\phi} - re^{i\theta}|^2}\right) d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} u_0(\phi)\left(\frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2}\right) d\phi. \end{aligned} \tag{13}$$

This result (13) is known as the **Poisson integral formula**, or just **Poisson's formula.**

Taking the negative sign in (12) gives

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left(\frac{1 - 2\zeta\overline{z} + |z|^2}{|\zeta - z|^2}\right) d\phi = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\left(1 + \frac{2i\Im(\overline{\zeta}z)}{|\zeta - z|^2}\right) d\phi,$$

which, on taking the imaginary part and using the Mean Value theorem (7), yields the complex conjugate function to u ($f = u + iv$)):

$$v(r,\theta) = v(0) + \frac{1}{\pi}\int_0^{2\pi} u_0(\phi)\frac{r\sin(\theta - \phi)}{1 - 2r\cos(\theta - \phi) + r^2} d\phi. \tag{14}$$

We can also obtain a formula for $f = u + iv$ in terms of the boundary data $u_0(\phi)$ by combining (13) and (14),

$$f(z) = iv(0) + \frac{1}{2\pi}\int_0^{2\pi} u_0(\phi)\frac{\zeta + z}{\zeta - z} d\phi. \tag{15}$$

That we are able to recover $f(z)$ everywhere within the unit disc, simply from knowledge of the values of its real part on the boundary, is a good illustration of the fact that we cannot arbitrarily specify both the real and the imaginary parts of an analytic function on the unit circle (or indeed on the boundary of a general domain).

We note also that, if u satisfies the Dirichlet problem (9), then its complex conjugate v satisfies a **Neumann problem**, in which the normal derivative is specified on the boundary of the unit disc. This is because the Cauchy-Riemann equations give

$$\frac{\partial u}{\partial s} = -\frac{\partial v}{\partial n} \quad \text{on } \partial D. \tag{16}$$

[Here, s denotes arclength along the boundary ∂D, measured so that s increases as we traverse the boundary with the domain on our left. The coordinate n measures distance along the outward normal vector to D, $\boldsymbol{n}$. If also $\boldsymbol{t}$ denotes the tangent vector to ∂D in the direction of increasing s then we have $\partial_s = \boldsymbol{t} \cdot \nabla$, and $\partial_n = \boldsymbol{n} \cdot \nabla$.] We will see that (16) is true in general in §4.2.4 later, but for now note that it is true for the boundary of the circle considered here because if we transform between cartesian and polar coordinates via

$$\partial_x = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta, \qquad \partial_y = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta,$$

then the Cauchy-Riemann equations transform to

$$u_r = \frac{1}{r}v_\theta, \quad \frac{1}{r}u_\theta = -v_r \tag{17}$$

so that, since $ds = d\theta$ and $\partial_n = \partial_r$ here,

$$\frac{\partial v}{\partial n} = -u_0'(\theta)$$

on the boundary $\zeta = e^{i\theta}$ $(r = 1)$. Hence, the procedure outlined above enables one to solve both Dirichlet and Neumann problems for the Laplace equation on the unit disc.

Example 1.23 *Use the Poisson integral formula to solve Laplace's equation* $\nabla^2 u = 0$ *on the unit disc* $r \le 1$*, with boundary data* $u = u_0(\theta) = 1$ *on* $r = 1$*.*

Of course, we know that the (unique) solution to this problem is just $u \equiv 1$, but it is instructive to see how the above results lead to this. We can either use (13) directly, giving

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)\, d\phi}{1 - 2r\cos(\theta - \phi) + r^2}, \tag{18}$$

an integral that can be evaluated directly[1] to obtain the result; or, more straightforwardly, we can use the complex form of the solution (15), noting that on ∂D, with $\zeta = e^{i\theta}$, $d\phi = d\zeta / i\zeta$, so that

$$f(z) = iv(0) + \frac{1}{2\pi} \int_{\partial D} \frac{(\zeta + z)\, d\zeta}{i\zeta(\zeta - z)}.$$

The integrand is singular at $\zeta = 0$, z (simple poles). The residues at these points are

$$\zeta = 0: \quad \lim_{\zeta \to 0} \frac{\zeta + z}{i(\zeta - z)} = -\frac{1}{i} = i,$$
$$\zeta = z: \quad \lim_{\zeta \to z} \frac{\zeta + z}{i\zeta} = \frac{2}{i} = -2i.$$

Thus, by the Residue theorem,

$$f(z) = iv(0) + \frac{1}{2\pi} 2\pi i(i - 2i) = 1 + iv(0).$$

The real part gives $u \equiv 1$, as it should.

Note that knowledge of an exact solution with given boundary data can, in some circumstances, give a way to evaluate certain integrals. For example, $u = r\cos\theta$ ($u \equiv x$) is an exact solution of Laplace's equation in polar coordinates, with boundary data $u(1,\theta) = u_0(\theta) = \cos\theta$. By the Poisson integral formula (13), we therefore have

$$\cos\theta = \frac{1 - r^2}{2\pi r} \int_0^{2\pi} \frac{\cos\phi\, d\phi}{1 - 2r\cos(\theta - \phi) + r^2},$$

an integral that is not straightforward to evaluate otherwise.

[1] see *e.g.* Gradshteyn & Ryzhik [5] for the result of the integral, or try it yourself – one way is to show that the integrals defining quantities $\partial u / \partial \theta$ and $\partial u / \partial r$ (differentiate under the integral sign in (18)) both evaluate to zero, so that u must be a constant, equal to the value on the boundary.

Remark Note that if we restrict the condition that the functions u, v, f have no singularities within the domain, then we lose the uniqueness. In the example above, if singularities of u within the unit disc are admitted then there are infinitely many solutions to the problem as stated. For example,

$$u(r,\theta) = 1 + C\log r + \sum_{n=1}^{\infty}\big(a_n(r^n - r^{-n})\cos n\theta + b_n(r^n - r^{-n})\sin n\theta\big)$$

solves the problem for arbitrary C, a_n, $b_n \in \mathbb{R}$, but is of course singular at the origin. This function may be recognized as

$$u = \Re\left(1 + C\log z + \sum_{n=1}^{\infty}[(a_n - ib_n)z^n - (a_n + ib_n)z^{-n}]\right).$$

1.4.2 Poisson integral formula for a half-space

Consider now the problem of solving Laplace's equation on a half-space, say the upper half-plane $y \geq 0$, with Dirichlet data $u(x,0) = u_0(x)$ specified all along the real axis. We know that $u = \Re(f(z))$ for some function $f(z)$ complex analytic on the upper half-plane, D^+. We restrict attention here to the case in which solutions decay at infinity, such that $|f(z)| \to 0$ uniformly as $|z| \to \infty$ with $z \in D^+$. As before, we take Cauchy's integral formula as our starting point, with contour C a large semicircle of radius R in the upper half-plane. Then, for $z \in D^+$ we have $\bar{z} \in D^-$ (the lower half-plane), and thus

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z}, \tag{19}$$

$$0 = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - \bar{z}}. \tag{20}$$

Similar to what was done before, we add and subtract these formulae, the aim being to obtain an integrand in which $f(\zeta)$ is multiplied by a purely real or purely imaginary combination, so that we can take real and imaginary parts and isolate the boundary data u_0 in the integral (remember, we do not know the values of $v = \Im(f)$ on the boundary). The assumption on the behavior of $|f|$ at infinity means that the integrals along the circular arc portions will go to zero as $R \to \infty$, and we will be left only with the integrals along the real axis. Thus, adding and subtracting gives, in the limit $R \to \infty$,

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} f(\zeta) \left(\frac{1}{\zeta - z} \pm \frac{1}{\zeta - \bar{z}} \right) d\zeta.$$

Writing $\zeta = \xi \in \mathbb{R}$ and $z = x + iy$, the $(+)$ sign above leads to

$$u(x,y) + iv(x,y) = \frac{1}{\pi i} \int_{-\infty}^{\infty} f(\xi) \frac{\xi - x}{(x-\xi)^2 + y^2} d\xi, \tag{21}$$

while the $(-)$ gives

$$u(x,y) + iv(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{y}{(x-\xi)^2 + y^2} d\xi. \tag{22}$$

Therefore, we can solve for both u and v in terms of the boundary data on u, by taking the imaginary part of (21), and the real part of (22), giving (respectively):

$$v(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} u_0(\xi) \frac{x - \xi}{(x-\xi)^2 + y^2} d\xi, \tag{23}$$

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} u_0(\xi) \frac{y}{(x-\xi)^2 + y^2} d\xi. \tag{24}$$

Again, when u is the solution to the Dirichlet problem, v satisfies the corresponding Neumann problem, via the Cauchy-Riemann equations, since

$$\left.\frac{\partial v}{\partial n}\right|_{\partial D} = -\left.\frac{\partial v}{\partial y}\right|_{y=0} = -\left.\frac{\partial u}{\partial x}\right|_{y=0} = -u_0'(x).$$

Example 1.24 *Solve Laplace's equation $\nabla^2 u = 0$ on $y \geq 0$, with $u(x,0) = x/(x^2+1)$.*

The boundary data decays as $|x| \to \infty$, so the Poisson integral formula (24) applies, and gives

$$\begin{aligned} u(x,y) &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\xi}{(\xi^2+1)((\xi-x)^2+y^2)} d\xi \\ &= \frac{x}{x^2 + (y+1)^2}. \end{aligned}$$

As before, the integral is nontrivial to evaluate, though it can be done! Note that the result as above is obtained only for $y > 0$; if $y < 0$ then one obtains $x/(x^2+(y-1)^2)$, this being the solution to the Laplace equation on the *lower* half plane with the same boundary data (see also the comments below).

As with the circular domain, there are other ways to obtain this solution; for example, by Fourier transform (see §2.3.1); or, for this simple case, by inspection. We know that the solution is the real part of a function $f(z$ analytic on $\Re(z) \geq 0$; and the boundary data gives

$$\frac{x}{x^2+1} \equiv \frac{x}{(x+i)(x-i)} = \frac{1}{2}(f(x) + \overline{f(x)}).$$

Therefore we might try a function of the form $f(z) = a/(bz + c)$, $a, b, c \in \mathbb{C}$. This gives

$$\frac{x}{(x+i)(x-i)} = \frac{1}{2}\left[\frac{a}{bx+c} + \frac{\overline{a}}{\overline{b}x+\overline{c}}\right] = \frac{(a\overline{b} + \overline{a}b)x + a\overline{c} + \overline{a}c}{2|bx+c|^2}.$$

Since the denominator in the left-hand side is exactly $|x+i|^2$, this suggests that $b = 1$, $c = i$ might work; and then matching the numerators gives $a = 1$. Therefore, the function f that is complex analytic on D^+, decays at infinity, and satisfies the boundary data for its real part, is $f(z) = 1/(z+i)$. Note that we could have equally well noted that the denominator in the left-hand side of the last equation is also equal to $|x-i|^2$, which would lead to $f(z) = 1/(z-i)$ as a candidate. However, this function fails the requirement for analyticity on D^+. If instead we had been asked to solve on the lower half plane, for the same boundary data, this would be the function we would use.

This inspection procedure then finally yields

$$u(x, y) = \Re\left(\frac{1}{z+i}\right) = \frac{x}{x^2 + (y+1)^2}.$$

1.5 Taylor series, Laurent series

Complex functions have convergent infinite series expressions (Taylor series) at points where they are analytic, just as real valued functions do.

Theorem 1.25 *(Taylor series) Suppose $f(z)$ is analytic for $|z - z_0| \leq R$. Then*

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)(z - z_0)^j}{j!}. \tag{25}$$

This is the Taylor series expansion of f about the point z_0, and it converges uniformly on $|z - z_0| \leq R$.

The power series expansion in this theorem contains only positive powers of $(z - z_0)$, reflecting the fact that the function f is analytic at the point z_0 (so the power series is well-behaved there). If we know only that f is analytic in some **annulus** about z_0, $R_1 \leq |z - z_0| \leq R_2$, then we can find a more general **Laurent series** expansion for $f(z)$ on the annulus, that contains both positive and negative powers of $(z - z_0)$.

Theorem 1.26 *(Laurent series) A function $f(z)$ analytic in an annulus $R_1 \leq |z - z_0| \leq R_2$ may be written as*

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n \tag{26}$$

in $R_1 \leq |z - z_0| \leq R_2$, where

$$C_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}}$$

and C is any simple closed contour lying within the region of analyticity and enclosing the inner boundary $|z - z_0| = R_1$. This series representation for f is unique, and converges uniformly to $f(z)$ for $R_1 < |z - z_0| < R_2$.

Remark The coefficient of the term $(z - z_0)^{-1}$ in the Laurent expansion, C_{-1}, is known as the **residue** of the function f at z_0.

Remark In the case that f is analytic on the whole disc $|z| \leq R_2$ it is easily checked, using formula (6) for $n \geq 0$ and Cauchy's theorem 1.14 for $n < 0$, that the Laurent expansion (26) for f reduces to the Taylor expansion (25).

1.6 Singular points

If the function $f(z)$ is analytic on $0 < |z - z_0| < R$ for some $R > 0$, but is not analytic at z_0, then z_0 is an **isolated singular point** of f. There are several types of such isolated singular points.

(i) If z_0 is an isolated singular point of f at which $|f|$ is bounded (*i.e.* there is some $M > 0$ such that $|f(z)| \leq M$ for all $|z| \leq R$) then z_0 is a **removable singularity** of f. Clearly, all coefficients C_n with $n < 0$ must be zero in the Laurent expansion (26), thus f in fact has a regular power series expansion $f(z) = \sum_{n=0}^{\infty} C_n(z - z_0)^n$ valid for $0 < |z - z_0| < R$. Since this power series converges at $z = z_0$ it follows that if we simply redefine $f(z_0) = C_0$ then f is analytic on the whole disc $|z - z_0| < R$, with Taylor series $f(z) = \sum_{n=0}^{\infty} C_n(z - z_0)^n$.

Example 1.27 *The function $f(z) = \sin z/z$ has a removable singularity at $z = 0$. Strictly speaking this function is undefined at zero, but it has Laurent series*

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}, \quad |z| > 0,$$

so $z = 0$ is a removable singularity of f, which we remove by defining $f(0) = 1$.

Example 1.28 *The function*

$$f(z) = \frac{e^{z^2} - 1}{z^2}$$

has a removable singularity at $z = 0$. It is undefined at $z = 0$, but has Laurent expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(n+1)!}, \quad |z| > 0,$$

so redefining $f(0) = 1$ removes the singularity.

(ii) If $f(z)$ can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^N}$$

where N is a positive integer and $g(z)$ is analytic on $|z - z_0| < R$ with $g(z_0) \neq 0$, then the point z_0 is a **pole of order** N (a **simple pole** if $N = 1$). Clearly $f(z)$ is unbounded as $z \to z_0$.

Example 1.29 *The function*

$$f(z) = \frac{e^{2z} - 1}{z^2}$$

has a simple pole at $z = 0$. Its Laurent expansion is given by

$$f(z) = \sum_{n=-1}^{\infty} \frac{2^{(n+2)} z^n}{(n+2)!},$$

so the residue at $z = 0$ is $C_{-1} = 2$.

Functions having poles as their only singularities are known as **meromorphic**.
(iii) An isolated singular point that is neither removable nor a pole is called an **essential singular point.** The Laurent expansion of the function about such a singular point is non-terminating for $n < 0$, that is, there is no positive integer N such that $C_{-n} = 0$ for all $n > N$.

Example 1.30 *The function*

$$f(z) = e^{1/z^2} \quad \textit{with Laurent expansion} \quad f(z) = \sum_{n=-\infty}^{0} \frac{z^{2n}}{|n|!}$$

has an essential singularity at $z = 0$. It is analytic on the rest of the complex plane, and the Laurent series converges uniformly everywhere except $z = 0$.

Other types of non-isolated singularities include **branch points** and **cluster points**.

Homework: Review your notes on branch points, branch cuts and multi-functions.

Homework: Ablowitz & Fokas, problems for section 2.1, question 3.
Ablowitz & Fokas, problems for section 3.2, question 2(b),(f), 6(c).
Ablowitz & Fokas, problems for section 3.3, question 3, 4(d).

1.6.1 Caserati-Weierstrass Theorem

Functions with essential singularities have remarkable properties in the neighborhood of these singularities, as the following theorem demonstrates.

Theorem 1.31 *(Caserati-Weierstrass, or Weierstrass-Caserati) If $f(z)$ has an essential singularity at $z = z_0$, then for any $w \in \mathbb{C}$, f becomes arbitrarily close to w in a neighborhood of z_0. That is, given $w \in \mathbb{C}$ and $\epsilon > 0$, $\delta > 0$, there exists a z such that*

$$|f(z) - w| < \epsilon$$

with $0 < |z - z_0| < \delta$.

Proof The result is proved by contradiction. We suppose that $|f(z)-w| > \epsilon$ whenever $|z - z_0| < \delta$ (where δ is small enough so that f is analytic on $0 < |z - z_0| < \delta$). Then in this region,

$$g(z) = \frac{1}{f(z) - w}$$

is analytic, and hence bounded; specifically, $|g(z)| < 1/\epsilon$. (Note that g is not identically constant, since this would mean f is constant and hence has no essential singularity at z_0.) The function g is thus representable by a power series about z_0

$$g(z) = \sum_{0}^{\infty} C_n (z - z_0)^n,$$

so its only possible singularity is removable, and we can ensure analyticity on $|z - z_0| < \delta$ by defining $g(z_0) = C_0$. It follows that

$$f(z) = w + \frac{1}{g(z)},$$

and thus $f(z)$ is either analytic with $g(z) \neq 0$ on the disc, or else $f(z)$ has a pole of order N, where c_N is the first nonzero coefficient in the Taylor series expansion of g about z_0 (we already know there must be a nonzero coefficient since g cannot be constant). In either case, we contradict the hypothesis that f has an essential singular point at z_0, and the theorem is proved.

Even more remarkable results about essential singularities may be proved. The best-known of these results is **Picard's theorem**, sometimes known as Picard's Great Theorem (to distinguish it from other results due to Picard, in particular, Picard's Little Theorem). We state this theorem without proof, since the proof relies on **Schottky's Theorem** which itself has a lengthy and technical proof.

Homework (Recommended!) Read up on the proof of Picard's Great Theorem for yourself.

Theorem 1.32 *(Picard's Theorem, or Picard's Great Theorem) Suppose $f(z)$ has an essential singularity at $z_0 \in \mathbb{C}$. Then, on any open set containing z_0, $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often.*

1.7 Analytic continuation

This is a key concept in many applications of complex analysis, which we will use later on in our work on free boundary problems. It may be expressed in many different ways, but the following theorems give common and useful statements of the analytic continuation property.

Theorem 1.33 *(Analytic continuation (1)) Suppose $f(z)$ and $g(z)$ are analytic in a common domain D. If f and g coincide in some subdomain $D' \subset D$, or on a curve $\Gamma \subset D$, then $f(z) = g(z)$ everywhere in D.*

Proof (Sketch) We consider the case in which f and g coincide on a subdomain D'. Let $z_0 \in D'$, and then take the largest circle $C \subset D$, within which both f and g are known to be analytic (see figure 1). By Taylor's theorem 25, both f and g have Taylor series expansions

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!} = \sum_{n=0}^{\infty} C_n(z_0)(z-z_0)^n, \tag{27}$$

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)(z-z_0)^n}{n!} = \sum_{n=0}^{\infty} E_n(z_0)(z-z_0)^n, \tag{28}$$

that converge inside C. Also, since D' is open, it contains an ϵ-neighborhood of z_0 (shaded in the sketch 1), and on which f and g are known to be identical.

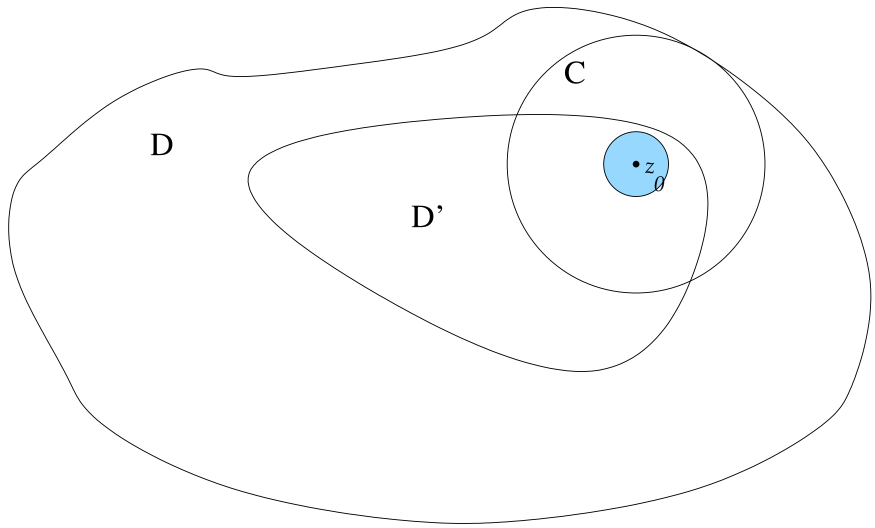

Figure 1: Definition sketch for the proof of theorem 1.33.

Within this neighborhood then, the Taylor expansions must be identical, by uniqueness of the Taylor series expansion for an analytic function. But the Taylor series within the ϵ-neighborhood are exactly (27), (28), and it follows that f and g are identical on the larger domain contained within C. We can now choose a new point z_1 within the extended domain on which f and g coincide, and show in the same way that the two functions coincide on a yet larger subdomain of D. Continuing in this manner we may cover the entire domain D.

Another theorem, which we state without proof, is:

Theorem 1.34 *(Analytic continuation (2)) A function that is analytic in some domain D is uniquely determined either by its values in some subdomain $D' \subset D$ or along some curve $\Gamma \subset D$.*

The analytic continuation concept may be illustrated by examples.

Example 1.35 *Consider the function $f(z)$ defined on $|z| < a$ by*

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n, \quad \textit{for } a > 0.$$

This series converges uniformly for $|z| < a$ and so defines an analytic function in that region, but it diverges for $|z| > a$ and so does not represent an analytic function there. However, the function

$$g(z) = \frac{a}{a - z}$$

is defined and analytic for all $z \neq a$, and moreover, for $|z| < a$ we have

$$g(z) = (1 - z/a)^{-1} = \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n \equiv f(z).$$

We say that $g(z)$ represents the **analytic continuation** of $f(z)$ outside the disc $|z| < a$. The representation $f(z)$, which is only valid on this subdomain, is sufficient to determine uniquely the function $g(z)$, valid in the entire complex plane minus the single point $z = a$.

Example 1.36 *What is the analytic continuation of the function that takes values $y^3 = y$ on the imaginary axis $z = iy$?*

An alternative way to phrase this question would be: Find the unique function $f(z)$, complex analytic in some neighborhood of the imaginary axis, such that $f(z) = y^3 - y$ on the imaginary axis $z = iy$.

To answer the question, we note that on $z = iy$, $y = -iz$, and so the boundary values of f may be re-expressed as

$$f(z) = iz^3 + iz \quad \text{on } \Re(z) = 0.$$

Since f as defined here is analytic in a neighborhood of the line, and is equal to the given values on the line, the analytic continuation property tells us that it is the unique analytic continuation of the boundary data away from the boundary curve.

Remark Contrast the analytic continuation property, in which both real *and* imaginary parts of an analytic function are specified on a given curve, with the solutions to Laplace's equation generated previously by our Poisson integral formulae. There we saw that specifying only the *real* part of a function (analytic on a given domain) on the boundary of the domain was sufficient to determine both real and imaginary parts of the function. The difference here is that the analytic continuation we generate by specifying both real and imaginary parts will, in general, have singularities in the domain of continuation. By contrast, any solutions generated by the Poisson formulae are guaranteed to be analytic in the chosen domain. Analytic continuation is an *ill-posed* technique mathematically (see, *e.g.* [8] § 2.3 in this context).

We will use this property later in applications. Another theorem on analytic continuation is proved below:

Theorem 1.37 *(Analytic continuation (3)) Let D_1 and D_2 be two disjoint domains, whose boundaries share a common contour Γ. Let $f(z)$ be analytic in $D_1 \cup \Gamma$, and $g(z)$ be analytic in $D_2 \cup \Gamma$, and let $f(z) = g(z)$ on Γ. Then the function*

$$H(z) = \begin{cases} f(z) & z \in D_1 \\ f(z) = g(z) & z \in \Gamma \\ g(z) & z \in D_2 \end{cases}$$

is analytic in $D = D_1 \cup \Gamma \cup D_2$. We say that g is the **analytic continuation** *of f into D_2, and vice-versa.*

Proof The proof uses **Morera's theorem** 1.15. Consider an arbitrary closed contour C in D. If C does not intersect Γ then it lies entirely in either D_1 or D_2, and thus $\oint_C H(z)dz = 0$. If C intersects Γ then (see figure 2) we have

$$\begin{aligned} \oint_C H(z)\,dz &= \int_{C_1} f(z)\,dz + \int_{C_2} g(z)\,dz \\ &= \int_{C_1} f(z)\,dz + \int_{\Gamma;a}^{b} f(z)\,dz + \int_{\Gamma;b}^{a} g(z)\,dz + \int_{C_2} g(z)\,dz \\ &= \int_{C_1\cup\Gamma} f(z)\,dz + \int_{C_2\cup\Gamma} g(z)\,dz \\ &= 0. \end{aligned}$$

Note that we can insert the extra two integrals along the curve Γ in the second line above, since they are of equal and opposite value (due to the equality of f and g along the common curve Γ). In the third line above, the integrals both vanish by Cauchy's theorem, as the contours $C_1 \cup \Gamma$ and $C_2 \cup \Gamma$ are each closed contours lying in the domain of analyticity of $H(z)$. Since the contour $C \in D$ is arbitrary, $H(z)$ is analytic in D by Morera's theorem, as claimed.

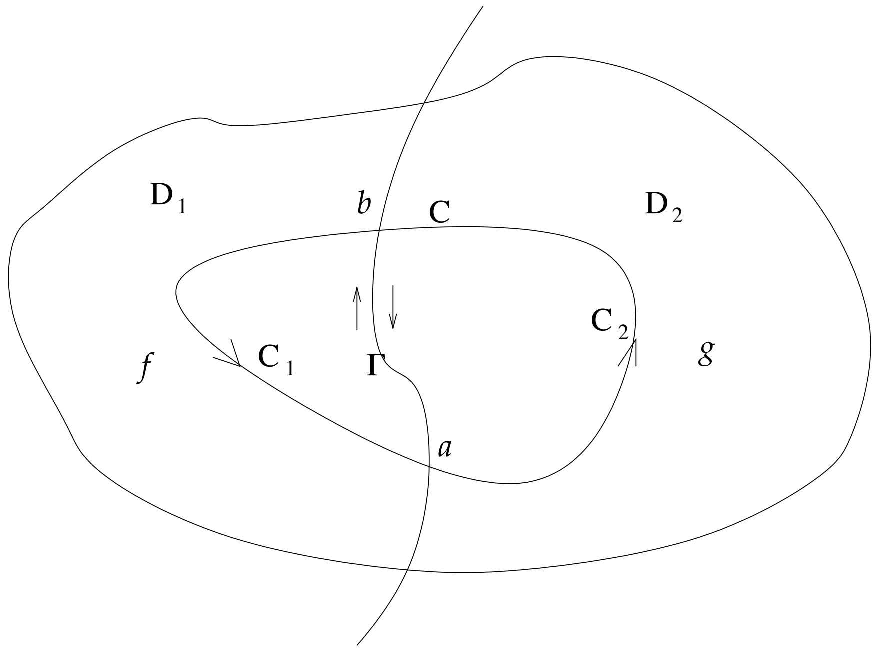

Figure 2: Definition sketch for the proof of theorem 1.37

Homework: Ablowitz & Fokas, problems for section 3.2, question 5. Interpret the results in terms of analytic continuations, relating to the theorems above.
What is the analytic continuation of the function that takes the value $\sin x$ at any point on the real axis, $z = x \in \mathbb{R}$?
What is the analytic continuation of the function that takes the value $x^2 - 2ix - 1$ at any point on the real axis, $z = x \in \mathbb{R}$?
What about the function that takes values e^y on the imaginary axis $z = iy$, $y \in \mathbb{R}$?
Find the analytic continuation of the function taking values $F(y)$ on the imaginary axis $z = iy$, $y \in \mathbb{R}$, if F is infinitely differentiable.

We give one final example, which sets the scene for some of our later work on free boundary problems.

Example 1.38 *The Schwarz function Any infinitely differentiable (analytic) curve* Γ *in the plane has a representation in terms of a unique function* $g(z)$*,* *complex analytic in some neighborhood of* Γ*:*

$$\overline{z} = g(z) \quad \textit{on } \Gamma. \tag{29}$$

Equation (29) defines the curve. Given the curve (e.g. as an algebraic equation $F(x,y) = 0$*), we know only the values of* g *on* Γ*. We can find these by solving*

$$F\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right)$$

for $\overline{z}$*. Analytic continuation will then tell us what the function* $g(z)$ *is at arbitrary points of the complex plane.*

To see how this works for some elementary examples, consider the unit circle $x^2 + y^2 = 1$. The above formula for extracting the Schwarz function gives

$$\frac{1}{4}(z+\overline{z})^2 - \frac{1}{4}(z-\overline{z})^2 = 1 \quad \rightarrow \quad z\overline{z} = 1, \quad \text{on } \Gamma.$$

(Of course, we could have written this down right away for this simple example.) Thus, $g(z) = 1/z$ on Γ, and hence $g(z) \equiv 1/z$, by analytic continuation (since $1/z$ is clearly analytic in a neighborhood of Γ).

Homework: Find the Schwarz function of the straight line $y = mx + c$, for any real constants m, c.

It is not usually so straightforward to extract the Schwarz function of a given curve! Usually in applications we want to find the Schwarz function of the curve bounding a given domain D. If D has a complicated boundary shape then one way to find its Schwarz function is to use the fact (which we will explore in detail later) that any simply-connected domain $D \subset \mathbb{C}$ may be recognized as the *image*, under a 1-1 analytic function ("conformal mapping"), of a nice domain such as the unit disc, $B(0;1)$. We can write

$$z \in D \quad \Rightarrow \quad z = f(\zeta), \quad |\zeta| \leq 1,$$

and further, boundary points map to boundary points, so that

$$z \in \partial D \quad \Rightarrow \quad z = f(\zeta), \quad |\zeta| = 1.$$

We need one more piece of useful information: given any analytic function $F(w)$, the function defined by taking $\overline{F(\overline{w})}$ is another analytic function, which we write as $\overline{F}(w)$: the *complex conjugate* function to F.

To determine the Schwarz function of the boundary curve ∂D, we need to know the values of $\overline{z}$ on the boundary. We know that a boundary point is the image of a point ζ on the boundary of the unit circle, $|\zeta| = 1$. Therefore,

$$\begin{aligned} z|_{\partial D} &= f(\zeta)|_{|\zeta|=1} = f\left(\frac{1}{\overline{\zeta}}\right)\bigg|_{|\zeta|=1} \\ \Rightarrow \quad \overline{z}|_{\partial D} &= \overline{f\left(\frac{1}{\overline{\zeta}}\right)}\Bigg|_{|\zeta|=1} = \overline{f}\left(\frac{1}{\zeta}\right)\bigg|_{|\zeta|=1}. \end{aligned}$$

As written, this identity holds only gives $g(z)$ on the boundary curve (∂D, or $|\zeta| = 1$), but we know that the function $\overline{f}(1/\zeta)$ is analytic, at least in some neighborhood of the boundary. Therefore, we may analytically continue the identity away from the boundary, and conclude that

$$g(z) = \overline{f}\left(\frac{1}{\zeta}\right),$$

or equivalently,

$$g(f(\zeta)) = \overline{f}\left(\frac{1}{\zeta}\right), \quad \text{or} \quad g(z) = \overline{f}\left(\frac{1}{f^{-1}(z)}\right)$$

(we know the inverse function f^{-1} exists since f is 1-1).

1.7.1 Natural barriers

There are some types of singularities (non-isolated) that preclude analytic continuation. We refer to these singularities as **natural barriers** of the function under consideration. A classic example of a function exhibiting a natural barrier is provided by

$$f(z) = \sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} z^{2^n}. \tag{30}$$

The ratio test guarantees convergence where $\lim_{n\to\infty} f_{n+1}/f_n < 1$, and

$$\frac{f_{n+1}}{f_n} = \frac{z^{2^{n+1}}}{z^{2^n}} = \frac{z^{2^n.2}}{z^{2^n}} = \frac{(z^{2^n})^2}{z^{2^n}} = z^{2^n}.$$

This limit is less than one only if $|z| < 1$, so the series converges on the open unit disc, but f clearly diverges for $z = 1$. Also,

$$f(z^2) = \sum_{n=0}^{\infty}(z^2)^{2^n} = \sum_{n=0}^{\infty} z^{2^{(n+1)}} = \sum_{n=1}^{\infty} z^{2^n} = \sum_{n=0}^{\infty} z^{2^n} - z = f(z) - z,$$

so f satisfies the functional equation

$$f(z) = f(z^2) + z. \tag{31}$$

Repeated application of this formula gives

$$f(z^2) \quad = \quad f(z^4) + z^2$$

and then substituting in (31) gives

$$f(z) = f(z^4) + z^2 + z. \tag{32}$$

Similarly, replacing z by z^4 in (31) gives

$$f(z^4) = f(z^8) + z^4,$$

and then substituting in (32) gives

$$f(z) = f(z^8) + z^4 + z^2 + z.$$

Continuing in this manner (prove by induction if you wish) we see that f satisfies

$$f(z) = f(z^{2^m}) + \sum_{n=0}^{m-1} z^{2^n}, \tag{33}$$

for any positive integer m. Since we know $f(1) = \infty$, this equation tells us that f is also singular at all points $z_{s,m}$ where $z_{s,m}^{2^m} = 1$, because at such points (33) gives

$$f(z_{s,m}) = f(1) + \sum_{n=0}^{m-1} z^{2^n} = \infty.$$

Therefore, f is singular at all points

$$z_{s,m} = e^{2\pi i s/2^m}, \qquad s = 1, 2, \ldots, 2^m, \quad \text{for any } m \in \mathbb{Z}.$$

In order to use the analytic continuation principle to extend the domain of analyticity of f into $|z| \geq 1$ we need (as a minimum) to find some arc of the unit circle (which bounds the known domain of analyticity, $|z| < 1$) on which f is analytic. However, any such arc segment, however small, will always contain a 2^mth root of unity, for some integer m, and thus a singularity of f. Hence there is no finite arc-length with $|z| = 1$ on which f is analytic.

Homework: Ablowitz & Fokas, problems for section 3.5, question 4.

2 Cauchy's Residue theorem (and others) and Applications

2.1 Cauchy's Residue Theorem

Knowing more about non-analytic functions, we are in a position to extend Cauchy's theorem 1.14 to situations where the function contains singularities within the contour of integration.

Theorem 2.1 *(Cauchy's Residue theorem) Let $f(z)$ be analytic inside and on a simple closed contour C, except for a finite number of isolated singular points $z_1, z_2, \ldots, z_N$ inside C. Then*

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{N} a_j,$$

where a_j is the residue of $f(z)$ at the point $z = z_j$, $a_j = Res(f(z); z_j)$.

This remarkable theorem has many applications, including the summation of infinite series, and inversion of Laplace transforms. Residues may be evaluated using the following theorem:

Theorem 2.2 *(Evaluating residues at poles) If $f(z)$ has a pole of order k at $z = z_0$ then*

$$Res(f(z), z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}}((z - z_0)^k f(z)) \tag{34}$$

Proof Direct manipulation of the Laurent expansion of f about z_0,

$$f(z) = \sum_{n=-k}^{\infty} C_n (z - z_0)^n$$

yields the result for $\mathrm{Res}(f(z), z_0) = C_{-1}$.

2.1.1 Application to summing series

The idea here is to recognise the series to be summed as a sum of residues at the singularities of an otherwise analytic function. By integrating around an appropriate contour on which the function decays suitably as the contour moves off to infinity, one can then apply the residue theorem to deduce the value of the sum of residues. The procedure is best illustrated by examples.

Example 2.3 *By evaluation of a suitable contour integral, find the infinite sum*

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The trick is to find a function whose residues will lead to the appropriate sum when the residue theorem is applied. Consider the function $f(z) = \cot \pi z / z^2$, which has simple poles at the points $z_n = n$ for $n \in \mathbb{Z}$, $(n \neq 0)$, and a triple pole at the origin. The residues at z_n are evaluated by applying (34):

$$\mathrm{Res}(f(z); z_n) = \lim_{z \to n} (z - n) f(z) = \frac{\cos n\pi}{n^2} \lim_{z \to n} \frac{(z - n)}{\sin \pi z} = \frac{1}{\pi n^2}.$$

The residue of the triple pole at $z = 0$ is also given by (34),

$$\mathrm{Res}(f(z); 0) = \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2}(z \cot \pi z) = \lim_{z \to 0} \pi \mathrm{cosec}^2 \pi z (\pi z \cot \pi z - 1) = -\frac{\pi}{3}.$$

The Residue theorem 2.1 states that, for a rectangular contour C_N composed of the straight lines $\Re(z) = \pm(N + 1/2)$ (so that we avoid the poles of the function) and $\Im(z) = \pm N$ (for $N \in \mathbb{N}$), we have

$$\oint_{C_N} f(z) dz = 2 \sum_{n=1}^{N} \frac{1}{\pi n^2} - \frac{\pi}{3}. \tag{35}$$

If we let $N \to \infty$ then we retrieve the sum S on the right-hand side, in terms of the contour integral.

To evaluate the contour integral we split it into four natural parts along each straight-line segment, and show that the size of each integral goes to zero as $N \to \infty$.

On the portion $\Re(z) = (N+1/2)$ we have $z = N+1/2+iy$, $-N < y < N$ and

$$|\cot \pi z| = \left|\frac{\cos[\pi(N+1/2)+i\pi y]}{\sin[\pi(N+1/2)+i\pi y]}\right| = \left|\frac{\sinh \pi y}{\cosh \pi y}\right| \leq 1,$$

and similarly on the portion $\Re(z) = -(N+1/2)$.

On the portion $\Im(z) = N$ we have $z = x+iN$, $-(N+1/2) < x < N+1/2$ and

$$\begin{aligned}
|\cot \pi z|^2 &= \left|\frac{\cos[\pi x + iN\pi]}{\sin[\pi x + iN\pi]}\right|^2 = \left|\frac{\cos \pi x \cosh N\pi - i \sin \pi x \sinh N\pi}{\sin \pi x \cosh N\pi + i \cos \pi x \sinh N\pi}\right|^2 \\
&= \frac{\cos^2 \pi x \cosh^2 N\pi + \sin^2 \pi x \sinh^2 N\pi}{\sin^2 \pi x \cosh^2 N\pi + \cos^2 \pi x \sinh^2 N\pi} \\
&= \frac{\cosh^2 N\pi + \sin^2 \pi x}{\cosh^2 N\pi + \cos^2 \pi x} \\
&\leq 1+\epsilon,
\end{aligned}$$

where we can take ϵ as small as we like by taking N sufficiently large (but it is enough for us that we bound it). We can similarly bound $|\cot \pi z|$ by $1+\epsilon$ on $\Im(z) = -N$.

We can therefore make $|\cot \pi z| \leq 2$ on all parts of the integration contour, for N sufficiently large, and

$$\left|\oint_{C_N} f(z)dz\right| \leq \oint_{C_N} |f(z)|\,|dz| \leq \frac{2}{N^2}\oint_{C_N} |dz| = \frac{4(4N+1)}{N^2} \to 0 \quad \text{as } N \to \infty.$$

Letting $N \to \infty$ in (35) then gives the result

$$\sum_{n=1}^{N} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This general method will work for any sum $\sum_{n=1}^{\infty} \phi(n)$, where ϕ has the following properties:

- $\phi(n) = \phi(-n)$ for $n = 1, 2, \ldots$;
- $\phi(n)$ is a rational function;
- $\phi(z) = O(|z|^{-2})$ as $|z| \to \infty$.

The procedure is to integrate $f(z) = \phi(z)\pi\cot\pi z$ around the same rectangular contour C_N and use the Residue theorem. The term $\pi\cot\pi z$ gives simple poles of f at each integer n (where ϕ is nonzero and analytic) of residue $\phi(n)$. The bound on ϕ ensures, as in the worked example above, that $\int_{C_N} f(z)\,dz \to 0$ as $N \to \infty$.

Under these same conditions on ϕ we can evaluate $\sum_{n=1}^{\infty}(-1)^n\phi(n)$ by integrating $f(z) = \phi(z)\pi\mathrm{cosec}\pi z$ around the same contour C_N. The cosec term leads to a simple pole of residue $(-1)^n\phi(n)$ at each integer n (again supposing ϕ is nonzero and analytic there). The integral around the contour will go to zero as $N \to \infty$, and the Residue theorem will give the result.

Homework: Ablowitz & Fokas, Problems for section 4.2, questions 8 and 9.

2.2 Principle of the argument & Rouché's theorem

Residue calculus can also be applied to deduce results about the number of zeros and poles of a meromorphic function in a given region.

Theorem 2.4 *(Argument principle) Let f be meromorphic inside and on a simple closed contour C, with N zeros inside C, P poles inside C (where multiple zeros and poles are counted according to multiplicity), and no zeros or poles on C. Then*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N - P = \frac{1}{2\pi}[\arg(f(z))]_C$$

(the last expression denotes the change in the argument of f as C is traversed once).

Proof Let z_i be a zero/pole of order n_i. Then

$$f(z) = (z - z_i)^{\pm n_i} g_i(z), \qquad (+\text{ for zero, } -\text{ for pole})$$

where $g_i(z_i) \neq 0$ and $g_i(z)$ is analytic in some neighborhood $B(z_i, \epsilon_i)$ of z_i. Thus in such a neighborhood we can write

$$\frac{f'(z)}{f(z)} = \frac{\pm n_i}{(z - z_i)} + \phi_i(z)$$

where $\phi_i(z) = g_i'(z)/g_i(z)$ is analytic in $B(z_i, \epsilon_i)$, and in the region D' made up of the interior of the curve C, with each $B(z_i, \epsilon_i)$ removed, f'/f is analytic. Application of Cauchy's theorem 1.14 to the region D' then gives

$$\begin{aligned}
0 &= \frac{1}{2\pi i} \oint_{\partial D'} \frac{f'(z)}{f(z)} dz \\
&= \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz - \frac{1}{2\pi i} \sum_i \oint_{\gamma(z_i,\epsilon_i)} \frac{f'(z)}{f(z)} dz \\
&= \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz - \frac{1}{2\pi i} \sum_i \oint_{\gamma(z_i,\epsilon_i)} \frac{\pm n_i}{(z - z_i)} + \phi_i(z) dz \\
\Rightarrow \quad & \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = \sum_{zeros} n_i - \sum_{poles} n_j = N - P,
\end{aligned}$$

where $\gamma(z_i, \epsilon_i)$ denotes the circle of center z_i and radius ϵ_i, and we used the residue theorem 2.1 in the last step. The quantities N, P are as defined in the theorem statement.

To demonstrate the final equality in the theorem, we parametrize the curve C so that

$$C = \{z(t) : t \in [a, b], z(a) = z(b)\}.$$

Then

$$\begin{aligned}
\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz &= \frac{1}{2\pi i} \int_a^b \frac{f'(z(t))}{f(z(t))} z'(t) dt = \frac{1}{2\pi i} [\log|f(z(t))| + i \arg(f(z(t)))]_{t=a}^b \\
&= \frac{1}{2\pi} [\arg(f(z))]_C
\end{aligned}$$

(which branch of the logarithm we choose is immaterial for the theorem result, but for definiteness you can assume the principal branch).

The theorem has an interesting geometrical interpretation. The function $w = f(z)$ represents a mapping from the complex z-plane to the complex w-plane, under which the curve C maps to some image curve C' in the w-plane (see later notes on conformal mappings). We then have $dw = f'(z)dz$, so that

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{C'} \frac{dw}{w} = \frac{1}{2\pi i}[\log(w)]_{C'} = \frac{1}{2\pi}[\arg(w)]_{C'}.$$

Remark This quantity $(1/2\pi)[\arg(w)]_{C'}$ is known as the **winding number** of the curve C' about the origin in the w-plane – the number of times that the closed curve C' encircles the origin. More generally we have:

Definition 2.5 *(Winding number) Let γ be a piecewise-smooth closed contour in the w-plane, and $w_0 \in \mathbb{C}$ a point in the w-plane. Then the quantity*

$$n(\gamma, w_0) = \frac{1}{2\pi i}\oint_\gamma \frac{dw}{(w - w_0)}$$

is known as the **winding number** *of the curve γ about the point w_0.*

Parametrizing γ as before by $t \in [a, b]$,

$$\gamma = \{w(t) : t \in [a, b], w(a) = w(b)\},$$

we have

$$n(\gamma, w_0) = \frac{1}{2\pi i}\int_a^b \frac{w'(t)dt}{(w(t) - w_0)} = \frac{1}{2\pi i}\left[\log(w(t) - w_0)\right]_a^b = \frac{1}{2\pi}[\arg(w - w_0)]_\gamma,$$

which, geometrically, represents the number of times that the curve γ encircles the point $w_0 \in \mathbb{C}$. (Note: for the sake of brevity this argument assumes that γ is smooth, but it is easily extended to the case of piecewise smooth γ.)

This concept provides us with an alternative way of defining a simply-connected domain:

Definition 2.6 *A domain $D \subset \mathbb{C}$ is called* **simply connected** *if it has the property that, for all $w_0 \notin D$ and for every piecewise-smooth closed contour $\gamma \in D$, we have $n(\gamma, w_0) = 0$.*

Example 2.7 *Find the number of zeros of the function $f(z) = z^5 + 1$ within the first quadrant.*

We use the principle of the argument to determine the change in $\arg(f)$ as we traverse an appropriate contour. Since f is analytic on the first quadrant of the z-plane, we may take the contour C to be a large quarter-circle, made up of the three portions $C = C_1 \cup C_2 \cup C_3$,

$$\begin{aligned}
C_1 &= \{z : z = x, \quad 0 \le x \le R\}, \\
C_2 &= \{z : z = Re^{i\theta}, \quad 0 \le \theta \le \pi/2\}, \\
C_3 &= \{z : z = iy, \quad 0 \le y \le R\}.
\end{aligned}$$

In the limit that R becomes arbitrarily large the contour will enclose the whole of the first quadrant. The argument of f satisfies $\arg(f) = \phi$, where $\tan\phi = \Im(f)/\Re(f)$. Along C_1 $f = 1 + x^5 \in \mathbb{R}^+$, and $\arg(f) = 0$. Along C_2 $f = 1 + R^5 e^{5i\theta} \approx R^5 e^{5i\theta}$, and the argument of f therefore increases from 0 to $5\pi/2$ (in the limit $R \to \infty$) as θ increases from 0 to $\pi/2$. Finally, on C_3 $f = 1 + iy^5$, and for $y = R \gg 1$ we have $\arg(f) \approx 5\pi/2$ (it must vary continuously from C_2 to C_3), while as y decreases from $R \gg 1$ to 0, $\tan\phi = \Im(f)/\Re(f)$ decreases (continuously) from $+\infty$ to 0^+ and thus $\arg(f)$ decreases (continuously) from $5\pi/2$ to 2π.

Thus, the net change in $\arg(f)$ as we traverse C is 2π. Applying theorem 2.4 then, since f has no singularities within the first quadrant (so $P = 0$), the number of zeros it has, N, satisfies

$$N = \frac{1}{2\pi}[\arg(f)]_C = \frac{2\pi}{2\pi} = 1.$$

Theorem 2.8 *(Rouché) Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed contour C. If $|f| > |g|$ on C, then f and $f + g$ have the same number of zeros inside C.*

This theorem makes sense intuitively, since we can think of g as being a perturbation to f, which will in turn perturb its zeros. If the size of the perturbation is bounded on the contour C, then (we can show) the size of the perturbation is bounded inside C as well (this is not surprising, given the maximum modulus principle for analytic functions). Therefore, we have not perturbed the zeros of f too much by adding g.

Proof Since $|f| > |g| \ge 0$ on C it follows that $|f| > 0$ on C and thus $f \ne 0$ on C. Moreover, $f(z) + g(z) \ne 0$ on C. Let

$$h(z) = \frac{f(z) + g(z)}{f(z)},$$

then h is analytic and nonzero on C, and the Argument Principle theorem 2.4 may be applied to deduce that

$$\frac{1}{2\pi i}\oint_C \frac{h'(z)}{h(z)}dz = \frac{1}{2\pi}[\arg(w)]_{C'},$$

where $w = h(z)$, and C' is the image of the curve C under this transformation in the w-plane. However,

$$w = h(z) = 1 + \frac{g(z)}{f(z)}$$

so, since $|g| < |f|$ on C, we have $|w - 1| < 1$ on C', so that all points of C' lie within the circle of radius 1 centered at $w = 1$. Thus, as we traverse the closed curve C' there is no net change in $\arg(w)$, because C' does not enclose the origin. It follows from the argument principle that $N_h - P_h = 0$, where N_h and P_h are the numbers of zeros and poles, respectively (counted according to multiplicity) of $h(z)$. Since f and g are analytic inside and on C, the poles of h coincide in location and multiplicity with the zeros of f, so that $P_h = N_f$. Also by analyticity of f, h is zero only where $f + g = 0$, and we have $N_h = N_{f+g}$. Thus,

$$N_h - P_h = 0 \qquad \Rightarrow \qquad N_f = N_{f+g},$$

and the theorem is proved.

Example 2.9 *Show that $4z^2 = e^{iz}$ has a solution on the unit disc $|z| \leq 1$.*

Take the contour C to be $|z| = 1$, so that points on C are given by $z = e^{it}$, $t \in [0, 2\pi]$. Then, with $f(z) = 4z^2$ and $g(z) = -e^{iz}$, we have $|f| = 4$ on C, while

$$|g(z)|_C = |e^{i(\cos t + i\sin t)}| = e^{-\sin t} \leq e < |f(z)|_C.$$

Thus Rouché's theorem applies, and $f + g$ has the same number of zeros as f on the unit disc. Clearly, $f(z) = 4z^2$ has exactly 2 zeros (both at $z = 0$ but we count according to multiplicity). It follows that $4z^2 = e^{iz}$ in fact has 2 solutions on the unit disc.

Homework: Ablowitz & Fokas, problems for section 4.4, questions 3(b), 5(a).

2.3 Integral transform methods

Two common and useful integral transforms that can be used to solve complicated partial differential equations and integro-differential equations are the Fourier Transform and the Laplace Transform. We shall briefly consider both of these transforms here, focusing particularly on the Laplace Transform, and use of the calculus of residues to invert this transform. However, we begin our discussion with the Fourier transform, of which the Laplace transform may be considered a special case.

2.3.1 Fourier Transform

Motivation (non-rigorous) Recall the definition of the **Fourier series** of a function f, defined on the real interval $-T \le t \le T$,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right)\right), \tag{36}$$

where a_n and b_n are given by

$$a_n = \frac{1}{T}\int_{-T}^{T} f(\tau)\cos\left(\frac{n\pi\tau}{T}\right)d\tau, \qquad b_n = \frac{1}{T}\int_{-T}^{T} f(\tau)\sin\left(\frac{n\pi\tau}{T}\right)d\tau,$$

so that (substituting for a_n, b_n in (36))

$$\begin{aligned} f(t) &= \frac{1}{2T}\int_{-T}^{T} f(\tau)d\tau + \sum_{n=1}^{\infty}\frac{1}{T}\int_{-T}^{T} f(\tau)\cos\left(\frac{n\pi(\tau-t)}{T}\right)d\tau \\ &= \sum_{-\infty}^{\infty}\frac{1}{2T}\int_{-T}^{T} f(\tau)\cos\left(\frac{n\pi(\tau-t)}{T}\right)d\tau. \end{aligned}$$

If we set $n\pi/T = k_n$, and $\pi/T = k_{n+1} - k_n = \delta k_n$, then the process $T \to \infty$ formally yields the result

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} f(\tau)\cos k(\tau - t)\, d\tau \tag{37}$$

a version of the **Fourier integral theorem.** Since the integrand is even in k, we can rewrite (37) as

$$f(t) = \frac{1}{\pi}\int_{0}^{\infty} dk \int_{-\infty}^{\infty} f(\tau)\cos k(\tau - t)\, d\tau. \tag{38}$$

Alternatively, since $\sin k(\tau - t)$ is odd in k, we can rewrite (37) as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} f(\tau) e^{ik(\tau - t)}\, d\tau. \tag{39}$$

Equations (37), (38) and (39) are equivalent. If we then define the **Fourier transform** of f by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik\tau} f(\tau) d\tau, \tag{40}$$

equation (39) provides the **inversion formula**

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikt} \hat{f}(k)\, dk. \tag{41}$$

For a large class of functions the Fourier transform $\hat{f}(k)$ defined in (40) is an analytic function of the complex variable k. We shall now prove the result (41) for a certain class of functions (though in fact the result extends to a far greater class than we shall prove it for).

Theorem 2.10 *(Fourier's Integral Theorem) Let $f(t)$ be a piecewise smooth function of the real variable t, with $|f|$ integrable on $(-\infty, \infty)$. Then*

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} f(\tau) e^{ik(\tau - t)} d\tau,$$

so that if the **Fourier transform** *$\hat{f}(k)$ is defined by*

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik\tau} f(\tau)\, d\tau,$$

then the **inversion formula** *is given by*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikt} \hat{f}(k)\, dk.$$

The proof will use the following Lemma:

Lemma 2.11 *For any $0 < T < \infty$ and piecewise smooth f,*

$$\lim_{L \to \infty} \int_{-T}^{T} f(t + \tau) \frac{\sin L\tau}{\tau} d\tau = \frac{\pi}{2} [f(t^+) + f(t^-)], \tag{42}$$

where $f(t^+)$, $f(t^-)$ denote the limits of $f(t + \epsilon)$, $f(t - \epsilon)$ as $\epsilon \to 0$ $(\epsilon > 0)$.

Proof To prove the lemma we split the range of integration on the left-hand side of (42) into positive and negative τ: $\int_{-T}^{0} + \int_{0}^{T}$. For the positive range we write

$$\int_0^T f(t+\tau)\frac{\sin L\tau}{\tau}d\tau = \int_0^T [f(t+\tau) - f(t^+)]\frac{\sin L\tau}{\tau}d\tau + f(t^+)\int_0^T \frac{\sin L\tau}{\tau}d\tau. \tag{43}$$

Then, in the second integral on the right-hand side above,

$$\lim_{L\to\infty} f(t^+)\int_0^T \frac{\sin L\tau}{\tau}d\tau = f(t^+)\lim_{L\to\infty}\int_0^{LT}\frac{\sin x}{x}dx = f(t^+)\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}f(t^+). \tag{44}$$

Returning to the first integral on the right-hand side of (43), the piecewise smoothness of f implies that $f'(t^+)$ exists, and thus given $\epsilon > 0$ there exists $h > 0$ such that for $0 < \tau < h$,

$$\frac{f(t+\tau) - f(t^+)}{\tau} = f'(t^+) + g(\tau),$$

with $|g(\tau)| < \epsilon$. Hence,

$$\int_0^T [f(t+\tau) - f(t^+)]\frac{\sin L\tau}{\tau}d\tau = \int_0^h [f'(t^+) + g(\tau)]\sin L\tau\, d\tau + \int_h^T [f(t+\tau) - f(t^+)]\frac{\sin L\tau}{\tau}d\tau. \tag{45}$$

We now prove the following

Claim 2.12 *For a function w piecewise continuous on an interval (a,b),*

$$\int_a^b w(\tau)\sin L\tau\, d\tau \to 0 \quad \textit{as } L\to\infty.$$

To show this let $\delta > 0$ be arbitrarily small, and divide up the interval (a,b) into $N(\delta)$ increments $\tau_k < \tau < \tau_{k+1}$ ($\tau_0 = a$, $\tau_N = b$) such that on each subinterval $\tau_k < \tau < \tau_{k+1}$, w can be approximated arbitrarily closely by some constant A_k, in the sense that, for the piecewise constant approximating function v defined by

$$v(\tau) = \begin{cases} A_1 & a = \tau_0 < \tau < \tau_1 \\ A_2 & \tau_1 < \tau < \tau_2 \\ \dots & \dots \\ A_N & \tau_{N-1} < \tau < \tau_N = b, \end{cases}$$

we have

$$\int_a^b |w(\tau) - v(\tau)| d\tau < \frac{\delta}{2}.$$

Then, for any L, we have

$$\begin{aligned} \left| \int_a^b w(\tau) \sin L\tau \, d\tau \right| - \left| \int_a^b v(\tau) \sin L\tau \, d\tau \right| &\leq \left| \int_a^b (w(\tau) - v(\tau)) \sin L\tau \, d\tau \right| \\ &\leq \int_a^b |w(\tau) - v(\tau)| \, d\tau < \frac{\delta}{2}. \end{aligned}$$

From this it follows that

$$\begin{aligned} \left| \int_a^b w(\tau) \sin L\tau \, d\tau \right| &< \frac{\delta}{2} + \left| \int_a^b v(\tau) \sin L\tau \, d\tau \right| \\ &= \frac{\delta}{2} + \left| \sum_{k=0}^{N-1} A_{k+1} \int_{\tau_k}^{\tau_{k+1}} \sin L\tau \, d\tau \right| \\ &\leq \frac{\delta}{2} + \frac{2NM}{L}, \end{aligned}$$

where $M = \max\{A_k\}$. For δ fixed and arbitrarily small (and N depends only on δ while M can be an independent bound), if we choose $L > 4MN/\delta$ then we have

$$\left| \int_a^b w(\tau) \sin L\tau \, d\tau \right| < \delta;$$

that is, the integral can be made arbitrarily small by choosing L sufficiently large, and hence it must go to zero as $L \to \infty$, which proves the claim.

Returning to (45), we see that the result of the claim applies to each of the integrals on the right-hand side, since the functions premultiplying $\sin L\tau$

are piecewise continuous on the range of integration in each case. It follows that each of these integrals tends to zero as $L \to \infty$, and hence

$$\int_0^T [f(t+\tau) - f(t^+)] \frac{\sin L\tau}{\tau} d\tau \to 0 \quad \text{as } L \to \infty. \tag{46}$$

Using (44) and (46) in (43), we obtain

$$\lim_{L\to\infty} \int_0^T f(t+\tau) \frac{\sin L\tau}{\tau} d\tau = \frac{\pi}{2} f(t^+).$$

A similar result holds for the integral over the negative range of τ-values,

$$\lim_{L\to\infty} \int_{-T}^0 f(t+\tau) \frac{\sin L\tau}{\tau} d\tau = \frac{\pi}{2} f(t^-),$$

and putting these two results together completes the proof of Lemma 2.11.

We are now in a position to prove theorem 2.10.

Proof (Fourier Integral Theorem) We shall prove that

$$\lim_{L\to\infty} \int_0^L dk \int_{-\infty}^{\infty} f(t+\tau) \cos k\tau \, d\tau = \frac{\pi}{2} [f(t^+) + f(t^-)] \tag{47}$$

which is equivalent to the identity (38), which in turn is equivalent to (37) and (39). Let $T_1 > T > 0$. Then

$$\begin{aligned} \int_0^L dk \int_{-T_1}^{T_1} f(t+\tau) \cos k\tau \, d\tau - \int_0^L dk \int_{-T}^{T} f(t+\tau) \cos k\tau \, d\tau = \\ \int_{-T_1}^{-T} f(t+\tau) \frac{\sin L\tau}{\tau} \, d\tau - \int_T^{T_1} f(t+\tau) \frac{\sin L\tau}{\tau} \, d\tau, \end{aligned} \tag{48}$$

performing the integral with respect to k. Since we assume $|f|$ is integrable, $K = \int_{-\infty}^{\infty} |f(\tau) d\tau| < \infty$, and thus from (48) above we have

$$\begin{aligned} &\left| \int_0^L dk \int_{-T_1}^{T_1} f(t+\tau) \cos k\tau d\tau - \int_0^L dk \int_{-T}^{T} f(t+\tau) \cos k\tau d\tau \right| \\ &\qquad = \left| \int_{-T_1}^{-T} f(t+\tau) \frac{\sin L\tau}{\tau} \, d\tau - \int_T^{T_1} f(t+\tau) \frac{\sin L\tau}{\tau} \, d\tau \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left|\int_{-T_1}^{-T} f(t+\tau)\frac{\sin L\tau}{\tau}\,d\tau\right| + \left|\int_{T}^{T_1} f(t+\tau)\frac{\sin L\tau}{\tau}\,d\tau\right| \\
&\leq \int_{-T_1}^{-T} \left|\frac{f(t+\tau)}{\tau}\right| d\tau + \int_{T}^{T_1} \left|\frac{f(t+\tau)}{\tau}\right| d\tau \\
&< \frac{1}{T}\int_{-\infty}^{\infty} |f(t+\tau)| d\tau = \frac{K}{T}.
\end{aligned}$$

Thus, letting $T_1 \to \infty$,

$$\left|\int_0^L dk \int_{-\infty}^{\infty} f(t+\tau)\cos k\tau\,d\tau - \int_0^L dk \int_{-T}^{T} f(t+\tau)\cos k\tau\,d\tau\right| < \frac{K}{T}. \quad (49)$$

The second expression on the left-hand side here can be written (as above, by performing the integral with respect to k) as

$$\int_0^L dk \int_{-T}^{T} d\tau = \int_{-T}^{T} f(t+\tau)\frac{\sin L\tau}{\tau} d\tau \to \frac{\pi}{2}[f(t^+)+f(t^-)] \quad \text{as } L \to \infty,$$

by Lemma 2.11. Using this in (49) gives, as $L \to \infty$,

$$\left|\lim_{L\to\infty}\int_0^L \int_{-\infty}^{\infty} f(t+\tau)\cos k\tau\,d\tau dk - \frac{\pi}{2}[f(t^+)+f(t^-)]\right| < \frac{K}{T}.$$

However T here is arbitrary, so letting it become large we establish equation (47).

Summary: We have proved that, if the Fourier transform of a piecewise smooth function $f(t)$, $t \in (-\infty, \infty)$, where also $|f|$ is integrable, is defined by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik\tau} f(\tau)\,d\tau \quad (50)$$

then the inversion formula gives $f(t)$ as

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikt}\hat{f}(k)\,dk. \quad (51)$$

As observed earlier however, in fact these formulae are good for a much wider class of functions $f(t)$.

The Fourier Transform and its inverse are sometimes also written as operators acting on the functions,

$$\mathcal{F}\left[f(t)\right] = \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik\tau} f(\tau)\, d\tau,$$
$$\mathcal{F}^{-1}\left[\hat{f}(k)\right] = f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikt}\hat{f}(k)\, dk.$$

Evaluation of the inverse Fourier transform can often be accomplished via complex contour integration.

Example 2.13 *Verify the result of the Fourier Integral theorem for the function f defined by*

$$f(t) = \begin{cases} e^{-at} & t > 0, \\ e^{bt} & t < 0, \end{cases} \tag{52}$$

where $a,\ b \in \mathbb{R}^+$.

Finding the Fourier transform is straightforward:

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{b\tau+ik\tau}\, d\tau + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-a\tau+ik\tau}\, d\tau \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{b\tau+ik\tau}}{b+ik}\right]_{-\infty}^{0} - \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-a\tau+ik\tau}}{a-ik}\right]_{0}^{\infty} \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a-ik} + \frac{1}{b+ik}\right).
\end{aligned}$$

To reconstruct f from $\hat{f}(k)$ we must evaluate the inversion integral (51), which gives

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikt}\left(\frac{1}{a-ik} + \frac{1}{b+ik}\right) dk, \tag{53}$$

and this may be done using the Residue theorem on a large closed semicircular contour if we note that, for $t > 0$, the integrand decays for $k = Re^{i\theta} = R(\cos\theta + i\sin\theta)$ where $\pi < \theta < 2\pi$, while for $t < 0$, the integrand decays for $k = Re^{i\theta} = R(\cos\theta + i\sin\theta)$ where $0 < \theta < \pi$. For values $t > 0$ we therefore use a large semicircular contour in the **lower**-half k-plane, C^-, while for $t < 0$ we use a large semicircle in the **upper**-half k-plane, C^+.

Considering each term in the inversion integral (53) separately we have, for $t > 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ikt}}{a-ik}dk = -\frac{1}{2\pi}\oint_{C^-}\frac{e^{-ikt}}{a-ik}dk + \frac{1}{2\pi}\int_{\Gamma^-}\frac{e^{-ikt}}{a-ik}dk, \tag{54}$$

where $\Gamma^- = \{k : k = Re^{i\theta} : \pi < \theta < 2\pi\}$ is the curved portion of C^-. For the integral around C^- the Residue theorem (2.1) applies, giving

$$\oint_{C^-}\frac{e^{-ikt}}{a-ik}dk = 2\pi i \mathrm{Res}(e^{-ikt}/(a-ik); -ia) = 2\pi i(-ie^{-at}) = 2\pi e^{-at};$$

the only singularity of the integrand inside C^- being at $k = -ia$; while on Γ^- we have

$$\begin{aligned}\left|\int_{\Gamma^-}\frac{e^{-ikt}}{a-ik}dk\right| &= \left|\int_{\pi}^{2\pi}\frac{e^{Rt(\sin\theta - i\cos\theta)}iRe^{i\theta}}{a-iRe^{i\theta}}d\theta\right| \\ &\leq \frac{R}{R-a}\int_{\pi}^{2\pi}e^{Rt\sin\theta}d\theta \\ &= \frac{R}{R-a}\int_{0}^{\pi}e^{-Rt\sin\phi}d\phi \\ &= \frac{2R}{R-a}\int_{0}^{\pi/2}e^{-Rt\sin\phi}d\phi.\end{aligned}$$

By Jordan's inequality we have $2\phi/\pi \leq \sin\phi \leq \phi$ for $0 \leq \phi \leq \pi/2$, and thus

$$\left|\int_{\Gamma^-}\frac{e^{-ikt}}{a-ik}dk\right| \leq \frac{2R}{R-a}\int_0^{\pi/2}e^{-2Rt\phi/\pi}d\phi = \frac{\pi}{t(R-a)}(1-e^{-Rt}) \to 0 \,\text{as}\, R \to \infty.$$

Returning to (54) then, for $t > 0$ the inverse Fourier transform of $1/(\sqrt{2\pi}(a-ik))$ is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ikt}}{a-ik}dk = e^{-at}, \qquad t > 0.$$

For $t < 0$ similar arguments may be applied to the inversion integral using the semicircular contour C^+ in the upper-half k-plane; here the singularity at $k = -ia$ lies outside the contour of integration and so

$$\oint_{C^+}\frac{e^{-ikt}}{a-ik}dk = 0,$$

giving the inverse Fourier transform of $1/(\sqrt{2\pi}(a-ik))$ for $t<0$ as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ikt}}{a-ik}dk=0, \qquad t<0.$$

For $t=0$ finally, we have (interpreting the inversion integral in the principal value sense, and again using the contour C^+ in the upper-half k-plane composed of the real line $-R\le\Re(k)\le R$ and the circular arc $\Gamma^+=\{k:k=Re^{i/\theta},0<\theta<\pi\}$)

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ikt}}{a-ik}dk &= \frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{R}\frac{1}{a-ik}dk\\
&= \frac{1}{2\pi}\lim_{R\to\infty}\left(\oint_{C^+}\frac{1}{a-ik}dk-\int_{\Gamma^+}\frac{1}{a-ik}dk\right)\\
&= -\frac{1}{2\pi}\lim_{R\to\infty}\int_0^{\pi}\frac{iRe^{i\theta}}{a-iRe^{i\theta}}d\theta\\
&= \frac{1}{2\pi}\lim_{R\to\infty}\int_0^{\pi}\frac{d\theta}{1+iae^{-i\theta}/R}\\
&= 1/2.
\end{aligned}$$

Thus the inverse Fourier transform of $1/(\sqrt{2\pi}(a-ik))$ over all ranges of t is given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ikt}}{a-ik}dk=\begin{cases}e^{-at} & t>0\\ 1/2 & t=0\\ 0 & t<0.\end{cases} \tag{55}$$

Exactly similar methods apply to the second term in the inversion integral (53) (**Exercise:** try doing this explicitly), giving the inverse Fourier transform of $1/(\sqrt{2\pi}(b+ik))$ as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-ikt}}{b+ik}dk=\begin{cases}0 & t>0\\ 1/2 & t=0\\ e^{bt} & t<0\end{cases} \tag{56}$$

(the pole in the inversion integrand here is at $k=ib$, inside C^+, so we pick up the nonzero residue contribution when doing this integral for the case $t<0$).

Note that the original function (52) is only defined away from $t=0$, but the operation of taking the Fourier transform and inverting gives a value $\mathcal{F}^{-1}\left[\mathcal{F}[f(t)]\right]_{t=0}=1$. This is in line with the result of theorem 2.10, which states that the operation of inverting the Fourier transform should lead to the average value, $(f(0^+)+f(0^-))/2$, at $t=0$.

Homework: Ablowitz & Fokas, problems for section 4.5, questions 1,2.

Some key properties of the Fourier Transform:

1. Linearity: $\mathcal{F}[c_1 f_1(t) + c_2 f_2(t)] = c_1 \mathcal{F}[f_1(t)] + c_2 \mathcal{F}[f_2(t)]$.

2. Fourier transform of a derivative:

$$\mathcal{F}\left[\frac{d^n f}{dt^n}(t)\right] = (-ik)^n \hat{f}(k). \tag{57}$$

   This follows from straightforward integration by parts, assuming that the derivatives decay appropriately at infinity:

$$\begin{aligned}
\mathcal{F}\left[\frac{d^n f}{dt^n}(t)\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik\tau} \frac{d^n f}{d\tau^n}(\tau)\, d\tau \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \left[e^{ik\tau} \frac{d^{n-1} f}{d\tau^{n-1}} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (ik) e^{ik\tau} \frac{d^{n-1} f}{d\tau^{n-1}}\, d\tau \right\} \\
&= (-ik) \mathcal{F}\left[\frac{d^{n-1} f}{dt^{n-1}}(t)\right].
\end{aligned}$$

   Repeating the argument leads to the result (57).

3. Fourier transform of a product: This is **not** the product of the Fourier transforms, but is the **convolution** of the Fourier transforms,

$$\begin{aligned}
\mathcal{F}\left[f(t) g(t)\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik\tau} f(\tau) g(\tau)\, d\tau \\
&= (\hat{f} * \hat{g})(k) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k - k') \hat{g}(k')\, dk'.
\end{aligned} \tag{58}$$

   The result is most easily demonstrated by using the inversion formula (51) on the convolution integral (58) to show we get the product of the functions f and g.

$$\begin{aligned}
\mathcal{F}^{-1}\left[(\hat{f} * \hat{g})(k)\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikt} (\hat{f} * \hat{g})(k)\, dk \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikt} \int_{-\infty}^{\infty} \hat{f}(k - k') \hat{g}(k')\, dk'\, dk
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k-k')t}\hat{f}(k-k')e^{-ik't}\hat{g}(k')\,dk'\,dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\kappa t}\hat{f}(\kappa)e^{-ik't}\hat{g}(k')\,d\kappa\,dk' \\
&= \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\kappa t}\hat{f}(\kappa)\,d\kappa\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ik't}\hat{g}(k')\,dk'\right) \\
&= f(t)g(t).
\end{aligned}
$$

4. In exactly the same way we can demonstrate that the Fourier transform of a convolution of two functions is the product of the Fourier transforms. If

$$(f * g)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau$$

then

$$\mathcal{F}\left[(f * g)(t)\right] = \hat{f}(k)\hat{g}(k).$$

Remark Note that the convolution integral (59) may also be expressed as

$$(f * g)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau.$$

Exercise: Show this, starting from (59).

Such results make the Fourier transform (and the closely-related Laplace transform which we discuss below) very useful for solving certain partial or ordinary differential equations. Consider the equation

$$\frac{d^2u}{dt^2} - \omega^2 u = -f(t), \qquad u \to 0 \quad \text{as } |t| \to \infty,$$

where $\omega > 0$. Taking the Fourier transform in t gives, using property 2 above,

$$(k^2+\omega^2)\hat{u}(k) = \hat{f}(k) \quad \Rightarrow \quad \hat{u}(k) = \frac{\hat{f}(k)}{k^2+\omega^2} = \hat{f}(k)\hat{g}(k),$$

a product of Fourier transforms, so the solution (by property 4 above) is the convolution of $f(t)$ with $g(t)$,

$$u(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau. \tag{59}$$

To find $g(t)$ note that

$$\hat{g}(k) = \frac{1}{k^2+\omega^2} = \frac{1}{2\omega}\left(\frac{1}{\omega - ik} + \frac{1}{\omega + ik}\right) = \frac{\sqrt{2\pi}}{2\omega}\mathcal{F}[G(t)], \tag{60}$$

where, by the result of example 2.13,

$$G(t) = \left\{ \begin{array}{ll} e^{-\omega t} & t > 0 \\ e^{\omega t} & t < 0 \end{array} \right\} = e^{-\omega|t|}. \tag{61}$$

Hence finally, by (59), (60) and (61), the solution is

$$u(t) = \frac{1}{2\omega}\int_{-\infty}^{\infty} f(t-\tau)e^{-\omega|\tau|}d\tau.$$

In this example we saved time by recognising the function $\hat{g}(k)$ appearing in the solution as the Fourier transform of a known function. In fact, since the Fourier transform and its inverse are one-to-one transformations, so that they form a unique pair, this is a very common method of inverting the transform. Extensive tables of Fourier transform pairs exist, examples of which may be found in [2, 5, 10].

Homework: Use the Fourier transform in x to solve the heat equation

$$u_t = \alpha u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$
$$u \to 0 \quad \text{as } |x| \to \infty, \quad u(x,0) = Be^{-\beta x^2}, \quad B, \beta \in \mathbb{R}^+.$$

Note: take care not to confuse the variables; here you are transforming with respect to x, not t!

2.3.2 Laplace Transform

Closely related to the Fourier transform is another widely-used exponential transform of a function $f(t)$, the **Laplace transform.**

Definition 2.14 *(Laplace Transform) The Laplace transform of a function $f(t)$ is defined by*

$$\mathcal{L}[f(t)] = \bar{f}(s) = \int_0^{\infty} e^{-st} f(t)\, dt. \tag{62}$$

This transform generally takes a real function of a real variable, $f(t)$, to a complex function of a complex variable, $\bar{f}(s)$. Note that if we identify s with $(-ik)$ and define

$$F(t) = \begin{cases} f(t) & t > 0, \\ 0 & t < 0, \end{cases}$$

then

$$\bar{f}(s) = \sqrt{2\pi}\hat{F}(is). \tag{63}$$

Therefore, we will be able to deduce results about the Laplace transform from those for the Fourier transform. We first prove that the Laplace transform is an analytic function of s, under appropriate conditions.

Theorem 2.15 *If $f(t)$ is piecewise continuous for $t \geq 0$, and there exist constants M, σ_0, $T_0 > 0$ such that*

$$|f(t)| < Me^{\sigma_0 t} \quad \forall t > T_0,$$

then $\bar{f}(s)$ exists and is analytic on $\Re(s) > \sigma_0$.

Remark Note that, of course, the domain of analyticity of $\bar{f}(s)$ may be larger than this. The theorem simply guarantees analyticity at least within this region.

Proof There are many ways of showing this. First we show that the transform exists when the conditions are satisfied. Defining $\bar{f}_T(s)$ by

$$\bar{f}_T(s) = \int_0^T e^{-st} f(t)\, dt$$

we have $\bar{f}(s) = \lim_{T\to\infty} \bar{f}_T(s)$, and

$$\begin{aligned}
|\bar{f}_T(s)| &= \left| \int_0^{T_0} e^{-st} f(t)\, dt + \int_{T_0}^{T} e^{-st} f(t)\, dt \right| \\
&\leq \left| \int_0^{T_0} e^{-st} f(t)\, dt \right| + \left| \int_{T_0}^{T} e^{-st} f(t)\, dt \right| \\
&\leq K_1 + \int_{T_0}^{T} |e^{-st}||f(t)|\, dt \\
&\leq K_1 + M \int_0^T e^{-(\Re(s)-\sigma_0)t}\, dt \\
&= K_1 + M \frac{1 - e^{-(\Re(s)-\sigma_0)T}}{\Re(s) - \sigma_0}.
\end{aligned}$$

Clearly then, for $\Re(s) > \sigma_0$, $\bar{f}_T(s)$ exists for all T, including in the limit $T \to \infty$, and thus the Laplace transform exists under the stated conditions.

To prove analyticity, we can demonstrate differentiability at all points. Let

$$D_\delta = \frac{\bar{f}_T(s+\delta) - \bar{f}_T(s)}{\delta} = \int_0^T f(t)e^{-st}\frac{(e^{-\delta t}-1)}{\delta}dt.$$

Then, for fixed T,

$$D := \lim_{\delta \to 0} D_\delta = \int_0^T f(t)e^{-st}(-t)dt,$$

wherever this integral converges. By arguments similar to those used above, we can show that D exists for $\Re(s) > \sigma_0$, for all T, because

$$\begin{aligned}
|D| &= \left|\int_0^{T_0} e^{-st}tf(t)\,dt + \int_{T_0}^{T} e^{-st}tf(t)\,dt\right| \\
&\leq \left|\int_0^{T_0} e^{-st}tf(t)\,dt\right| + \left|\int_{T_0}^{T} e^{-st}tf(t)\,dt\right| \\
&\leq K_2 + \int_{T_0}^{T} t|e^{-st}||f(t)|\,dt \\
&\leq K_2 + M\int_0^T te^{-(\Re(s)-\sigma_0)t}\,dt \\
&= K_2 + M\left(\frac{(-T)e^{-(\Re(s)-\sigma_0)T}}{\Re(s)-\sigma_0} + \frac{(1-e^{-(\Re(s)-\sigma_0)T})}{(\Re(s)-\sigma_0)^2}\right),
\end{aligned}$$

where integration by parts was used to derive the last equality. Thus, the derivative $\bar{f}_T'(s)$ exists for any $T > 0$, including in the limit $T \to \infty$. It follows that $\bar{f}'(s)$ exists for any $\Re(s) > \sigma_0$, and hence that $\bar{f}(s)$ is analytic on $\Re(s) > \sigma_0$.

Example 2.16 *Find the Laplace transform of $f(t) = t^n$, for $n \in \mathbb{N}$.*

Let

$$I_n = \bar{f}(s) = \int_0^\infty e^{-st}t^n\,dt.$$

Then, for $\Re(s) > 0$, integration by parts gives

$$I_n = \left[\frac{t^n e^{-st}}{(-s)}\right]_0^\infty + \int_0^\infty \frac{nt^{n-1}}{s} e^{-st}\, dt = \frac{n}{s} I_{n-1}.$$

Repeating the argument we find

$$I_n = \frac{n!}{s^n} I_0 = \frac{n!}{s^{n+1}} = \bar{f}(s). \tag{64}$$

This is analytic for $\Re(s) > 0$, as we expect from theorem 2.15.

Example 2.17 *Find the Laplace transform of $f(t) = \sin at$.*

We again integrate by parts, for $\Re(s) > 0$, to obtain

$$\begin{aligned} I &= \int_0^\infty e^{-st} \sin at\, dt = \left[-\frac{e^{-st}}{a} \cos at\right]_0^\infty - \int_0^\infty \frac{se^{-st}}{a} \cos at\, dt \\ &= \frac{1}{a} - \frac{s}{a}\left\{\left[\frac{e^{-st}}{a} \sin at\right]_0^\infty + \int_0^\infty \frac{se^{-st}}{a} \sin at\, dt\right\} \\ &= \frac{1}{a} - \frac{s^2}{a^2} I. \end{aligned}$$

Rearranging, we find that

$$I = \bar{f}(s) = \frac{a}{a^2 + s^2}.$$

2.3.3 Properties of Laplace transform

1. Linearity: $\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1 \mathcal{L}[f_1(t)] + c_2 \mathcal{L}[f_2(t)]$.

2. Shifting in s: $\mathcal{L}[e^{at} f(t)] = \bar{f}(s - a)$.

$$\mathcal{L}[e^{at} f(t)] = \int_0^\infty f(t) e^{-(s-a)t}\, dt = \bar{f}(s - a).$$

3. Laplace transform of a derivative: If f is n-times differentiable on $t > 0$ and $|f(t)e^{-st}| \to 0$ as $t \to \infty$ for $\Re(s) > \sigma_0$ (some $\sigma_0 \in \mathbb{R}$), then for $\Re(s) > \sigma_0$ we have

$$\mathcal{L}\left[\frac{d^n f}{dt^n}(t)\right] = s^n \bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0). \tag{65}$$

Proof: Let $I_n(s) = \mathcal{L}[f^{(n)}(t)]$. Then

$$I_n(s) = \int_0^\infty f^{(n)}(t)e^{-st}\,dt = \left[f^{(n-1)}(t)e^{-st}\right]_0^\infty + \int_0^\infty sf^{(n-1)}(t)e^{-st}\,dt.$$

The behavior of $|f(t)e^{-st}|$ as $t \to \infty$ will also guarantee that $f^{(j)}(t)e^{-st} \to 0$ as $t \to \infty$ for $\Re(s) > \sigma_0$, for $j \geq 1$, and thus

$$\begin{aligned}
I_n(s) &= -f^{(n-1)}(0) + sI_{n-1}(s) \\
&= -f^{(n-1)}(0) + s[-f^{(n-2)}(0) + sI_{n-2}(s)] \\
&= \cdots = -f^{(n-1)}(0) - sf^{(n-2)}(0) - s^2 f^{(n-3)}(0) \cdots + s^n I_0(s).
\end{aligned}$$

Since $I_0(s)$ is just $\bar{f}(s)$, the result follows.

4. Derivative of a Laplace transform: Where $\bar{f}(s)$ is defined and analytic we have

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n \bar{f}}{ds^n}(s). \tag{66}$$

Proof: Since $\bar{f}(s)$ is analytic we can differentiate in the definition to obtain

$$\frac{d^n \bar{f}}{ds^n}(s) = \int_0^\infty f(t)\frac{d^n}{ds^n}(e^{-st})\,dt = \int_0^\infty (-t)^n f(t)e^{-st}\,dt = (-1)^n \mathcal{L}[t^n f(t)],$$

as claimed.

5. Laplace transform of an integral: Where $g(t) = \int_0^t f(t)\,dt$ is defined for all t, and $|g(t)e^{-st}| \to 0$ as $t \to \infty$ for $\Re(s) > \sigma_0$ (some $\sigma_0 \in \mathbb{R}$), then for $\Re(s) > \sigma_0$ we have

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{\bar{f}(s)}{s}. \tag{67}$$

Proof: Integrating by parts in the definition gives

$$\begin{aligned}
\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] &= \int_0^\infty \left(\int_0^t f(\tau)\,d\tau\right) e^{-st}\,dt \\
&= \left[\left(\int_0^t f(\tau)\,d\tau\right)\frac{e^{-st}}{(-s)}\right]_0^\infty + \int_0^\infty f(t)\frac{e^{-st}}{s}\,dt \\
&= \frac{\bar{f}(s)}{s}.
\end{aligned}$$

6. Laplace transform of a convolution:

$$\mathcal{L}\left[(f * g)(t)\right] = \mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)\,d\tau\right] = \bar{f}(s)\bar{g}(s). \qquad (68)$$

Proof: Assume f and g are such that Fubini's theorem is applicable to $f(t-\tau)g(\tau)e^{-st}$ on the region $D = \{(\tau, t) : 0 < \tau < t < \infty\}$ in (τ, t)-space.[2] Then the order of the integration may be interchanged, and thus

$$\begin{aligned}\mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)\,d\tau\right] &= \int_0^\infty \left(\int_0^t f(t-\tau)g(\tau)\,d\tau\right) e^{-st}\,dt \\ &= \int_0^\infty \int_\tau^\infty f(t-\tau)e^{-s(t-\tau)}g(\tau)e^{-s\tau}\,dt d\tau.\end{aligned}$$

Set $\tilde{t} = t - \tau$ in the first t-integral, so that $d\tilde{t} = dt$. Then

$$\begin{aligned}\mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)\,d\tau\right] &= \int_0^\infty \left(\int_0^\infty f(\tilde{t})e^{-s\tilde{t}}\,d\tilde{t}\right) g(\tau)e^{-s\tau}\,d\tau \\ &= \int_0^\infty \bar{f}(s)g(\tau)e^{-s\tau}\,d\tau = \bar{f}(s)\bar{g}(s).\end{aligned}$$

Example 2.18 *Solve the following integro-differential equation, subject to boundary conditions $f(0) = 0$, $f(\pi) = \sqrt{2}$.*

$$\int_0^t f(\tau)\,d\tau + 4tf(t) = \frac{df}{dt}(t).$$

Taking the Laplace transform we obtain

$$\frac{\bar{f}(s)}{s} - 4\frac{d\bar{f}}{ds}(s) = s\bar{f}(s) - f(0),$$

using the properties listed above. With the given boundary condition on $f(0)$ we have a separable equation that we can solve for $\bar{f}(s)$,

$$\begin{aligned} 4\int \frac{d\bar{f}}{\bar{f}} &= \int \left(\frac{1}{s} - s\right) ds \\ \Rightarrow \qquad \bar{f}(s) &= ks^{1/4}e^{-s^2/8},\end{aligned}$$

where k is an as yet undetermined constant of integration.

For this procedure to be useful, we clearly need to be able to **invert** the Laplace transform, to recover the desired function $f(t)$.

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[2]This requires that $\iint_D |f(t-\tau)g(\tau)e^{-st}|d(\tau,t) < \infty$.

Homework: Ablowitz & Fokas, problems for section 4.5, questions 13(a), (b).

2.3.4 Inversion of Laplace transform

We now exploit further the relation between the Laplace and Fourier transforms (seen in (63)) to derive the inversion formula for the Laplace transform. If we write $s = \sigma - ik$ (σ, $k \in \mathbb{R}$) in the definition (62) then

$$\begin{aligned}
\bar{f}(s) = \bar{f}(\sigma - ik) &= \int_0^\infty e^{ikt} e^{-\sigma t} f(t)\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ikt} (\sqrt{2\pi} e^{-\sigma t} f(t))\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{ikt} p(t)\, dt \equiv \hat{p}(k),
\end{aligned}$$

where

$$p(t) = \begin{cases} \sqrt{2\pi} e^{-\sigma t} f(t) & t > 0 \\ 0 & t < 0. \end{cases}$$

Thus, we identify the Laplace transform of f with the Fourier transform of p, and p may be recovered by the Fourier inversion theorem 2.10, which gives

$$p(t) = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} e^{-ikt} \hat{p}(k)\, dk \tag{69}$$

For $t > 0$ then, (69) gives

$$\begin{aligned}
\sqrt{2\pi} e^{-\sigma t} f(t) &= \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} e^{-ikt} \hat{p}(k)\, dk \\
&= \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{\sigma+iR}^{\sigma-iR} e^{st} e^{-\sigma t} \bar{f}(s) i ds \\
\Rightarrow \quad f(t) &= \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\sigma-iR}^{\sigma+iR} e^{st} \bar{f}(s)\, ds. \qquad (70)
\end{aligned}$$

In applying this result we can in fact choose σ to be any real number such that the line γ from $\sigma - i\infty$ to $\sigma + i\infty$ lies to the right of all singularities of $\bar{f}(s)$ in the complex s-plane.

This result is formalized in the following theorem:

Theorem 2.19 *(Laplace transform inversion) If $\mathcal{L}[f(t)] = \bar{f}(s)$ exists for $\Re(s) > \sigma_0$, and $f(t)$ is piecewise smooth, then the inverse Laplace transform of $\bar{f}(s)$ is given, for $t > 0$, by*

$$\frac{1}{2}\left[f(t^+) + f(t^-)\right] = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\sigma-iR}^{\sigma+iR}\bar{f}(s)e^{st}ds, \qquad \sigma > \sigma_0. \tag{71}$$

(Clearly if $f(t)$ is continuous then the left-hand side in (71) is just $f(t)$.) This theorem guarantees that any continuous and piecewise smooth function is uniquely determined by its Laplace transform.

The integral in (71) is called the **Bromwich integral**. If f has no branch points then this integral can be evaluated by closing off the contour to the left with a large semicircle (this closed contour is called the **Bromwich contour**; see figure 3), which in the limit $R \to \infty$ will enclose all singularities of the integrand, and applying the Residue Theorem. If f has a branch point, *e.g.* at $s = 0$, with a branch-cut going off from $s = 0$ to $s = -\infty$, then the contour can be "wrapped around" the branch cut (see figure 4), the contributions from either side of the branch cut, and from the integral around the branch point, being evaluated explicitly.

Remark As for Fourier transforms, the other (very common) way to invert Laplace Transforms is "by inspection", that is, recognise the answer as the Laplace transform of a known function, usually by reference to a Table of Laplace transforms (many books contain such tables, *e.g.* Gradshteyn & Ryzhik [5], Spiegel & Liu [10], Abramowitz & Stegun [2]; this last reference [2] is available online at http://www.math.sfu.ca/˜ cbm/aands/).

Example 2.20 *(Function with a branch-point) Find $f(t)$, given that its Laplace transform $\bar{f}(s) = 1/\sqrt{s}$.*

Here $\bar{f}(s)$ has a branch point at the origin, and no other singularities. We cannot integrate this function around the standard Bromwich contour of figure 3 as we require a branch-cut to make $\bar{f}(s)$ single-valued, so we take a branch-cut along the negative real axis from $s = 0$ to $s = -\infty$, and define a single-valued function $\bar{f}(s)$ in the cut plane via

$$\bar{f}(s) = \frac{1}{\sqrt{|s|}}\exp(-i\mathrm{Arg}(s)/2), \qquad \mathrm{Arg}(s) \in (-\pi, \pi].$$

We can integrate this function around the "keyhole" Bromwich contour of figure 4, considering each portion of the curve γ_L, γ_{1R}, γ_+, γ_ϵ, γ_-, γ_{2R},

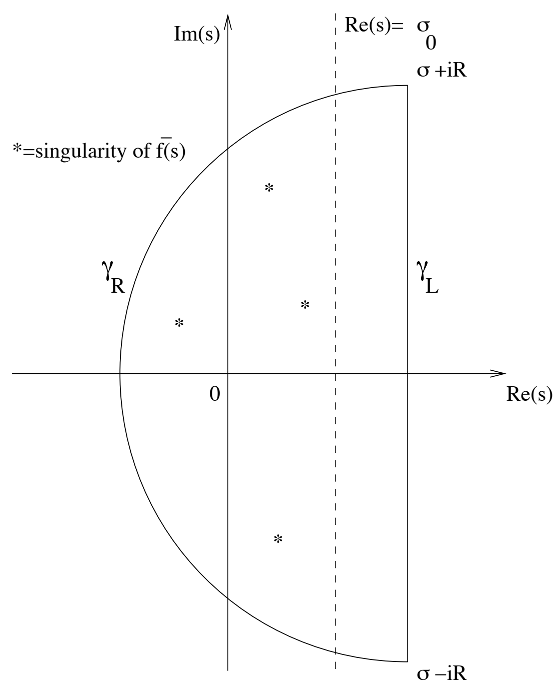

Figure 3: The Bromwich contour to evaluate an inverse Laplace transform where $\bar{f}(s)$ has no branch points.

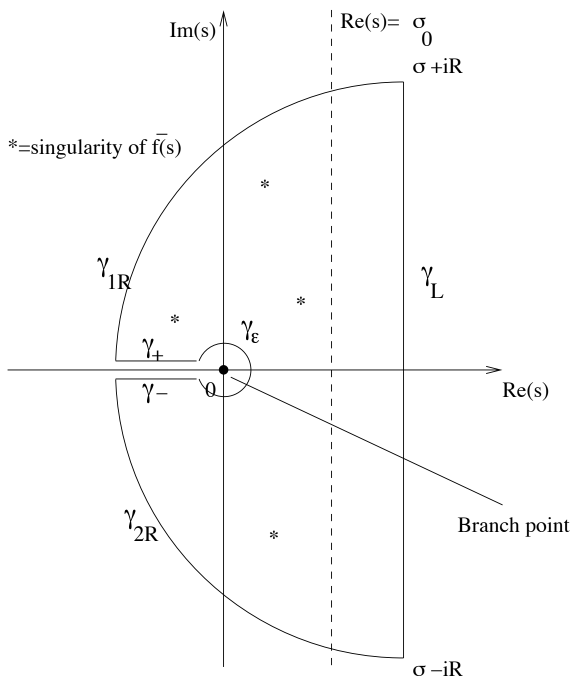

Figure 4: The Bromwich contour, modified to evaluate an inverse Laplace transform where $\bar{f}(s)$ has a branch point at the origin with a branch cut along the real negative s-axis.

separately. Since $\bar{f}(s)$ is analytic within the total closed Bromwich contour C we have

$$\oint_C \bar{f}(s)e^{st}\,ds = \left(\int_{\gamma_L} + \int_{\gamma_{1R}} + \int_{\gamma_+} + \int_{\gamma_\epsilon} + \int_{\gamma_-} + \int_{\gamma_{2R}}\right)\bar{f}(s)e^{st}\,ds = 0$$

and, since the portion we require is the integral along γ_L, it follows from the Inversion theorem that

$$f(t) = \frac{-1}{2\pi i}\left(\int_{\gamma_{1R}} + \int_{\gamma_+} + \int_{\gamma_\epsilon} + \int_{\gamma_-} + \int_{\gamma_{2R}}\right)\bar{f}(s)e^{st}\,ds. \tag{72}$$

For the integral along γ_{R1} we have $s = \sigma + iRe^{i\theta}$, $0 < \theta < \pi/2$, so that $ds = -Re^{i\theta}\,d\theta$, and

$$\begin{aligned}\left|\int_{\gamma_{1R}} \bar{f}(s)e^{st}\,ds\right| &= \left|\int_0^{\pi/2} (\sigma + iRe^{i\theta})^{-1/2}e^{\sigma t}e^{-Rt\sin\theta}e^{iR\cos\theta}Re^{i\theta}\,d\theta\right| \\ &\le \int_0^{\pi/2} R^{-1/2}(1 + O(\frac{1}{R}))e^{\sigma t}e^{-Rt\sin\theta}R\,d\theta.\end{aligned}$$

Clearly the integrand decays exponentially in R as $R \to \infty$ where $\theta > 0$, so we just have to concern ourselves with whether the contribution from the endpoint $\theta = 0$ decays. Considering the portion $0 < \theta < \delta$ for $0 < \delta \ll 1$, we have

$$\int_0^{\delta} R^{1/2}e^{\sigma t}e^{-Rt\sin\theta}\,d\theta \approx \int_0^{\delta} R^{1/2}e^{\sigma t}e^{-Rt\theta}\,d\theta = \frac{e^{\sigma t}}{t\sqrt{R}}(1 - e^{-Rt\delta}) \to 0 \quad \text{as} \quad R \to \infty.$$

[We note, as an aside, that this endpoint contribution would go to zero whenever the integrand $\bar{f}(s)$ is of order R^{-K} for some $K > 0$.] Hence the total integral over γ_{1R} goes to zero as $R \to \infty$. Similar arguments show that the integral over γ_{2R} goes to zero as $R \to \infty$.

For the integral along γ_ϵ we have $s = \epsilon e^{i\theta}$, $-\pi < \theta < \pi$, so that $ds = i\epsilon e^{i\theta}\,d\theta$, and

$$\begin{aligned}|\bar{f}(s)\,ds| &= |(\epsilon e^{i\theta})^{-1/2}e^{\epsilon t(\cos\theta + i\sin\theta)}(i\epsilon e^{i\theta})\,d\theta| \\ &= \epsilon^{1/2}e^{\epsilon t\cos\theta}\,d\theta.\end{aligned}$$

Hence the integral along this portion goes to zero as $\epsilon \to 0$.

On the curves γ_+, γ_-, we have $s = xe^{i\pi^+}$, $s = xe^{-i\pi^-}$, respectively, and so

$$\left(\int_{\gamma_+} + \int_{\gamma_-}\right) \bar{f}(s)e^{st}\,ds = \int_\infty^0 \frac{1}{i\sqrt{x}}e^{-xt}(-dx) + \int_0^\infty \frac{1}{-i\sqrt{x}}e^{-xt}(-dx)$$
$$= -2i\int_0^\infty \frac{e^{-xt}}{\sqrt{x}}dx.$$

We evaluate this integral by setting $xt = X^2$, so that $dx = 2XdX/t$, and

$$\left(\int_{\gamma_+} + \int_{\gamma_-}\right) \bar{f}(s)e^{st}\,ds = -\frac{4i}{\sqrt{t}}\int_0^\infty e^{-X^2}\,dX = -2i\sqrt{\frac{\pi}{t}}.$$

Putting all this together then, from (72) we obtain

$$f(t) = \frac{1}{\sqrt{\pi t}}.$$

Homework: Ablowitz & Fokas, problems for section 4.5, questions 13(c), 14, 18.

2.3.5 Laplace transform applied to partial differential equations

The properties described in §2.3.3 mean that the Laplace transform (like the Fourier transform) can be useful to solve partial differential equations, often governing a quantity that varies both in space and time. For example, the diffusion, or heat, equation

$$u_t = u_{xx} \tag{73}$$

where t is time and x is position in space governs (among many other things!) the flow of heat in one spatial dimension (*e.g.* heat flow along a bar). This equation is first-order in time and second-order in space, and requires two spatial **boundary conditions** and an **initial condition** in time. So for example we must specify $u(a,t)$, $u(b,t)$ at two known points $x = a$, $x = b$, and the initial data $u(x,0)$. If we take the Laplace transform of (73) with respect to time $t \geq 0$ then with $\bar{u}(x;s) = \int_0^\infty u(x,t)e^{-st}dt$ we have

$$s\bar{u}(x;s) - u(x,0) = \frac{d^2\bar{u}}{dx^2}(x;s), \tag{74}$$

a second order **ordinary** differential equation for $\bar{u}$ in x, completed by knowledge of the initial condition. The transformed boundary conditions

$$\bar{u}(a;s) = \int_0^\infty u(a,t)e^{-st}\,dt, \qquad \bar{u}(b;s) = \int_0^\infty u(b,t)e^{-st}\,dt,$$

enable (74) to be solved for $\bar{u}(x;s)$. Once this is done, inversion needs to be performed to recover the solution $u(x,t)$ to the original PDE. We consider this general procedure ((i) transform PDE plus boundary conditions; (ii) solve resulting ODE to obtain Laplace transform of the desired solution; (iii) invert to recover the solution) applied to the following specific example.

Example 2.21 *Solve the partial differential equation (a linear* **wave equation***)*

$$u_{tt} = u_{xx}, \qquad 0 \le x \le l, \quad t > 0, \tag{75}$$

with initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = \sin(\pi x/l), \tag{76}$$

and boundary conditions

$$u(0,t) = 0 = u(l,t). \tag{77}$$

(Note that as this PDE is second order in time t we require 2 initial conditions.)

Laplace-transforming the PDE (75) and boundary conditions (77) in t, using property 3 of the Laplace transform (equation (65)) together with the initial conditions (76), leads to the ODE problem

$$\begin{aligned} &s^2\bar{u}(x;s) - su(x,0) - u_t(x,0) = \bar{u}_{xx}(x;s) \\ \Rightarrow \quad &\bar{u}_{xx}(x;s) - s^2\bar{u}(x,s) = -\sin\left(\frac{\pi x}{l}\right), \qquad \bar{u}(0;s) = 0 = \bar{u}(l;s). \end{aligned} \tag{78}$$

The general solution to the ODE (78) is

$$\bar{u}(x;s) = A(s)e^{sx} + B(s)e^{-sx} + \frac{\sin(\pi x/l)}{\pi^2/l^2 + s^2},$$

and $A(s)$, $B(s)$ must be determined from the boundary conditions, which yield $A(s) = 0 = B(s)$. We next have to invert $\bar{u}(x;s) = \sin(\pi x/l)/(\pi^2/l^2 +$

s^2). We can do this either by using the result of example 2.17, which "by inspection" gives

$$u(x,t) = \frac{1}{\pi}\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi t}{l}\right),$$

or, more systematically, we can apply the inversion theorem, which tells us that

$$u(x,t) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\sigma-iR}^{\sigma+iR} \bar{u}(x;s)e^{st}\,dt, \qquad t>0.$$

The integrand has just two simple poles, at $s = \pm i\pi/l$ and no branch points, hence we evaluate it by applying the Residue theorem to a standard Bromwich contour (figure 3) for any real $\sigma > 0$.

The semicircular part of the contour γ_R may be parametrized by $s = \sigma + iRe^{i\theta}$, for $0 \le \theta \le \pi$, so that for this part of the integral we have

$$\begin{aligned} I_R &= \int_{\gamma_R} \bar{u}(x;s)e^{st}\,dt = \int_0^\pi \frac{-e^{\sigma t}e^{-Rt\sin\theta}Re^{i\theta}\,d\theta}{\lambda^2+(\sigma+iRe^{i\theta})^2} \\ &= \frac{1}{R}\int_0^\pi \frac{e^{\sigma t}e^{-Rt\sin\theta}e^{iRt\cos\theta}e^{-i\theta}\,d\theta}{\left(1-\frac{i\sigma}{R}e^{-i\theta}\right)^2 - \frac{\lambda^2}{R^2}e^{-2i\theta}} \\ \Rightarrow \quad |I_R| &\le \frac{1}{R}\int_0^\pi \frac{e^{\sigma t}e^{-Rt\sin\theta}\,d\theta}{\left|\left|1-\frac{\sigma}{R}\right|^2 - \frac{\lambda^2}{R^2}\right|} \to 0 \quad \text{as } R\to\infty. \end{aligned}$$

The total integral around the closed Bromwich contour C is then equal just to the portion along the straight line, γ_L, which is exactly what we require for the Laplace inversion theorem. Thus, applying the Residue theorem we have

$$u(x,t) = [\text{Residue of } \bar{u}(x;s)e^{st} \text{ at } i\pi/l] + [\text{Residue of } \bar{u}(x;s)e^{st}\text{at } -i\pi/l].$$

We evaluate the residues using (34) (theorem 2.2), which gives

$$\begin{aligned} \operatorname{Res}(\bar{u}(x;s)e^{st}; \pm i\pi/l) &= \lim_{s\to\pm i\pi/l}(s\mp i\pi/l)\frac{\sin(\pi x/l)e^{st}}{(s+i\pi/l)(s-i\pi/l)} \\ &= \pm\frac{\sin(\pi x/l)e^{\pm i\pi t/l}}{2\pi i/l} \end{aligned}$$

so that, adding the residues,

$$u(x,t) = \frac{l}{\pi}\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi t}{l}\right),$$

as found by inspection.

Example 2.22 *(Diffusion equation) Solve*

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t \geq 0, \tag{79}$$
$$u_x(0,t) = 0, \quad u(1,t) = u_1, \tag{80}$$
$$u(x,0) = u_0, \tag{81}$$

where u_1, u_0 are constants.

We take the Laplace transform as before (using (65) and (64)) to transform the PDE (79) and initial condition (81) into a second-order ODE for $u(x;s)$,

$$\bar{u}_{xx} - s\bar{u} = -u_0, \tag{82}$$

with boundary conditions from transforming those for the PDE (80),

$$\bar{u}_x(0;s) = 0, \quad \bar{u}(1;s) = \frac{u_1}{s}. \tag{83}$$

Equation (82) has solution

$$\bar{u}(x;s) = A(s)e^{x\sqrt{s}} + B(s)e^{-x\sqrt{s}} + \frac{u_0}{s},$$

and the boundary conditions give

$$A(s) = B(s) = \frac{u_1 - u_0}{s(e^{\sqrt{s}} + e^{-\sqrt{s}})}.$$

Hence the solution to the original PDE is given by inverting

$$\begin{aligned} \bar{u}(x;s) &= \frac{(u_1 - u_0)}{s}\frac{\cosh x\sqrt{s}}{\cosh\sqrt{s}} + \frac{u_0}{s} \\ \Rightarrow \quad u(x,t) &= (u_1 - u_0)\mathcal{L}^{-1}\left(\frac{\cosh x\sqrt{s}}{s\cosh\sqrt{s}}\right) + u_0\mathcal{L}^{-1}\left(\frac{1}{s}\right). \end{aligned} \tag{84}$$

The function $1/s$ is easily inverted by inspection (it is the Laplace Transform of 1), but for the other part we need the Inversion theorem. We thus consider the inversion integrand $\bar{g}(s)e^{st} = \frac{e^{st}\cosh x\sqrt{s}}{s\cosh\sqrt{s}}$, which has a simple pole at $s = 0$, and other singularities at the zeros of $\cosh\sqrt{s} = \cos i\sqrt{s}$. Since $\cosh\sqrt{s}$ is an analytic function with the Taylor series

$$\cosh\sqrt{s} = \sum_{n=0}^{\infty} \frac{s^n}{(2n)!},$$

it is in fact single-valued on $\mathbb{C}$, and has only simple zeros at the points $s_n = -(2n+1)^2\pi^2/4$, on the negative real axis, corresponding to simple poles of the inversion integrand. We therefore consider a standard Bromwich contour C as in figure 3, and check that the integral around the semicircular portion γ_R tends to zero as $R \to \infty$. On γ_R, $s = \sigma + iRe^{i\theta}$, $0 \le \theta \le \pi$, and the integral I_R around γ_R satisfies, for R sufficiently large,

$$\begin{aligned} |I_R| &= \left| \int_0^{\pi} \frac{\cosh(x(\sigma + iRe^{i\theta})^{1/2})e^{(\sigma+iRe^{i\theta})t}(-Re^{i\theta}d\theta)}{(\sigma + iRe^{i\theta})\cosh((\sigma + iRe^{i\theta})^{1/2})} \right| \\ &\le M_0 \int_0^{\pi} \left| \frac{\cosh(x(\sigma + iRe^{i\theta})^{1/2})}{\cosh((\sigma + iRe^{i\theta})^{1/2})} \right| e^{\sigma t - Rt\sin\theta} d\theta \\ &= M_0 \int_0^{\pi} |G(\theta)|\, d\theta. \end{aligned}$$

Splitting the range of integration, for $0 < \theta < \pi/2$ we have

$$\begin{aligned} |G(\theta)| &= \frac{\cosh\left(x\sqrt{R}\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{R}\right)\right)\right)}{\cosh\left(\sqrt{R}\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{R}\right)\right)\right)} e^{\sigma t - Rt\sin\theta} \\ &= \left[\frac{e^{\left(-2x\sqrt{R}\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)(1+O\left(\frac{1}{R}\right))\right)}+1}{e^{\left(-2\sqrt{R}\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)(1+O\left(\frac{1}{R}\right))\right)}+1}\right] \frac{e^{\left(x\sqrt{R}\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)(1+O\left(\frac{1}{R}\right))\right)}}{e^{\left(\sqrt{R}\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)(1+O\left(\frac{1}{R}\right))\right)}} e^{\sigma t - Rt\sin\theta}. \end{aligned}$$

For $0 < \theta < \pi/2$, $\cos(\frac{\theta}{2}+\frac{\pi}{4}) > 0$, and so for $x > 0$ the term in square brackets is bounded as $R \to \infty$ (by M_1, say) and we have

$$\begin{aligned} |G(\theta)| &\le M_1 e^{\sigma t} \exp\left\{-Rt\sin\theta - (1-x)\sqrt{R}\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\left(1+O\left(\frac{1}{R}\right)\right)\right\} \\ &\to 0 \quad \text{as } R \to \infty. \end{aligned}$$

It follows that, for $0 < x < 1$,

$$\int_0^{\pi/2} |G(\theta)|\, d\theta \to 0 \quad \text{as } R \to \infty.$$

Similar arguments show that

$$\int_{\pi/2}^{\pi} |G(\theta)|\, d\theta \to 0 \quad \text{as } R \to \infty,$$

and thus I_R, the contribution to the Bromwich integral around the semicircle γ_R, goes to zero as $R \to \infty$.

The Residue theorem is therefore directly applicable to recover the inversion integral along the straight-line portion γ_L of the Bromwich contour C for $a > 0$:

$$\frac{1}{2\pi i} \lim_{R\to\infty} \int_{\sigma-iR}^{\sigma+iR} \bar{g}(x;s) e^{st}\, ds = \sum_{\text{all poles}} \text{Res}\left(\frac{e^{st}\cosh(x\sqrt{s})}{s\cosh\sqrt{s}} \right).$$

For the pole at $s = 0$ we have (applying (34))

$$\text{Res}\left(\frac{e^{st}\cosh(x\sqrt{s})}{s\cosh\sqrt{s}}, s = 0 \right) = 1.$$

For the pole at $s_n = -(2n+1)^2\pi^2/4$ we write $s = s_n + \epsilon$, $|\epsilon| \ll 1$, and note that

$$\begin{aligned}
\cosh\sqrt{s} &= \cos(i\sqrt{s}) = \cos\left(\frac{\pi}{2}(2n+1)\left(1 - \frac{2\epsilon}{\pi^2(2n+1)^2} + \cdots\right)\right) \\
&= \sin\left(\frac{\pi}{2}(2n+1)\right)\sin\left(\frac{\epsilon}{\pi(2n+1)}\right) + \cdots = \frac{(-1)^n\epsilon}{\pi(2n+1)} + \cdots
\end{aligned}$$

so that

$$\lim_{s\to s_n} \frac{s - s_n}{\cosh\sqrt{s}} = (-1)^n\pi(2n+1).$$

Finally then,

$$\text{Res}\left(\frac{e^{st}\cosh(x\sqrt{s})}{s\cosh\sqrt{s}}, s_n \right) = \frac{4(-1)^{n+1}}{\pi(2n+1)} \cos\left(\frac{\pi x(2n+1)}{2}\right) \exp\left(-\frac{(2n+1)^2\pi^2 t}{4}\right),$$

giving the solution to the original PDE, from (84), as

$$u(x,t) = u_1 + \frac{4}{\pi}(u_1 - u_0) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)} \cos\left(\frac{\pi x(2n+1)}{2}\right) \exp\left(-\frac{(2n+1)^2\pi^2 t}{4}\right).$$

Homework: Ablowitz & Fokas, problems for section 4.6, questions 2, 6(b), 7.

3 Conformal Mappings

3.1 Motivation

Many problems that arise in physical applied mathematics are free boundary problems, in that the domain on which governing equations have to be solved is itself unknown, and has to be determined as part of the solution process. Often, in fact, the shape of the domain is the main information that is desired. Many problems in fluid dynamics are of this kind, for example, when solving for free surface flows of water, the equations of hydrodynamics have to be solved on the domain occupied by the water, which (apart from any rigid containing boundaries) cannot be specified, but will depend on the various forces acting on the flowing fluid. Moreover, many fluid flows can be described in terms of solutions of Laplace's equation (2), which (in two space dimensions) we have already seen can be described in terms of analytic functions of a complex variable.

The fact that the domain on which equations must be solved is unknown greatly increases the difficulty of solving the problem. One possible way around this extra difficulty, when we are solving a 2D problem for which there is a complex variable representation for its solutions, is to **transform** the unknown domain onto some known domain, of simple geometry (for example, the unit disc). This can be accomplished by means of **conformal mapping**. Conformal mapping can also be very useful even when we know the domain that we have to solve on. If the domain has a very complicated shape then it can be difficult to apply boundary conditions and solve the PDE, but if we can map this complicated domain onto one of simple geometry then it is often easy to solve the related PDE in the new complex plane.

We shall see that it is possible to map any simply-connected domain onto any other via a conformal transformation (the **Riemann mapping theorem**), meaning that, instead of solving a given PDE (for example, Laplace's equation) on a domain of difficult and/or unknown geometry, one can instead solve a related PDE on (*e.g.*) the unit disc. In the particular case of Laplace's equation, the PDE itself is in fact **invariant** under conformal transformation, so that one only has to worry about how the boundary conditions transform to the new complex plane, and about the details of the conformal mapping required to accomplish the transformation.

We shall begin by discussing the general principles underlying conformal mapping, and then study some specific examples of conformal transformations, before moving onto applications (drawn from fluid dynamics) for which conformal mapping can be a very useful tool.

3.2 Conformality: general principles

Consider a curve $C \in \mathbb{C}$, and let $z = f(\zeta)$, where f is some analytic function of the complex variable $\zeta = \xi + i\eta$, define a change of variable from the complex variable ζ to the complex variable z. Under such a transformation the curve C in the ζ-plane will be mapped to another curve C' in the z-plane; and we shall refer to such transformations as **mappings** of the complex plane.

Suppose ζ_0 is a fixed point on C, and $\zeta_1 = \zeta_0 + \delta\zeta$ is another nearby point on C (so that $|\delta\zeta| \ll 1$). The corresponding points on C' are then $z_0 = f(\zeta_0)$, and

$$z_1 = f(\zeta_1) = f(\zeta_0 + \delta\zeta) = z_0 + (\delta\zeta) f'(\zeta_0) + \frac{(\delta\zeta)^2}{2} f''(\zeta_0) + O(|\delta\zeta|)^3, \quad (85)$$

by Taylor's theorem 1.25. Suppose now that the derivative $f'(\zeta_0) \neq 0$.

Definition 3.1 *If the mapping function f is defined and analytic on a domain $D \subset \mathbb{C}$, and furthermore has nonzero derivative everywhere on D, then we say that f is* **conformal** *on the domain D.*

Then the two image points and the two preimage points are related as follows:

$$z_1 - z_0 = (\zeta_1 - \zeta_0) f'(\zeta_0) + O((\zeta_1 - \zeta_0)^2). \quad (86)$$

If we take ζ_1 sufficiently close to ζ_0 then the curve C is approximately linear between the two points, represented by

$$\begin{aligned} C = \{\zeta : \zeta = \zeta_0 + s\exp(i\arg(\zeta_1 - \zeta_0)), \quad s \in [0, |\zeta_1 - \zeta_0|]\}, \\ \text{and} \quad \zeta_1 - \zeta_0 = |\zeta_1 - \zeta_0| \exp[i(\arg(\zeta_1 - \zeta_0)]. \end{aligned}$$

At z_1 we have, by (86),

$$z_1 = z_0 + |\zeta_1 - \zeta_0||f'(\zeta_0)| \exp(i[\arg(\zeta_1 - \zeta_0) + \arg(f'(\zeta_0))]). \quad (87)$$

Thus locally, a conformal transformation magnifies distances between points by a factor $|f'(\zeta_0)|$, and shifts angles by an additive constant $\arg(f'(\zeta_0))$. If the original curve is continuous and differentiable then under the conformal transformation, the image curve will be also.

Definition 3.2 *If $f'(\zeta_0) = 0$ at some point ζ_0 then the analytic transformation f is not conformal at that point. Such a point ζ_0 is called a* **critical point** *of the map f.*

To see what happens local to a critical point, we must continue the expansion (85) as far as necessary, to the first non-vanishing derivative, say the nth. Then, noting that

$$(\delta\zeta)^n = (|\delta\zeta|e^{i\arg\delta\zeta})^n = |\delta\zeta|^n e^{in\arg\delta\zeta},$$

it follows that locally two neighboring points on the image curve are related by

$$\begin{aligned} z_1 - z_0 &= \frac{(\delta\zeta)^n}{n!} f^{(n)}(\zeta_0) + O(|\delta\zeta|^{n+1}) \\ &= \frac{|\delta\zeta|^n}{n!} |f^{(n)}(\zeta_0)| \exp(i[n\arg(\delta\zeta) + \arg(f^{(n)}(\zeta_0))]) + O(|\delta\zeta|^{n+1}). \quad (88) \end{aligned}$$

Thus here, in addition to angles being shifted through angle $\arg(f^{(n)}(\zeta_0))$, they are first multiplied by n. Considering a second small line segment, from ζ_0 to the point ζ_2 in the preimage plane, we obtain a similar relation to (88) for its image point z_2. It follows that when two neighboring infinitesimal line segments of equal length $|\delta\zeta|$ are transformed in this way, the angle between them in the transformed plane (given by $\arg(z_2 - z_0) - \arg(z_1 - z_0)$) is equal to the angle between the original line segments, multiplied by the factor n. This is because

$$\begin{aligned} z_1 - z_0 &= \frac{|\delta\zeta|^n}{n!} |f^{(n)}(\zeta_0)| \exp(i[n\arg(\delta\zeta_1) + \arg(f^{(n)}(\zeta_0))]) + O(|\delta\zeta|^{n+1}), \\ z_2 - z_0 &= \frac{|\delta\zeta|^n}{n!} |f^{(n)}(\zeta_0)| \exp(i[n\arg(\delta\zeta_2) + \arg(f^{(n)}(\zeta_0))]) + O(|\delta\zeta|^{n+1}), \\ &\Rightarrow \quad \arg(z_2 - z_0) - \arg(z_1 - z_0) = n(\arg(\delta\zeta_2) - \arg(\delta\zeta_1)). \end{aligned}$$

Remark Note that even if the preimage curve in the ζ-plane does not self-intersect, there is nothing to stop self-intersection of the image curve in the z-plane. However, if the analytic mapping function f is also one-to-one, then the image curve will not self-intersect.

Definition 3.3 *f is* **locally one-to-one** *at $\zeta_0 \in D$ if $\exists\delta > 0$ such that for any ζ_1, $\zeta_2 \in B(\zeta_0, \delta)$ we have $f(\zeta_1) \neq f(\zeta_2)$.*

Definition 3.4 *f is* **locally one-to-one** *throughout D if f is locally one-to-one at each $\zeta \in D$.*

Definition 3.5 *f is one-to-one, or* **schlicht** *on D if for every distinct ζ_1, $\zeta_2 \in D$, $f(\zeta_1) \neq f(\zeta_2)$.*

Definition 3.6 *If f is one-to-one and analytic on a domain D then it is said to be* **univalent** *on D (it takes no value more than once in D).*

This last property will be useful (and necessary) later in applications. Note (no proof) that univalency guarantees **invertibility** of f on D. The image of a closed curve C under a univalent map F is another closed curve C'; and if f is univalent on the domain D contained within C then D will be mapped under f to the domain D' contained within C'.

3.3 Simple examples of conformal mappings

The best way to see the effect that a given conformal map has on the shape of a curve is to try out some examples. Once you have an idea of the basics, more complicated examples can be built up. The representation (87) makes it clear how to construct some very simple conformal maps such as rotations, scalings, and translations. Such simple examples may be directly related to linear transformations in the real (x, y)-plane.

Simple scaling & rotation The mapping

$$z = f(\zeta) = a\zeta$$

will magnify the original curve by the factor $|a| = |f'(\zeta)|$, and rotate it through angle $\arg(a)$. Thus if $a = 2i$ and we consider a rectangular shape made of four linear portions $\Im(\zeta) = 0$, $\Im(\zeta) = 2$, $\Re(\zeta) = 0$, $\Re(\zeta) = 1$, then the image shape will be a rectangle that is double the linear size of the original, and rotated through angle $\pi/2$, the images of the four lines being (respectively): $\Re(z) = 0$, $\Re(z) = -4$, $\Im(z) = 0$, $\Im(z) = 2$. Figure 5 shows both rectangles. To translate the rotated and scaled rectangle, one

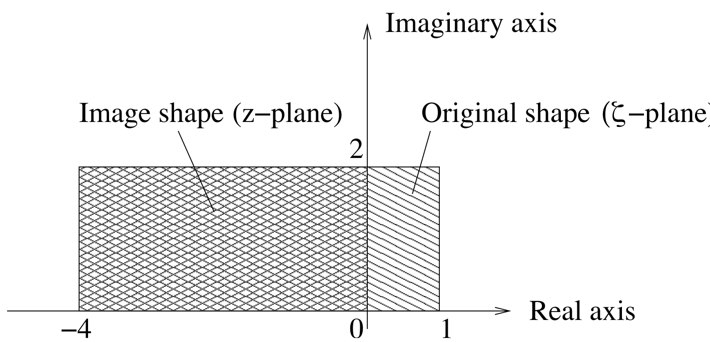

Figure 5: Simple conformal mapping that scales and rotates an initial shape. Both the image (z) and preimage (ζ) planes are shown on the same axes.

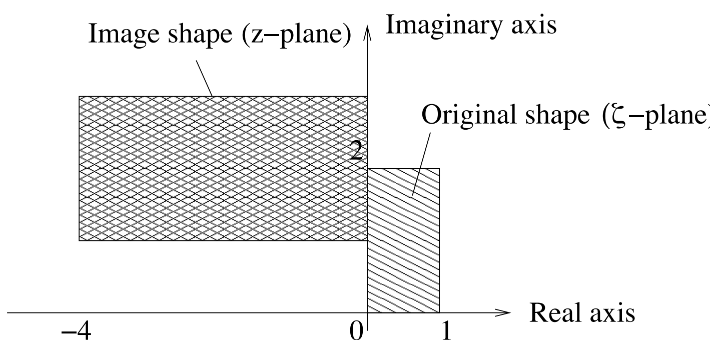

Figure 6: Simple conformal mapping that scales, rotates and translates an initial shape. Both the image (z) and preimage (ζ) planes are shown on the same axes.

can simply add a constant to the conformal map:

$$z = f(\zeta) = a\zeta + b.$$

Taking $b = i$ shifts the image rectangle in figure 5 up one unit on the imaginary axis (see figure 6).

Expansion of a sector We saw above how non-conformality of f at a point ζ_0 leads to magnification of angles locally at that point. Consider the image of the sector (angular domain) $0 \leq \arg(\zeta) \leq \phi$ under the mapping $f(\zeta) = \zeta^n$ for $n > 0$. The boundary $\arg(\zeta) = 0$ is represented by $z = r$, $r \in \mathbb{R}^+$, and maps to

$$z = r^n \in \mathbb{R}^+,$$

while the boundary $\arg(\zeta) = \phi$ is represented by $z = re^{i\phi}$, $r \in \mathbb{R}^+$, and maps to

$$z = r^n e^{in\phi},$$

a straight line with constant argument $n\phi$. So the image of the sector is another sector, with the angle at the apex multiplied by n. Note that, while the original result represented by (88) was demonstrated only for integers n, we can see from this example that the property of angles being multiplied locally by a factor of n holds more generally for positive real n. In particular, the choice $n = \pi/\phi$ takes the sector to a half-plane.

Homework: Investigate the effect of the mapping $z = \zeta^{-n}$, $n > 0$, on a sector $0 \leq \arg(\zeta) \leq \phi$.

Exponential mapping The mapping function $f(\zeta) = e^\zeta$ is conformal on $\mathbb{C}$. It takes straight lines with constant imaginary part, $\zeta = \xi + i\eta_0$, to straight lines of constant argument η_0,

$$z = e^\xi e^{i\eta_0},$$

so that a strip $\eta_0 < \Im(\zeta) < \eta_1$ is mapped to a sector $\eta_0 < \arg(z) < \eta_1$. On a strip of width greater than 2π the map is no longer one-to-one (univalent), mapping to a self-overlapping sector as indicated in figure 7 for $\eta_0 = 0$, $\eta_1 = 9\pi/4$.

Straight lines of constant real part, $\zeta = \xi_0 + i\eta$, are mapped to curves of constant absolute value $|z| = e^{\xi_0}$,

$$z = e^{\xi_0} e^{i\eta},$$

so that as η varies we traverse circles of radius e^{ξ_0}. Thus a strip $\xi_0 < \Re(\zeta) < \xi_1$ is mapped to an annulus $e^{\xi_0} < |z| < e^{\xi_1}$. Note however that, similar to the case above, unless we restrict η to lie in the range $0 \leq \eta < 2\pi$ (or $-\pi \leq \eta < \pi$ or some other equivalent range of length 2π) we cover the annulus more than once as η varies.

With the relevant range restrictions in place the map is **univalent** and therefore invertible, with inverse given by a specific branch of the logarithm function, *e.g.*

$$\zeta = \ln(z), \qquad \arg(z) \in [-\pi, \pi).$$

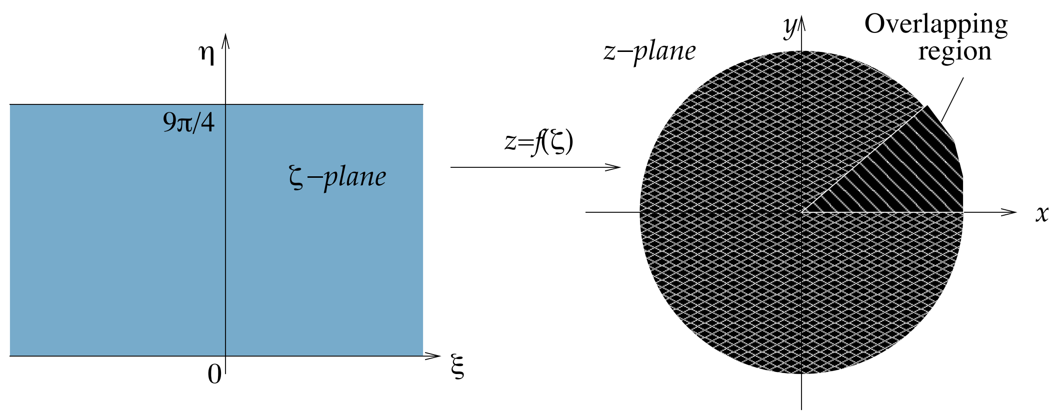

Figure 7: The exponential mapping $z = f(\zeta) = e^{\zeta}$ applied to a strip parallel to the y-axis, of width greater than 2π. The image is a self-overlapping sector.

Homework: By considering the image of the unit circle, $\zeta = e^{i\theta}$, under the exponential mapping $z = f(\zeta) = e^{\zeta}$, find the image of the unit circle under this mapping function. (Note: a sketch of the image domain is enough; total accuracy is not required.)

Trigonometric mappings (related to exponentials) Consider now the mapping

$$z = f(\zeta) = c \sin \zeta, \qquad c \in \mathbb{R}^+.$$

The general point $\zeta = \xi + i\eta$ maps to

$$\begin{aligned} x + iy &= c \sin(\xi + i\eta) = c(\sin \xi \cosh \eta + i \cos \xi \sinh \eta) \\ \Rightarrow \qquad & x = c \sin \xi \cosh \eta, \qquad y = c \cos \xi \sinh \eta. \end{aligned}$$

Curves $\Im(\zeta) = \text{constant} = \eta_0$ are therefore mapped to ellipses with

$$\frac{x^2}{c^2 \cosh^2 \eta_0} + \frac{y^2}{c^2 \sinh^2 \eta_0} = 1,$$

which in the limit $\eta_0 \to 0$ collapse to the straight line segment $y = 0$, $|x| \leq c$. The strip $\eta_1 \leq \eta \leq \eta_2$ is mapped to the region between the two ellipses

$$\frac{x^2}{c^2 \cosh^2 \eta_1} + \frac{y^2}{c^2 \sinh^2 \eta_1} = 1 \quad \text{and} \quad \frac{x^2}{c^2 \cosh^2 \eta_2} + \frac{y^2}{c^2 \sinh^2 \eta_2} = 1,$$

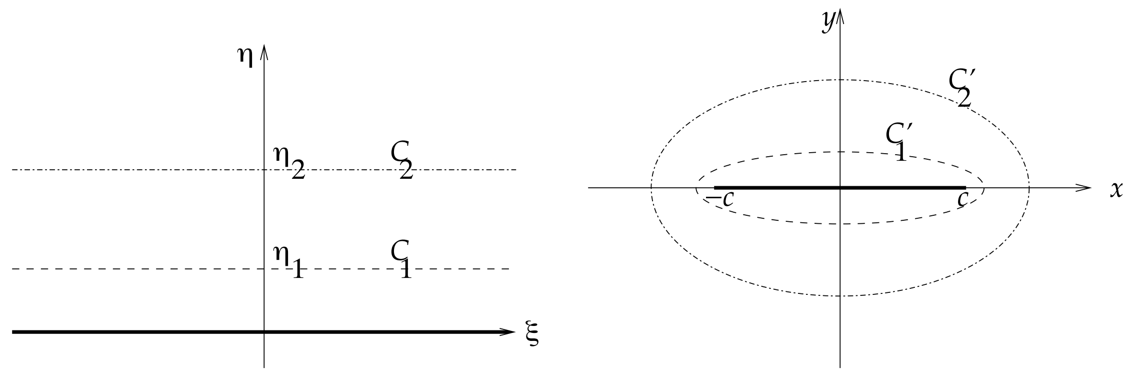

Figure 8: Mapping of curves $\Im(\zeta)$ = constant under the conformal transformation $z = c \sin \zeta$.

(see figure 8), this region being covered more than once unless ξ is restricted to an interval of length 2π. As all values $\eta_0 \geq 0$ are used, the whole of the z-plane is covered by the ellipses. Values $\eta_0 < 0$ generate the same family of ellipses.

Curves $\Re(\zeta)$ = constant = ξ_0 are mapped to hyperbolae

$$\frac{x^2}{c^2 \sin^2 \xi_0} - \frac{y^2}{c^2 \cos^2 \xi_0} = 1.$$

The value $\xi_0 = 0$ maps to the y-axis, while values $\xi_0 = \mp\pi/2$ map to lines $y = 0$, $x < -c$ and $y = 0$, $x > c$, respectively; see figure 9. As values ξ_0 are taken in the range $-\pi/2 \leq \xi_0 \leq \pi/2$ the whole z-plane is covered by the hyperbolae; values outside this range will generate repeated coverings of the region by the same family of curves.

Remark Note that, while over most of the domain we will require conformality (and often univalency) of the mapping function, often it is useful to have points of non-conformality or even singularities in the mapping function. This need arises when we are solving a physical problem on an irregular domain, and we would like to map the problem onto a nice regular-shaped domain on which it is easier to solve, for example. We have already seen how a sector $0 \leq \arg(\zeta) \leq \phi$ can be expanded to a half-space by the power mapping $z = \zeta^{\pi/\phi}$, which is not conformal at $\zeta = 0$.

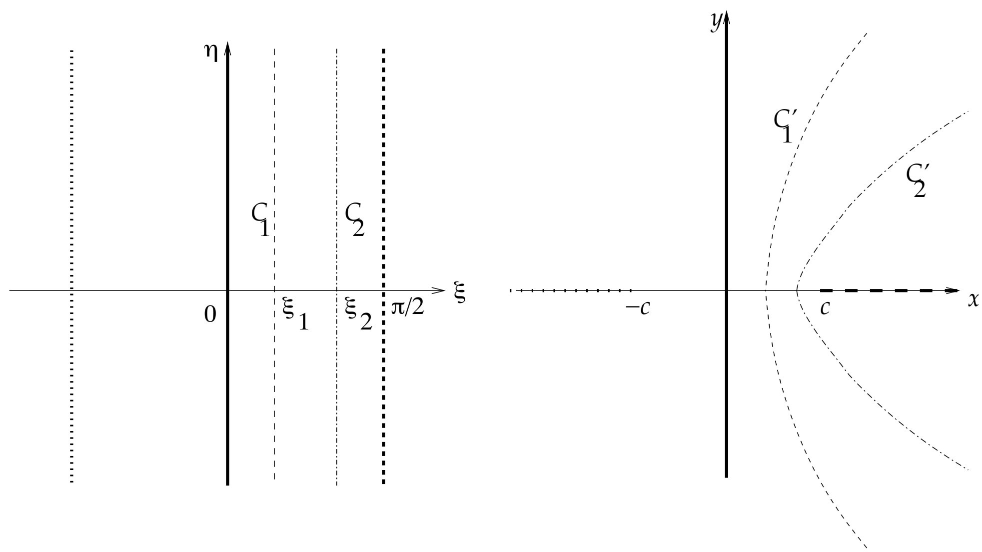

Figure 9: Mapping of curves $\Re(\zeta)$ = constant under the conformal transformation $z = c\sin\zeta$.

Example 3.7 *The mapping $z = f(\zeta) = 1/\zeta$ takes the interior of the unit disc, $|\zeta| \leq 1$, to the exterior of the unit disc, $|z| \geq 1$, because if $\zeta = re^{i\theta}$ with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$ then*

$$z = \frac{e^{-i\theta}}{r},$$

so as r varies in $[0, 1]$ and θ varies in $[0, 2\pi)$ the entire domain exterior to $|z| = 1$ is covered.

In this example f is conformal on the unit disc except at $\zeta = 0$, which is the singular point mapped to infinity in the image domain.

Example 3.8 Coordinate transformations as conformal mappings

We can recognize certain coordinate transformations as conformal mappings when we view them correctly. Consider the bipolar coordinate transformation, which relates cartesian coordinates (x, y) to coordinates (u, v), with the change of variables defined by

$$x = \frac{a\sinh v}{\cosh v - \cos u}, \quad y = \frac{a\sin u}{\cosh v - \cos u}.$$

Coordinate lines u = constant, v = constant each map to circles in the (x, y)-plane, since

$$x^2 + (y - a\cot u)^2 = a^2\mathrm{cosec}^2 u,$$

which for constant u represent circles with center $(0, a\cot u)$ and radius $a\mathrm{cosec}u$. All such circles intersect the x-axis at points $x = \pm a$. Also,

$$(x - a\mathrm{coth}v)^2 + y^2 = a^2\mathrm{cosech}^2 v,$$

which for constant v represent circles with center $(a\mathrm{coth}v, 0)$ and radius $a\mathrm{cosech}v < a\mathrm{coth}v$ (so for $v > 0$ all such circles are in the right half-plane $x > 0$ and vice-versa for $v < 0$).

It is far from obvious that this transformation is equivalent to a conformal mapping, but if we form $z = x + iy$, we can show that

$$z = x + iy = \frac{a(\sinh v + i\sin u)}{\cosh v - \cos u} = ia\cot\left(\frac{\zeta}{2}\right),$$

where $\zeta = u + iv$.

This is most easily demonstrated by showing that the final equality above holds. With $\zeta = u + iv$,

$$\begin{aligned}
ia\cot\left(\frac{\zeta}{2}\right) &= ia\frac{\cos\frac{1}{2}(u+iv)}{\sin\frac{1}{2}(u+iv)} \\
&= ia\frac{\cos\frac{u}{2}\cosh\frac{v}{2} - i\sin\frac{u}{2}\sinh\frac{v}{2}}{\sin\frac{u}{2}\cosh\frac{v}{2} + i\cos\frac{u}{2}\sinh\frac{v}{2}} \\
&= ia\frac{(\cos\frac{u}{2}\cosh\frac{v}{2} - i\sin\frac{u}{2}\sinh\frac{v}{2})(\sin\frac{u}{2}\cosh\frac{v}{2} - i\cos\frac{u}{2}\sinh\frac{v}{2})}{\sin^2\frac{u}{2}\cosh^2\frac{v}{2} + \cos^2\frac{u}{2}\sinh^2\frac{v}{2}} \\
&= ia\frac{(\frac{1}{2}\sin u\cosh^2\frac{v}{2} - \frac{1}{2}\sin u\sinh^2\frac{v}{2} - \frac{i}{2}\sinh v\cos^2\frac{u}{2} - \frac{i}{2}\sinh v\sin^2\frac{u}{2})}{\cosh^2\frac{v}{2}(1-\cos^2\frac{u}{2}) + \cos^2\frac{u}{2}(\cosh^2\frac{v}{2} - 1)} \\
&= \frac{ia}{2}\frac{\sin u - i\sinh v}{\cosh^2\frac{v}{2} - \cos^2\frac{u}{2}} \\
&= ia\frac{\sin u - i\sinh v}{(2\cosh^2\frac{v}{2} - 1) - (2\cos^2\frac{u}{2} - 1)} \\
&= a\frac{(\sinh v + i\sin u)}{\cosh v - \cos u}.
\end{aligned}$$

Hence the result is proved.

If we are prepared to put up with a nonlinear rescaling of distance, then standard polar coordinates could be thought of in a similar manner, as a complex transformation $\zeta = \log z$ (conformal away from the origin, with an appropriate branch-cut). The real part of ζ is $\log |z| = \log r$, rather than the normal polar distance r; while $\Im(\zeta)$ is the standard polar angle θ.

Sometimes before attempting a conformal transformation between two given domains, it can help to spend some time thinking of the most appropriate geometrical description of the regions, as the following example illustrates.

Example 3.9 *Find a mapping from the interior of the unit disc to the upper-half z-plane.*

Such a map necessarily has a point of non-analyticity, since we are transforming from a finite domain to an infinite one. In this case an easy way to find the desired transformation is to note that the defining characteristic of the unit disc is that it is the set of points $\zeta \in \mathbb{C}$ such that $|\zeta| \leq 1$, while a convenient way to define the upper half-plane is that it is exactly the set of points $z \in \mathbb{C}$ that are closer to i than $-i$, and therefore $|z - i| < |z + i|$. It follows that the map

$$\zeta = \frac{z - i}{z + i}$$

takes the upper-half z-plane onto the unit disc $|\zeta| \leq 1$. We are asked for the inverse of this map, so we simply rearrange the above relation to find that

$$z = f(\zeta) = \frac{i(1 + \zeta)}{(1 - \zeta)} \tag{89}$$

is a mapping function that will do the trick.[3] This map is singular at the point $\zeta = 1$ on the boundary of the unit disc; this is the point that is mapped to ∞ on the "boundary" of the upper half plane.

[3] It is not the only one that would work; for example we could first rotate the unit disc through any arbitrary angle before applying the above transformation to the rotated disc. We will say more later about conditions that guarantee uniquenss of a conformal map between given domains.

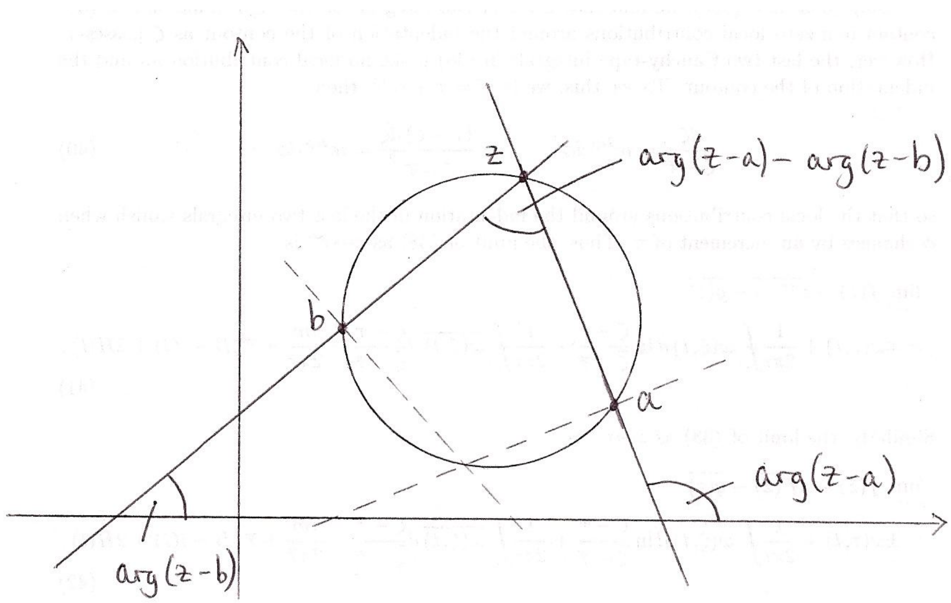

Figure 10: A circular arc in the complex plane passing through points a, b, with defining angle μ.

Homework: 1. Use the method above to find a conformal transformation from the unit disc $|\zeta| \leq 1$ onto the half-space $\Re(z) \geq 1$. 2. Alternatively, combine the transformation (89) above with a subsequent rotation and translation to perform the same task as in 1.
Also: Ablowitz & Fokas, problems for section 5.2, questions 3,6. Problems for section 5.3, question 4.

Here is another example where an appropriate description of the domain leads to an easy way to map it to a simple domain:

Example 3.10 *Mapping of circular arcs.*

Recall from Math 656 that, as a consequence of simple circle theorems, circular arcs passing through two points a, b in the complex z-plane, have representations

$$\arg(z-a) - \arg(z-b) = \mu$$

(see figure 10). Equivalently then,

$$\arg\left(\frac{z-a}{z-b}\right) = \mu.$$

This remarkably simple observation gives us an easy way to transform circular arcs into straight lines through the origin! More remarkably, the crescent- or lens-shaped region between two circular arcs that intersect at points a, b, can be transformed into a sector, simply by applying the transformation

$$w = \frac{z-a}{z-b}.$$

If the first arc has defining angle μ_1 and the second $\mu_2 > \mu_1$, then the region between them will be transformed into the sector

$$\mu_1 < \arg(w) < \mu_2.$$

The point $z = a$ maps to the origin in the w-plane, while $w = b$ maps to infinity.

3.4 Detour: Laplace's equation under conformal transformation

Many physical problems of interest (problems in inviscid fluid flow, potential theory problems in electrostatics and gravitation, and many more) are governed by Laplace's equation on a given domain D'

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{for } (x, y) \in D', \tag{90}$$

subject to appropriate boundary conditions on the boundary of the domain, ∂D (for example, the values of u on $\partial D'$ may be specified). If the boundary of D' has an irregular shape then (90) can be very difficult to solve; however there are known techniques (*e.g.* Green's functions, separation of variables) for solving Laplace's equation on canonical "nice" domains such as the unit circle or a half-space. Suppose we are able to map conformally from such a "nice" domain D (say the unit disc, to be specific) in the ζ-plane, onto the irregular domain D' in the z-plane via $z = f(\zeta)$.

By Lemma 1.8 u must be the real part of an analytic function, $u(x, y) = \Re(g(z))$ on D'. We can then define an analytic function $G(\zeta)$ on D via

$$G(\zeta) = g(f(\zeta)) \equiv (g \circ f)(\zeta),$$

(the composition of two analytic functions is another analytic function) and if $U(\xi,\eta) = \Re(G(\zeta))$ then U is a harmonic function of (ξ,η), and at corresponding points in the two complex planes we have

$$U(\xi,\eta) = \Re(G(\zeta)) = \Re(g(f(\zeta))) = \Re(g(z)) = u(x,y).$$

Thus, if we know the values of $U(\xi,\eta)$ (which we find by solving Laplace's equation on the "nice" domain), and the conformal map f that takes points (ξ,η) to points (x,y), then we have solved the original problem for $u(x,y)$. We have already mentioned that it is known how to solve Laplace's equation with given boundary data on nice canonical domains, therefore the crucial part of the above procedure is whether the desired conformal mapping can be found. That it can is guaranteed by the following theorem, which we state first informally:

Theorem 3.11 *(Riemann mapping theorem — informal statement) Any two simply-connected domains, each having more than one boundary point, can be mapped conformally one on the other.*

Unfortunately, neither the theorem, nor its proof, give any hint as to how one might construct the desired mapping in a given situation. However, there is an important special case in which one can construct a conformal mapping explicitly: when one is mapping from a simple canonical domain (such as a half-space or the unit disc) onto the interior of an arbitrary polygon. The associated conformal mapping is called the **Schwarz-Christoffel transformation**, and is outlined in Appendix A.

3.5 Riemann mapping theorem

Definition 3.12 *(Conformal equivalence) We call two regions D and D'* **conformally equivalent** *if there exists a function f univalent on D such that $D' = f(D)$, that is, if there exists a conformal one-to-one mapping of D' onto D.*

An equivalent statement of the Riemann mapping theorem is:

Theorem 3.13 *(Riemann mapping theorem) Every simply-connected region D in the plane is conformally equivalent to the unit disc.*

The mapping of any one region to any other is not unique. In order to explore the freedom in the mapping it is instructive first to consider how many different ways the unit disc may be mapped to itself.

Definition 3.14 *A one-to-one conformal mapping of any region to itself is called an* **automorphism** *of the region.*

As a preliminary to proving results about general automorphisms of the unit disk, we first state and prove:

Lemma 3.15 *(Schwarz Lemma) Let $f : B(0,1) \to B(0,1)$ be analytic, with $f(0) = 0$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|\ \forall z \in B(0,1)$. If $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \in B(0,1)\backslash\{0\}$ then $f(z) = e^{i\alpha}z$ for some $\alpha \in \mathbb{R}$.*

Proof Since $f(0) = 0$ the function defined by

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & z \neq 0, \\ f'(0) & z = 0, \end{cases}$$

is analytic in $B(0,1)$ (a simple consequence of Taylor's theorem 25). We claim that $|g(z)| \leq 1$ for all $z \in B(0,1)$. For, suppose that $|g(z^*)| = t > 1$ for some $z^* \in B(0,1)$, $|z^*| < 1$. Let $r_n = 1 - 1/n$, then for $r_n > |z^*|$ (that is, $n > 1/(1 - |z^*|)$) the maximum modulus principle applied to g on $B(0, r_n)$ states that the maximum of $|g|$ must occur on the boundary, $|z| = r_n$. Since we know $|g(z^*)| = t$ and $z^* \in B(0, r_n)$, we therefore have some point z_n with $|z_n| = r_n$ and $|g(z_n)| \geq t$. But since $t > 1$ is fixed and r_n can be made arbitrarily close to 1, we can also choose n large enough that $r_n t > 1$ and then

$$|g(z_n)| \geq t \quad \Rightarrow \quad |f(z_n)| \geq t|z_n| = tr_n > 1,$$

which contradicts the assumption that $f : B(0,1) \to B(0,1)$.

This proves the claim that $|g(z)| \leq 1$ for all $z \in B(0,1)$, and there are now two possibilities. Either g is a constant of modulus 1 or, by the maximum modulus theorem 1.10, we have $|g(z)| < 1$ for all $z \in B(0,1)$. If $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \in B(0,1)\backslash\{0\}$ then the latter possibility cannot hold, and so $g(z) = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$, giving $f(z) = e^{i\alpha}z$.

Lemma 3.16 *(Automorphisms of the unit disc that fix the origin) The only automorphisms of the unit disc with $f(0) = 0$ are given by $f(\zeta) = e^{i\alpha}\zeta$, some $\alpha \in \mathbb{R}$.*

Proof Write $\zeta_1 = f(\zeta)$ for the mapping from the unit disc in the ζ-plane to the unit disc in the ζ_1-plane. By the Schwarz lemma 3.15, $|\zeta_1| = |f(\zeta)| \leq |\zeta|$ for $|\zeta| < 1$. Since f is 1-1 it is also invertible; and $\zeta = f^{-1}(\zeta_1)$ is another automorphism of the unit disc with $f^{-1}(0) = 0$, so also $|\zeta| = |f^{-1}(\zeta_1)| \leq |\zeta_1|$. It follows that $|\zeta_1| = |\zeta|$, or $|f(\zeta)| = |\zeta|$, and thus $f(\zeta) = e^{i\alpha}\zeta$.

Lemma 3.17 *(Automorphisms of the unit disc) The automorphisms of the unit disc are of the form $f(\zeta) = e^{i\alpha}\left(\frac{\zeta - a}{1 - \bar{a}\zeta}\right)$, where $\alpha \in \mathbb{R}$ and $|a| < 1$.*

Proof Any automorphism of the unit disc must map some point $a \in B(0,1)$ to the origin. Let $w = g(\zeta) = \frac{\zeta - a}{1 - \bar{a}\zeta}$. Then g is analytic and single-valued on the unit disc, and the point $\zeta = a$ within $B(0,1)$ maps to $w = 0 \in B(0,1)$. Also,

$$|w|^2 = \frac{|\zeta|^2 - \bar{a}\zeta - a\bar{\zeta} + |a|^2}{1 - \bar{a}\zeta - a\bar{\zeta} + |a|^2|\zeta|^2} = 1, \quad \text{for } |\zeta| = 1.$$

By the maximum modulus theorem 1.10 it follows that $|g(\zeta)| \leq 1$, and g is a mapping from $B(0,1)$ to $B(0,1)$. The function g is also 1-1 from $B(0,1)$ to itself, since

$$w = g(\zeta) \quad \Rightarrow \quad \zeta = \frac{w + a}{1 + \bar{a}w} = g^{-1}(w)$$

(a function of exactly the same type as g), which shows that to any image point $|w| \leq 1$ there corresponds a preimage point $\zeta \in B(0,1)$ (exactly the same argument as for g proves that $|g^{-1}(w)| \leq 1$ for $w \in B(0,1)$). Therefore g is a 1-1 mapping of the unit disc to itself, that is, an automorphism.

Suppose now that f is another automorphism of the unit disc such that $f(a) = 0$. Then $h = f \circ g^{-1}$ is another automorphism of $B(0,1)$, and $h(0) = f(g^{-1}(0)) = f(a) = 0$. By lemma 3.16 above then, $h(\zeta) = e^{i\alpha}\zeta$; and with $w = g(\zeta)$, $\zeta = g^{-1}(w)$ we have

$$h(w) = (f \circ g^{-1})(w) \quad \Rightarrow \quad f(\zeta) = h(g(\zeta)) = e^{i\alpha}g(\zeta) = e^{i\alpha}\left(\frac{\zeta - a}{1 - \bar{a}\zeta}\right).$$

Lemma 3.18 *(Uniqueness of mappings between domains) Suppose D_1 and D_2 are simply-connected domains, that $f : D_1 \to D_2$ is a 1-1 conformal mapping between them, and that g is an automorphism of D_2. Then*
(a) Any other 1-1 conformal mapping $F : D_1 \to D_2$ is of the form $g \circ f$.
(b) Any automorphism h of D_1 is of the form $f^{-1} \circ g \circ f$.

Proof (a) If f and F are both 1-1 conformal mappings of D_1 onto D_2, then $F \circ f^{-1}$ is an automorphism g of D_2, that is, $F \circ f^{-1} = g$, so that $F = g \circ f$.
(b) If h is an automorphism of D_1 then $f \circ h \circ f^{-1}$ is an automorphism of D_2, since (applying the maps in sequence) f^{-1} is a 1-1 mapping of D_2 to D_1, h is an automorphism of D_1, and f is a 1-1 mapping of D_1 to D_2. So, $f \circ h \circ f^{-1} = g$, and thus $h = f^{-1} \circ g \circ f$.

Homework: On the following theorem.

Theorem 3.19 *(Automorphisms of the upper half-plane) The automorphisms of the upper half-plane D^+ are of the form*

$$f(\zeta) = \frac{a\zeta + b}{c\zeta + d}, \tag{91}$$

with $a, b, c, d \in \mathbb{R}$, and $ad - bc > 0$.

To do: You should first verify that f maps D^+ to D^+ by showing that the boundary maps to the boundary and that an interior point maps to an interior point. Next check for the one-to-one (invertible) nature of the map. Then use the result of example 3.9 together with lemmas 3.18 and 3.17 to prove the theorem.

Remark The mapping (91) is known as a **Mobius transformation** or **bilinear mapping**, and will be studied in generality in §3.7.

Theorem 3.20 *(Uniqueness of map in Riemann mapping theorem) The conformal mapping f from $|\zeta| \leq 1$ onto the simply connected region D is uniquely specified by any of the following:*

1. *Specifying the image z_0 of $\zeta = 0$ under f, and specifying $f'(0) \in \mathbb{R}^+$; or*

2. *Specifying the image z_0 of $\zeta = 0$ under f, and specifying that ζ_1 on $|\zeta| = 1$ maps to a given point z_1 on ∂D; or*

3. *Choosing three arbitrary points $\zeta_1, \zeta_2, \zeta_3$ on $|\zeta| = 1$ and specifying the three image points z_1, z_2, z_3 on ∂D (the two sets of points appearing in the same order as the boundary is traversed in the same sense).*

Proof (of uniqueness)

1. Let f_1, f_2 be two mappings with property 1, then $F = f_1^{-1} \circ f_2$ is an automorphism of $B(0,1)$. Also,

   $$F(0) = f_1^{-1}(f_2(0)) = f_1^{-1}(z_0) = 0,$$

   so by lemma 3.16 $F(\zeta) = e^{i\alpha}\zeta$, where $\alpha \in \mathbb{R}$. But then

   $$\begin{aligned} e^{i\alpha} = F'(\zeta) &= \frac{d}{d\zeta}\left(f_1^{-1}(f_2(\zeta))\right) = f_1^{-1'}(f_2(\zeta))f_2'(\zeta) \\ \Rightarrow \quad e^{i\alpha} &= F'(0) = f_1^{-1'}(z_0)f_2'(0) = \frac{f_2'(0)}{f_1'(0)} \in \mathbb{R}^+, \end{aligned}$$

   where in the last line we used the fact that $f^{-1'} = 1/f'$.
   Exercise: Prove this last result.
   It follows that $\alpha = 0$ (modulo 2π), and thus that $F(\zeta) = \zeta$, and $f_2 = f_1$.

2. Again let f_1, f_2 be two mappings with property 2, then as above $F = f_1^{-1} \circ f_2 = e^{i\alpha}\zeta$. Also,

   $$e^{i\alpha}\zeta_1 = F(\zeta_1) = f_1^{-1}(f_2(\zeta_1)) = f_1^{-1}(z_1) = \zeta_1,$$

   thus again, $\alpha = 0$ (modulo 2π), and $f_2 = f_1$.

3. As above, $F = f_1^{-1} \circ f_2$ is an automorphism of $B(0,1)$, and so by lemma 3.17,

   $$F(\zeta) = e^{i\alpha}\frac{\zeta - a}{1 - \overline{a}\zeta},$$

   and F has three distinct fixed points ζ_1, ζ_2, ζ_3. But any fixed point ζ_* of F satisfies

   $$\overline{a}\zeta_*^2 + \zeta_*(e^{i\alpha} - 1) - ae^{i\alpha} = 0,$$

   a quadratic equation with at most two distinct roots, unless $a = 0$ and $\alpha = 0$ (modulo 2π). The result $f_1 = f_2$ again follows.

Homework: Read up on the proof of the Riemann mapping theorem (non-examinable).

3.6 Joukowski transformation

One mapping transformation that is very important in inviscid flow applications (in particular, aerodynamics) is the Joukowski transformation, defined by

$$z = f(\zeta) = \zeta + \frac{a^2}{\zeta}, \qquad a \in \mathbb{R}^+. \tag{92}$$

Noting that $f'(\zeta) = 1 - a^2/\zeta^2$, the map is conformal except at the points $\zeta = 0$, $\zeta = \pm a$. Circles $|\zeta| = r$ transform via

$$z = x + iy = \left(r + \frac{a^2}{r}\right)\cos\theta + i\left(r - \frac{a^2}{r}\right)\sin\theta$$

hence the circle $|\zeta| = r > a$ is mapped to the ellipse

$$\frac{x^2}{(r + a^2/r)^2} + \frac{y^2}{(r - a^2/r)^2} = 1,$$

and the region exterior to the circle $|\zeta| = r > a$ is mapped to the region exterior to this ellipse.

The circle of radius a is a special case, containing 2 points of non-conformality, and maps to the **slit** $|x| \le 2a$, $y = 0$.

Consider next what happens to circles not centered on the origin under (92); in particular, what happens to a circle that passes through just one of the points of non-conformality. Such a circle is given by

$$|\zeta - \delta| = a + \delta, \quad \delta > 0,$$

which passes through $\zeta = -a$. Clearly the resulting shape will be smooth except for a sharp point at the image of $\zeta = -a$; and for $0 < \delta \ll 1$ (where we anticipate a slender shape close to the straight line segment $|x| \le 2a$) we can set $\zeta = \delta + (a + \delta)e^{i\theta}$ and expand in powers of δ to find an approximate expression for the image shape in terms of the parameter θ:

$$\begin{aligned} z = x + iy &= 2a\cos\theta + 2\delta\sin\theta(\sin\theta + i(1 + \cos\theta)) + O(\delta^2) \\ \Rightarrow &\quad \left\{ \begin{array}{l} x = 2a\cos\theta + 2\delta\sin^2\theta + O(\delta^2) \\ y = 2\delta\sin\theta\cos^2(\theta/2) + O(\delta^2). \end{array} \right. \end{aligned} \tag{93}$$

Restricting to the range of interest $0 \le \theta < 2\pi$ we find that

$$\frac{dx}{d\theta} = -2a\sin\theta + 4\delta\sin\theta\cos\theta + O(\delta^2),$$
$$\frac{dy}{d\theta} = 4\delta\cos(\theta/2)\cos(3\theta/2) + O(\delta^2)$$

so that $dx/d\theta = 0$ only at $\theta = 0$, π, while $dy/d\theta = 0$ at $\theta = \pi/3$, π, $2\pi/3$. $\theta = \pi$ corresponds to the critical point of the map at the "rear"; the turning-point in x at $\theta = 0$ corresponds to the blunt "front" of the shape; and $\theta = \pi/3$, $2\pi/3$ are the turning-points $\pm y_{max}$ in y marking the thickest point of the shape. The thickness is found by substituting $\theta = \pi/3$ into (93), giving

$$y_{max} = \frac{3\delta\sqrt{3}}{4} \quad \text{at} \quad x = a + \frac{3\delta}{2} + O(\delta^2).$$

The rear of the airfoil is at $(-2a, 0)$ (the image of $\theta = \pi$), and the front is at $(2a, 0)$ (the image of $\theta = 0$). Near the critical point at the rear we have $\theta = \pi + \epsilon$, where $|\epsilon| \ll 1$, and a local analysis yields

$$x = -2a + \epsilon^2(a + 2\delta) + O(\delta^2, \epsilon^4), \qquad y = -\delta\epsilon^3 + O(\delta^2, \epsilon^5),$$
$$\Rightarrow \qquad x \approx -2a + a\left(\frac{|y|}{\delta}\right)^{2/3},$$

giving a sharply-pointed rear, as anticipated. The result of the mapping is an "airfoil" type shape shown in figure 11. The entire domain exterior to the circle is mapped to the domain exterior to the pointed airfoil. This transformation enables inviscid flow round the airfoil shape to be calculated exactly.

Finally, we note without proof that if we slightly offset the circle centre from the real as well as the imaginary axis in the ζ-plane, then the airfoil shape loses its symmetry about the x-axis and becomes "cambered", an even more realistic shape for applications (figure 12).

3.7 Mobius transformations (bilinear transformations)

One very important class of conformal mapping functions consists of the Mobius transformations (or bilinear transformations), which have the general form

$$z = f(\zeta) = \frac{a\zeta + b}{c\zeta + d}, \qquad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

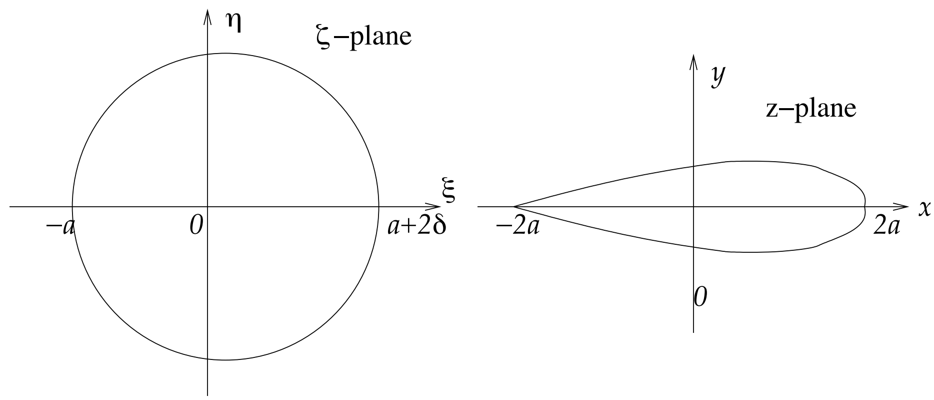

Figure 11: The Joukowski transformation (92) takes the circle $|\zeta - \delta| = a + \delta$ to the "airfoil" shape indicated.

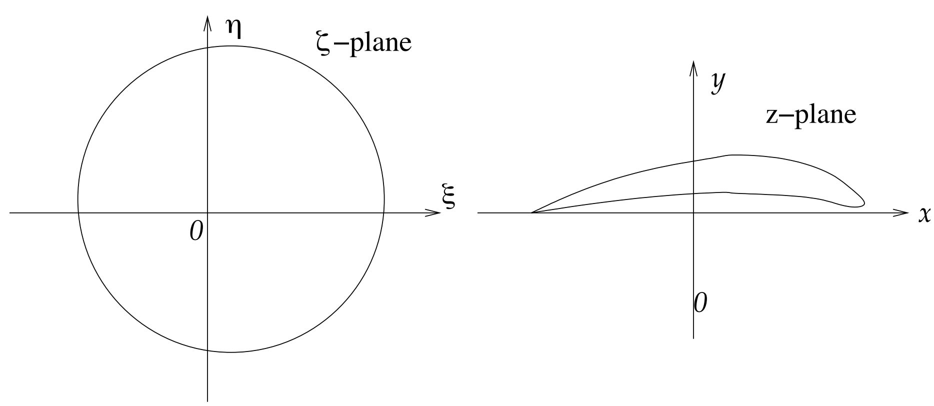

Figure 12: The Joukowski transformation (92) takes the circle with centre offset from both real and imaginary axes to the "cambered airfoil" shape indicated.

We have already come across such mappings in examples 3.7 and 3.9 (where we mapped the interior of a circle to the exterior of a circle, and the interior of a circle to a half-plane), and in theorem 3.19. We shall see below that, if we consider a straight line as being the limiting form of a circle, then Mobius transformations always map circles to circles.

Section 5.7 in Ablowitz & Fokas gives a very detailed overview of Mobius transformations; here we summarize the important points for our purposes.

1. Mobius transformations are **conformal** away from $\zeta = -d/c$ (the sole point where the map is nonanalytic), since

$$f'(\zeta) = \frac{ad - bc}{(c\zeta + d)^2}$$

   and $ad - bc \neq 0$.

2. Mobius transformations may be decomposed into the three simpler transformations

$$\begin{aligned} w = f_1(\zeta) = c\zeta + d, \quad u = f_2(w) = \frac{1}{w}, \\ z = f_3(u) = \frac{a}{c} + \frac{(bc - ad)}{c} u, \end{aligned} \tag{94}$$

   so that

$$z = f(\zeta) = f_3(f_2(f_1(\zeta))) = (f_3 \circ f_2 \circ f_1)(\zeta).$$

   f_1 and f_3 represent a combination of scaling/rotation and translation (as we saw earlier), while f_2 represents inversion in the unit circle (this transformation takes the interior of the unit circle to its exterior).

3. Mobius transformations are one-to-one, and therefore invertible, if we include the point infinity in each of the complex ζ- and z planes. With

$$z = f(\zeta) = \frac{a\zeta + b}{c\zeta + d}$$

   we have

$$\zeta = f^{-1}(z) = \frac{-dz + b}{cz - a}.$$

   Thus the inverse transformation is another Mobius transformation.

4. Mobius transformations have at most two fixed points. Suppose $\zeta_0 = f(\zeta_0)$, then

$$\zeta_0(c\zeta_0 + d) = a\zeta_0 + b \qquad \Rightarrow \qquad c\zeta_0^2 + (d-a)\zeta_0 - b = 0.$$

   Since this is a quadratic equation it has at most two roots, or just one in the exceptional case $(d-a)^2 + 4bc = 0$.

5. **Theorem 3.21** *A bilinear transformation is uniquely determined by specification of three distinct preimage points in the ζ-plane, and the corresponding (distinct) image points in the z-plane.*

   The **proof** we will give uses the following

   Lemma 3.22 *The cross-ratio of four preimage points ζ_j, and four corresponding image points z_j ($j = 1, 2, 3, 4$) is an invariant, that is,*

$$\frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)} = \frac{(\zeta_1 - \zeta_4)(\zeta_3 - \zeta_2)}{(\zeta_1 - \zeta_2)(\zeta_3 - \zeta_4)}. \tag{95}$$

   Proof (Lemma) Define the cross-ratio function $X(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ to be the right-hand side of (95). Since we know the bilinear transformation can be decomposed into linear transformations and inversions, as in (94), it is enough to show that these sub-transformations leave X unchanged.

   Considering the sub-transformation $w = f_1(\zeta)$ in (94), we have

$$\frac{(w_1 - w_4)(w_3 - w_2)}{(w_1 - w_2)(w_3 - w_4)} = \frac{c(\zeta_1 - \zeta_4)c(\zeta_3 - \zeta_2)}{c(\zeta_1 - \zeta_2)c(\zeta_3 - \zeta_4)} = \frac{(\zeta_1 - \zeta_4)(\zeta_3 - \zeta_2)}{(\zeta_1 - \zeta_2)(\zeta_3 - \zeta_4)}.$$

   Under the second sub-transformation (inversion) f_2 we have

$$\begin{aligned} \frac{(u_1 - u_4)(u_3 - u_2)}{(u_1 - u_2)(u_3 - u_4)} &= \frac{(1/w_1 - 1/w_4)(1/w_3 - 1/w_2)}{(1/w_1 - 1/w_2)(1/w_3 - 1/w_4)} \times \frac{w_1 w_2 w_3 w_4}{w_1 w_2 w_3 w_4} \\ &= \frac{(w_4 - w_1)(w_2 - w_3)}{(w_2 - w_1)(w_4 - w_3)}, \end{aligned}$$

   which is exactly the cross-ratio of the four points w_j $j = 1, 2, 3, 4$.

   The third sub-transformation f_3 in (94) is again linear, so the result holds as for f_1. Hence the lemma is proved.

Proof (Theorem 3.21) Replace ζ_4 by ζ and z_4 by z, then (95) defines a bilinear transformation $z = f(\zeta)$ that takes distinct points ζ_1, ζ_2, ζ_3 to z_1, z_2, z_3. Suppose there exists another such bilinear transformation $F \neq f$ taking ζ_1, ζ_2, ζ_3 to z_1, z_2, z_3, then since F and f are different transformations $\exists\, \zeta_4 \in \mathbb{C}$ such that $Z_4 = F(\zeta_4) \neq z_4 = f(\zeta_4)$. But by invariance of the cross-ratio of four points (95) for each of the transformations F and f, we must have

$$\begin{aligned} & \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)} = \frac{(z_1 - Z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - Z_4)} \\ \Rightarrow \quad & Z_4(z_1 - z_3) = z_4(z_1 - z_3) \quad \Rightarrow \quad Z_4 = z_4, \end{aligned}$$

since the points are all distinct. This contradicts the assumption, and the bilinear transformation is unique, as claimed.

Corollary 3.23 *The unique bilinear mapping sending* ζ_1, ζ_2, ζ_3 *to the points* 0, ∞, 1 *(respectively), is given by*

$$z = f(\zeta) = \frac{(\zeta - \zeta_1)(\zeta_3 - \zeta_2)}{(\zeta - \zeta_2)(\zeta_3 - \zeta_1)}. \tag{96}$$

Proof It is clear that (96) is a bilinear mapping that takes ζ_1, ζ_2, ζ_3 to the points 0, ∞, 1 (respectively). Uniqueness follows from theorem 3.21.

Remark To construct the mapping given in the proof above, note that setting $z_1 = 0$, $z_2 = \infty$, $z_3 = 1$, $z_4 = z$ and $\zeta_4 = \zeta$ in (95) gives

$$\frac{-z}{1 - z} = \frac{(\zeta_1 - \zeta)(\zeta_3 - \zeta_2)}{(\zeta_1 - \zeta_2)(\zeta_3 - \zeta)},$$

which rearranges trivially to (96).

Returning to example 3.9, we now recognise it as the unique Mobius transformation taking the points $\zeta_1 = -1$, $\zeta_2 = 1$ and $\zeta_3 = -i$ (all of which lie on the unit circle) to the points 0, ∞ and 1, respectively.

6. Mobius transformations map circles (or lines) into circles (or lines).
Proof: To show this, it is enough to show that each of the three subtransformations f_1, f_2, f_3 in (94) map circles or lines to circles or lines.

First, suppose that C is a circle, with center at $\alpha \in \mathbb{C}$ and radius $r > 0$. The equation in the ζ-plane is

$$|\zeta - \alpha| = r \qquad \Rightarrow \qquad \zeta = \alpha + re^{i\theta}, \qquad \theta \in [0, 2\pi),$$

and if $c = |c|e^{i\arg(c)}$ then a general point on the image curve under f_1 is

$$w = f_1(\zeta) = \alpha c + d + |c|re^{i(\theta+\arg(c))}$$

which lies on the circle with center $\alpha c + d$ and radius $r|c|$. If C is a straight line then it can be represented in the ζ-plane by equation[4]

$$\Re(\beta\zeta) = \gamma, \qquad \text{for } \beta \in \mathbb{C},\ \gamma \in \mathbb{R}.$$

Since $\zeta = (w - d)/c$ the equation in the w-plane is

$$\Re\left(\frac{\beta}{c}(w-d)\right) = \gamma \qquad \Rightarrow \qquad \Re\left(\frac{\beta w}{c}\right) = \gamma + \Re\left(\frac{\beta d}{c}\right),$$

so that if $\Re(\beta d/c) \neq -\gamma$ then

$$\Re\left(\frac{\beta w}{c(\gamma + \Re(\beta d/c))}\right) = 1,$$

which is the equation of a straight line; while if $\Re(\beta d/c) = -\gamma$ then $\Re(\beta w/c) = 0$, also the equation of a straight line.

If C is the same circle in the w-plane then

$$|w - \alpha|^2 = r^2 \qquad \Rightarrow \qquad w\bar{w} - \bar{\alpha}w - \alpha\bar{w} + |\alpha|^2 = r^2. \tag{97}$$

To find the image in the u-plane we note that $w = 1/u$ and thus provided the circle does not pass through the origin (so that $|\alpha| \neq r$) we have

$$\frac{1}{u\bar{u}} - \frac{\bar{\alpha}}{u} - \frac{\alpha}{\bar{u}} + |\alpha|^2 = r^2 \tag{98}$$

$$\Rightarrow \qquad u\bar{u} + \frac{\alpha u}{r^2 - |\alpha|^2} + \frac{\bar{\alpha}\bar{u}}{r^2 - |\alpha|^2} = \frac{1}{r^2 - |\alpha|^2}.$$

[4] If $\beta = \beta_1 + i\beta_2$ and $\zeta = \xi + i\eta$ then $\Re(\beta\zeta) = \gamma$ is equivalent to $\beta_1\xi - \beta_2\eta = \gamma$.

Comparing with the form of equation (97) we see that u lies on a circle with center A and radius R given by

$$A = \frac{\bar{\alpha}}{|\alpha|^2 - r^2}, \qquad R = \frac{r}{||\alpha|^2 - r^2|}$$

respectively. If the circle does pass through the origin, $|\alpha| = r$, and equation (98) for u is simpler:

$$2\Re(\alpha u) = 1 \qquad \Rightarrow \qquad 2(\alpha_1 u_1 - \alpha_2 u_2) = 1,$$

if $\alpha = \alpha_1 + i\alpha_2$ and $u = u_1 + iu_2$. This is clearly the equation of a straight line in the u-plane.

To check that straight lines map to straight lines or circles under f_2 let C be the straight line

$$\Re(\beta w) = \gamma$$

in the w-plane. Then, since $w = 1/u$, $\Re(\beta/u) = \gamma$ and hence

$$\frac{\beta_1 u_1 + \beta_2 u_2}{u_1^2 + u_2^2} = \gamma \quad \Rightarrow \quad \left(u_1 - \frac{\beta_1}{2\gamma}\right)^2 + \left(u_2 - \frac{\beta_2}{2\gamma}\right)^2 = \frac{\beta_1^2 + \beta_2^2}{4\gamma^2},$$

the equation of a circle in the u-plane (a straight line if $\gamma = 0$).

Finally, if C is the circle or straight line in the u-plane then, since f_3 is a simple linear map like f_1, we have already shown that the image of C in the z-plane is another circle or straight line.

Homework A circle or straight line in $\mathbb{C}$ divides the complex plane into two **complementary regions**, one on either side of the boundary (for a circle the two regions are the circle interior and exterior; for a line we have the two half-spaces on either side of the line). Show that a Mobius transformation takes the complementary regions of the preimage circle or line to the complementary regions of the image circle or line. [**Hint:** Consider the images of the complementary regions under the sub-transformations (94).]
Also: Ablowitz & Fokas, problems for section 5.7, question 1,

3.8 Composition of conformal maps

In order to map a given domain onto another desired domain, it is often convenient to use a series of conformal mappings to intermediate domains, and compose all the maps at the end to obtain the final transformation.

Example 3.24 *Find a conformal mapping from the strip* $0 < \Im(\zeta) < 1$ *to the circle* $|z - a| = b$.

There are many ways of approaching this problem, but we choose to map via an intermediate domain that we choose to be the upper half plane. We know that we can map the strip onto sectors (and the upper half plane is a special case of a sector) via an exponential mapping function, and the corollary 3.23 above then gives us a constructive method to map the half-space onto the interior of the given circle.

The mapping $u = e^{\alpha\zeta}$ takes the strip $0 < \Im(\zeta) < 1$ to the sector $0 < \arg u < \alpha$, so choosing $\alpha = \pi$ takes the strip to the upper half u-plane.

We then want to find a Mobius transformation between the upper half u-plane and the disc $|z - a| < b$. By (96), the mapping

$$u = \frac{(1-i)(z-(a-b))}{2(z-(a+ib))}$$

takes the points $z_1 = a-b$, $z_2 = a+ib$, $z_3 = a+b$ to $u = 0$, ∞, 1, respectively. Inverting this map gives

$$z = \frac{2u(a+ib)-(1-i)(a-b)}{2u-(1-i)},$$

so that the final mapping desired is given by the composition,

$$z = \frac{2e^{\pi\zeta}(a+ib)-(1-i)(a-b)}{2e^{\pi\zeta}-(1-i)}.$$

Remark As a final check it is always a good idea to ensure that an interior point in the preimage domain maps to one in the image domain. The point $z = a$ within the circle corresponds to $u = (1+i)/2$ in the upper half-plane, and this in turn corresponds to the point $\zeta = -\log|\sqrt{2}| + i\pi/4$, which does indeed lie in the appropriate strip. When mapping boundaries to boundaries, there is always the possibility that one is mapping to the **exterior** of the desired domain. With the constructive method presented for Mobius transformations, a useful rule to remember is that the **sense** of the domain will be preserved under the mapping. Thus, in the mapping $z(u)$ above, marching along the boundary from $u_1 = 0$ to $u_3 = 1$ to $u_2 = \infty$, we keep the domain (the upper half-plane) to our left. This is also the case in the z-plane, hence the image points, $z_1 = a - b$, $z_3 = a + b$, $z_2 = a + ib$ are good choices, because moving from z_1 to z_3 to z_2 takes us around the circle anticlockwise, keeping its interior to the left. Thus, the circle's interior will be the image domain of the upper half plane.

Homework: Ablowitz & Fokas, problems for section 5.7, questions 2,3,6.

4 Applications of conformal mapping

Many of the common applications of conformal mapping and other complex variable techniques in applied mathematics arise in fluid dynamics problems. We will consider several such problems here. A knowledge of the equations of fluid dynamics is not required to understand the complex analysis methods we will use, although for those interested, a good readable introduction is given in the book by Acheson [3], or (less readable!) in the classic text by Batchelor [4]. Ablowitz & Fokas also give some of the fluid-dynamical background in chapter 5.

4.1 2D potential flow of an inviscid fluid

The physical state of a fluid may be characterized by its **velocity field** $\boldsymbol{u}(\boldsymbol{x}, t)$. If the flow is two-dimensional, taking place in the (x, y)-plane, and **irrotational** (meaning that its **vorticity** $\boldsymbol{\omega}$ is everywhere zero, $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u} = \boldsymbol{0}$ throughout the flow domain), then its velocity may be written as the gradient of a **velocity potential**, $\boldsymbol{u} = \nabla\phi$. See Appendix B) for further details of the equations of fluid dynamics, or the book by Ockendon & Ockendon [7] for a full discussion. If the fluid is also **incompressible** then the velocity field $\boldsymbol{u} = (u, v)$ is divergence-free, which means that the velocity potential satisfies Laplace's equation within the fluid domain:

$$0 = \nabla \cdot \boldsymbol{u} = \nabla^2\phi. \tag{99}$$

Therefore, solutions to the Laplace equation, subject to appropriate boundary conditions, represent realizations of 2D incompressible inviscid flow.

Boundary conditions must be imposed at any boundaries within the flow domain, for example, if there is a rigid obstacle with boundary ∂D within the flow domain then the appropriate condition to impose is that fluid cannot flow through ∂D, that is

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \Rightarrow \quad \boldsymbol{n} \cdot \nabla\phi = \frac{\partial \phi}{\partial n} = 0, \tag{100}$$

where $\boldsymbol{n}$ is a vector normal to ∂D.

We shall restrict ourselves to studying steady inviscid flows, so that $\boldsymbol{u}$ and ϕ are independent of time t. Since $\phi(x, y)$ is harmonic, it has a harmonic conjugate $\phi(x, y)$ such that $w(z) = \phi + i\psi$ is analytic (a **complex potential** for the flow). Differentiating the complex potential we have

$$\frac{dw}{dz} = \phi_x + i\psi_x = \phi_x - i\phi_y = u - iv, \tag{101}$$

relating w to the "complex conjugate velocity". For a steady flow, fluid particles follow the velocity vector field (u, v). If we consider the curves along which $\psi(x, y) = \psi_0 =$ constant, then (basic geometry/vector calculus) the vector $\nabla\psi = (\psi_x, \psi_y)$ is perpendicular to these curves, and we note that

$$\boldsymbol{u} \cdot \nabla\psi = (\phi_x, \phi_y) \cdot (\psi_x, \psi_y) = \phi_x\psi_x + \phi_y\psi_y = -\phi_x\phi_y + \phi_y\phi_x = 0,$$

using the Cauchy-Riemann equations. Thus, fluid particles follow the level curves of ψ, $\psi(x, y) =$ constant; and $\psi(x, y)$ is known as the **streamfunction** of the flow.

We begin with some very basic examples of inviscid flows, before considering how conformal mapping may be used to generate more complicated flow patterns.

Example 4.1 *(Uniform linear flow) Find the complex potential and streamlines for a fluid moving with constant speed $u_0 > 0$ at an angle α with the positive x-axis.*

Here the velocity components will be $\boldsymbol{u} = (u, v) = u_0(\cos\alpha, \sin\alpha)$, so that $\phi = u_0(x\cos\alpha + y\sin\alpha)$, and (101) gives

$$\frac{dw}{dz} = u_0 e^{-i\alpha} \quad \Rightarrow \quad w(z) = u_0 z e^{-i\alpha}. \tag{102}$$

The streamfunction $\psi = \Im(w)$ is given by

$$\psi = u_0(y\cos\alpha - x\sin\alpha)$$

so that streamlines $\psi =$ constant satisfy

$$y = x\tan\alpha + c,$$

where c is a real constant.

Example 4.2 *(Point source or sink in two dimensions) Find the complex potential of a point (line) source or sink, in an unbounded fluid domain, of constant volume flux Q.*

It is convenient to work with polar coordinates (r, θ), with origin at the source/sink, in which the general representation of a velocity field is $\boldsymbol{u} = u\boldsymbol{e}_r + v\boldsymbol{e}_\theta$ (where $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are unit vectors in the polar coordinate directions). For this example the flow will be purely radial, so $v \equiv 0$, and $u = u(r)$. The constant volume flux Q satisfies

$$Q = \oint_C u(r)\,ds,$$

where C is any closed curve surrounding the source/sink. In particular this holds for C a circle of arbitrary radius r, and thus

$$Q = \int_0^{2\pi} u(r)r\,d\theta.$$

This must hold for constant Q independently of r, which implies that $u(r)r$ is a constant, k say. Then $Q = 2\pi k$, giving

$$u(r) = \frac{Q}{2\pi r}.$$

The velocity potential ϕ, which satisfies

$$\boldsymbol{u} = \nabla\phi = \frac{\partial\phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\boldsymbol{e}_\theta,$$

will also be a function only of r,

$$\phi(r) = \frac{Q}{2\pi}\log r,$$

corresponding to a complex potential

$$w(z) = \frac{Q}{2\pi} \log z \tag{103}$$

and a streamfunction $\psi = \Im(w) = Q\theta/(2\pi)$. Streamlines are thus lines on which $\theta =$ constant, that is, fluid particles follow straight lines emanating from the source or sink, as we would expect. (The case $Q > 0$ describes a point source, with fluid moving outwards from the origin, while for $Q < 0$, $u(r) < 0$, giving flow into the origin which acts as a point sink.)

Example 4.3 *(Rotating flow with constant circulation) Find the complex potential for a flow in uniform rotation about the origin, with constant (nonzero)* **circulation** Γ_0 *around any closed curve* C *enclosing the origin, defined by*

$$\Gamma = \oint_C \boldsymbol{u} \cdot \boldsymbol{ds}.$$

A flow in pure uniform rotation about the origin is of the form

$$\boldsymbol{u} = v(r)\boldsymbol{e}_\theta,$$

and if the circulation is constant about any closed curve then in particular it is constant around any circle about the origin, on which $\boldsymbol{ds} = \boldsymbol{e}_\theta ds = \boldsymbol{e}_\theta r d\theta$, so that

$$\Gamma_0 = \int_0^{2\pi} v(r)\boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta r\, d\theta = \int_0^{2\pi} v(r) r\, d\theta.$$

As above, this must hold independently of the circle radius r, so that now the azimuthal velocity $v(r)$ satisfies $rv(r) =$ constant $= c$, say, where $\Gamma_0 = 2\pi c$. The velocity field is then

$$\boldsymbol{u} = \frac{\Gamma_0}{2\pi r}\boldsymbol{e}_\theta = \frac{\partial \phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\boldsymbol{e}_\theta,$$

so that the velocity potential ϕ satisfies

$$\frac{\partial \phi}{\partial r} = 0, \quad \frac{\partial \phi}{\partial \theta} = \frac{\Gamma_0}{2\pi} \quad \Rightarrow \phi = \frac{\Gamma_0 \theta}{2\pi}.$$

The complex potential is therefore

$$w(z) = \frac{i\Gamma_0}{2\pi} \log z, \tag{104}$$

and the streamfunction is $\psi = \Im(w) = (\Gamma_0/2\pi) \log r$. Streamlines are thus curves $r =$ constant, that is, circles about the origin, as we would expect. This flow-field is known as a **point vortex** at the origin, of circulation Γ_0.

4.1.1 Obstacles in the flow

Often one wishes to consider flow around rigid obstacles in the flow. For example, in the study of flight, it is useful to know how inviscid fluid (air in this case) flows around a wing-shaped cross section such as that represented by the Joukowski transformation of (92). From the analytical representation of the velocity field, one can calculate the forces exerted by the air on the body, and hence investigate how the lift on the wing changes as flow parameters are varied.

A very useful observation when considering flow around obstacles is that, if one has a complex potential $w(z) = \phi + i\psi$ containing closed streamlines, any one of the closed streamlines can represent a rigid obstacle in the flow. This is because on such a streamline γ we have ψ = constant, so that $\nabla\psi$ (evaluated on γ) is a vector $\boldsymbol{n}$ normal to γ. Therefore, on γ,

$$\boldsymbol{u} \cdot \boldsymbol{n} = \nabla\phi \cdot \nabla\psi = \phi_x\psi_x + \phi_y\psi_y = \psi_y\psi_x - \psi_x\psi_y = 0,$$

using the Cauchy-Riemann equations. Thus, on a streamline γ we have $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, which is exactly the boundary condition we require at a rigid boundary in the flow with normal vector $\boldsymbol{n}$ (see (100)). A closed streamline in inviscid flow is equivalent to the boundary of a rigid obstacle within the flow, or conversely, the boundary of any rigid obstacle in an inviscid flow is always a streamline for the flow.

Example 4.4 *(Flow around a circular cylinder) Show that the complex potential*

$$w(z) = u_0\left(z + \frac{a^2}{z}\right) \tag{105}$$

for u_0, $a \in \mathbb{R}^+$, represents inviscid flow, uniform at infinity, around a circular cylinder of radius a centered at the origin.

The complex potential is analytic in $|z| \geq a$, so will represent some flow in that region. The flow is uniform at infinity, because

$$w(z) \sim u_0 z \quad \text{for } |z| \gg 1 \quad \Rightarrow \quad (u, v) \sim (u_0, 0).$$

We now need to check that the cylinder boundary $|z| = a$ is a streamline of the flow. On the boundary, $z = ae^{i\theta}$, and

$$w(z) = a(e^{i\theta} + e^{-i\theta}) = 2a\cos\theta \quad \Rightarrow \quad \psi = 0,$$

as required. Thus the complex potential does indeed represent flow exterior to the cylinder $|z| = a$, uniform at infinity, with real potential and stream-function given in $r \geq a$ by

$$\phi(r, \theta) = u_0 \left(r + \frac{a^2}{r} \right) \cos\theta, \quad \psi(r, \theta) = u_0 \left(r - \frac{a^2}{r} \right) \sin\theta.$$

From this representation it is clear that the cylinder boundary $r = a$ and the horizontal axis $\theta = 0, \pi$ are streamlines; and that streamlines are symmetric about the horizontal axis (if $\psi(r, \theta) = \psi_0$ is a streamline in $\theta > 0$ then $\psi(r, -\theta) = -\psi_0$ is another streamline).

To obtain the velocity field for this flow, either take the gradient of ϕ, or differentiate $w(z)$:

$$\frac{dw}{dz} = u - iv = u_0 \left(1 - \frac{a^2}{z^2} \right).$$

There are two **stagnation points** of the flow – points where the velocity is zero – at $z = \pm a$; the "front" and "rear" of the cylinder, with respect to the flow. The streamlines along $\theta = 0, \pi$ are dividing streamlines for the flow. All fluid above $\theta = \pi$ in the oncoming flow passes above the cylinder, while all fluid below it passes below the cylinder.

Remark This complex potential is exactly the same as that due to a superposition of a uniform flow and a point dipole at the origin. When the strength of the dipole is chosen correctly, the resulting flow will always have a circular streamline – which represents the cylinder boundary in our example here.

Remark Note that, if $w_1(z)$ and $w_2(z)$ are two complex potentials that each have a common closed streamline γ, then γ is a streamline for any flow w represented by a linear combination of w_1 and w_2. Flows around the same rigid obstacle can be superposed.

Example 4.5 *(Flow around a circular cylinder with circulation) Example 4.4 above gave a complex potential having $|z| = a$ as a streamline, and we also saw earlier that the point vortex complex potential (104) has circular streamlines. Thus, a vortex of arbitrary strength Γ_0 can be added onto (105) above and we obtain a new complex potential for* **flow around a cylinder with circulation,**

$$w(z) = u_0 \left(z + \frac{a^2}{z} \right) + \frac{i\Gamma_0}{2\pi} \log z.$$

The velocity potential and streamfunction for this flow are

$$\phi = u_0 \left(r + \frac{a^2}{r} \right) \cos\theta - \frac{\Gamma_0 \theta}{2\pi},$$
$$\psi = u_0 \left(r - \frac{a^2}{r} \right) \sin\theta + \frac{\Gamma_0}{2\pi} \log r.$$

Note that, if a given point (r, θ) lies on a streamline $\psi = \psi_0$, then the point $(r, \pi - \theta)$ also lies on this streamline. Thus, this flow has fore-aft symmetry about the cylinder. As above, we can study the velocity field of this flow most easily by differentiating the complex potential:

$$u - iv = \frac{dw}{dz} = u_0 \left(1 - \frac{a^2}{z^2} \right) + \frac{i\Gamma_0}{2\pi z}.$$

The stagnation points of this flow are therefore given by the solutions z^* of the quadratic

$$z^2 + \frac{i\Gamma_0}{2\pi u_0} - a^2 = 0,$$

which are

$$z^* = -\frac{i\Gamma_0}{4\pi u_0} \pm \left(a^2 - \left(\frac{\Gamma_0}{4\pi u_0} \right)^2 \right)^{1/2}.$$

For $0 \leq \Gamma_0 < 4\pi u_0 a$ we have a real square-root, and hence solutions for z^* have the same imaginary part, and equal but opposite real parts. For $\Gamma_0 = 0$ the stagnation points lie at the front and rear of the cylinder, $z^* = \pm a$, as we know they must; but as Γ_0 increases the stagnation points move. Note that in this regime we have

$$|z^*|^2 = a^2 - \left(\frac{\Gamma_0}{4\pi u_0} \right)^2 + \left(\frac{\Gamma_0}{4\pi u_0} \right)^2 = a^2,$$

so the stagnation points lie **on** the cylinder, below the x-axis.

When $\Gamma_0 = 4\pi u_0 a$ then $z^* = -ia$, a repeated root; the stagnation points coincide at the bottom of the cylinder. For $\Gamma_0 > 4\pi u_0 a$

$$z^* = -\frac{i\Gamma_0}{4\pi u_0} \pm i \left(\left(\frac{\Gamma_0}{4\pi u_0} \right)^2 - a^2 \right)^{1/2},$$

both roots pure imaginary, and

$$|z^*| = a\left\{\frac{\Gamma_0}{4\pi u_0 a} \pm \left(\left(\frac{\Gamma_0}{4\pi u_0 a}\right)^2 - 1\right)^{1/2}\right\}.$$

With $\Gamma_0 > 4\pi u_0 a$, $\{\cdot\} > 1$ with the $+$ sign, and $\{\cdot\} < 1$ with the $-$ sign, so now one stagnation point lies inside the cylinder, and is physically irrelevant. The other lies in the flow on the y-axis directly below the cylinder. As in the previous example, stagnation points are associated with separatrices of the flow.

Related to example 4.4 above is an interesting and useful theorem due to Milne-Thompson.

Theorem 4.6 *(Milne-Thompson's circle theorem) Suppose the complex potential $w_0(z)$ represents a flow in an unbounded region, with no singularities in $|z| \le a$. Then the function*

$$w(z) = w_0(z) + \overline{w}_0\left(\frac{a^2}{z}\right) \tag{106}$$

represents a flow around a circular cylinder of radius a, and having the same singularities as the original flow $w_0(z)$.

Remark The **complex conjugate function** $\overline{w}_0$ is defined in terms of w_0 by

$$\overline{w}_0(z) = \overline{w_0(\overline{z})}.$$

If w_0 is analytic on the domain D, then $\overline{w}_0$ defines a function analytic on the domain $\tilde{D} = \{z : \overline{z} \in D\}$.

"Singularities" in the statement of theorem 4.6 could refer, e.g., to sources or sinks in the flow domain (see example 4.2).

Proof of theorem 4.6. $w_0(z)$ is analytic in $|z| \le a$ and thus $\overline{w}_0(a^2/z)$ has no singularities in $|z| > a$, so adding this part to $w_0(z)$ adds no new singularities to the flow. Moreover, on $|z| = a$ we have $z\overline{z} = a^2$ and so

$$w(z) = w_0(z) + \overline{w}_0\left(\frac{a^2}{z}\right) = w_0(z) + \overline{w}_0(\overline{z}) = w_0(z) + \overline{w_0(z)} \in \mathbb{R},$$

on $|z| = a$. Thus, $|z| = a$ is a streamline of the new flow represented by w. Also, as $|z| \to \infty$,

$$w(z) \sim w_0(z) + \overline{w}_0(0),$$

so since $\overline{w}_0(0)$ is just a constant, the new flow has the same far-field behavior as the original flow.

Example 4.7 *(Flow around a cylinder, linear at infinity) We apply Milne-Thomson's theorem to the the simple linear flow represented by (102).*

Setting $w_0(z) = u_0 z e^{-i\alpha}$ in (106) leads to

$$w(z) = u_0 z e^{-i\alpha} + \frac{u_0 a^2 e^{i\alpha}}{z} = u_0 \left(z e^{-i\alpha} + \frac{a^2}{z} e^{i\alpha} \right). \tag{107}$$

This represents oncoming flow inclined at an angle α to the positive x-axis (when $\alpha = 0$ it is exactly example 4.4 given above). The straight line $y = x \tan \alpha$ through the center of the cylinder, aligned with the oncoming flow, is a streamline, since on this line $z = s e^{i\alpha}$ for $s \in \mathbb{R}$, and

$$w(z) = u_0 \left(s + \frac{a^2}{s} \right) \in \mathbb{R}.$$

Example 4.8 *Find the complex potential for uniform linear flow, at angle α to the x-axis, around a circular cylinder, with a point source located at $z = b > a$ on the real axis.*

The complex potential for the unbounded flow (no cylinder) is simply the superposition of the uniform linear flow with the point source flow:

$$w_0(z) = u_0 z e^{-i\alpha} + \frac{Q}{2\pi} \log(z - b).$$

To add the cylinder, we take

$$\begin{aligned} w(z) &= w_0(z) + \overline{w}_0\left(\frac{a^2}{z}\right) \\ &= u_0 z e^{-i\alpha} + \frac{Q}{2\pi} \log(z-b) + \overline{u_0 \frac{a^2}{\bar{z}} e^{-i\alpha}} + \overline{\frac{Q}{2\pi} \log\left(\frac{a^2}{\bar{z}} - b\right)} \\ &= u_0 \left(z e^{-i\alpha} + \frac{a^2}{z} e^{i\alpha} \right) + \frac{Q}{2\pi} \left[\log(z-b) + \log(a^2 - bz) - \log z \right]. \end{aligned}$$

Note that this last, simplified, expression again makes clear the decomposition of the flow into its constituent singularities, and we see that the equivalent flow could be generated by superposition of a uniform linear flow with a point dipole at the origin (of appropriate strength and direction), a point source at $z = b$, a point source at $z = a^2/b$, and a point sink at $z = 0$. The new "additional" singularities all lie within the circular cylinder $|z| \leq a$.

4.1.2 Method of images

Milne-Thompson's circle theorem is an example of an "image" method for finding flow in the presence of a rigid boundary. The point $\tilde{z} = a^2/\overline{z}$ that occurs in the definition of the second potential $\overline{w}_0(a^2/z)$ is the image or inverse point to z in the circle $|z| = a$, in the sense that for points z **on** the circle, the image point $\tilde{z}$ coincides with z. For points z inside the circle, $\tilde{z}$ lies outside the circle, and vice-versa.

The simplest application of images in inviscid flow problems with walls is when the wall is planar. For a wall lying along the imaginary axis, $\Re(z) = 0$, the image point of z in the wall is $-\overline{z}$; and for a wall along the real axis, $\Im(z) = 0$, the image point of z is $\overline{z}$. Consider the latter case first. Suppose we have a complex potential $w_0(z)$ representing flow in an unbounded region, that has no singularities in $\Im(z) < 0$. Then, the complex potential

$$w(z) = w_0(z) + \overline{w}_0(z) \equiv w_0(z) + \overline{w_0(\overline{z})} \tag{108}$$

has exactly the same singularities as w_0 in $\Im(z) > 0$, and it has $\Im(z) = 0$ as a streamline, since on $\Im(z) = 0$ $z = x \in \mathbb{R}$, and

$$w(z) = w_0(x) + \overline{w}_0(x) \equiv w_0(x) + \overline{w_0(x)} \in \mathbb{R}.$$

Thus, the potential given in (108) represents a flow in the upper half plane with an impermeable wall along $\Im(z) = 0$.

Similarly, if $w_0(z)$ represents flow in an unbounded region with no singularities in $\Re(z) < 0$, then the complex potential

$$w(z) = w_0(z) + \overline{w}_0(-z) \equiv w_0(z) + \overline{w_0(-\overline{z})} \tag{109}$$

represents a flow in the right half plane $\Re(z) > 0$, with an impermeable wall along $\Re(z) = 0$. It has has exactly the same singularities as w_0 in $\Re(z) > 0$, and on $z = iy$, $y \in \mathbb{R}$,

$$w(z) = w_0(iy) + \overline{w}_0(-iy) \equiv w_0(iy) + \overline{w_0(iy)} \in \mathbb{R}.$$

Each of (108) and (109) are analogous to Milne-Thompson's result (106).

Homework: Try generalizing these results for a flow on one side of a wall $y = mx$, $m \in \mathbb{R}$.

Example 4.9 *Find the complex potential for a point vortex of circulation* Γ *at* $z = c \in \mathbb{C}$ *(where* $\Re(c) > 0$*), adjacent to a wall along* $\Re(z) = 0$.

The complex potential $w_0(z) = i\Gamma/(2\pi)\log(z - c)$ represents a vortex of circulation Γ at $z = c$ in an unbounded flow domain (see example 4.3). By (109),

$$w(z) = w_0(z) + \overline{w}_0(-z) = \frac{i\Gamma}{2\pi}\log(z-c) - \frac{i\Gamma}{2\pi}\log(z+\overline{c}) + \text{constant}$$

represents a flow in the right-half plane, with a single vortex at $z = c$, and a wall along $\Re(z) = 0$. To verify this directly note that $z = c$ is the only singularity of w in the right-half plane (since $\Re(c) > 0$), and on the wall $z = iy$ we have

$$\begin{aligned} w(iy) &= \frac{i\Gamma}{2\pi}\log(iy-c) - \frac{i\Gamma}{2\pi}\log(iy+\overline{c}) + \text{constant} \\ &= \frac{i\Gamma}{2\pi}\log(iy-c) - \frac{i\Gamma}{2\pi}\log(\overline{-iy+c}) + \text{constant} \\ &= \frac{i\Gamma}{2\pi}\log(iy-c) - \frac{i\Gamma}{2\pi}\log(\overline{iy-c}) + \text{constant} \\ &= \frac{i\Gamma}{2\pi}\log(iy-c) + \overline{\frac{i\Gamma}{2\pi}\log(iy-c)} + \text{constant} \\ &= \text{real} + \text{constant}, \end{aligned}$$

so that $z = iy$ is a streamline for the flow.

Flow in a corner Suppose we need to solve for flow, with specified driving singularities, in a right-angled corner (the first quadrant, say). In this case, a given singularity in the flow will require 3 images in order for its effect to cancel appropriately on both walls. If we take just two images, one in each wall, then the resulting complex potential will not be symmetric about both walls. In this case the appropriate combination is found by taking

$$\begin{aligned} w(z) &= w_0(z) + \overline{w_0(\overline{z})} + \overline{w_0(-\overline{z})} + w_0(-z) \\ &= w_0(z) + \overline{w}_0(z) + \overline{w}_0(-z) + w_0(-z), \end{aligned}$$

where $w_0(z)$ is the complex potential for the unbounded flow with no walls. It is easily checked that this complex potential satisfies all requirements, since:

1. $w_0(z)$ is complex analytic, with singularities (specified) only on the first quadrant.
2. $\overline{w_0(\bar{z})}$ is complex analytic, with singularities only in the 4th quadrant (corresponding to those of w_0, reflected in the x-axis).
3. $\overline{w_0(-\bar{z})}$ is complex analytic, with singularities only in the 2nd quadrant (corresponding to those of w_0, reflected in the y-axis).
4. $w_0(-z)$ is complex analytic, with singularities only in the 3rd quadrant (corresponding to those of w_0, rotated through π).
5. On the wall $z = x \in \mathbb{R}^+$,

$$w(z) = w_0(x) + \overline{w_0(x)} + \overline{w_0(-x)} + w_0(-x) \in \mathbb{R}.$$

6. On the wall $z = iy$, $y \in \mathbb{R}^+$,

$$w(z) = w_0(iy) + \overline{w_0(-iy)} + \overline{w_0(iy)} + w_0(-iy) \in \mathbb{R}.$$

Homework: 1. Find the complex potential for an array of sources or sinks of arbitrary strengths Q_i at positions z_i in the right half plane $\Re(z) > 0$, with a wall along $\Re(z) = 0$. As above, verify directly that the imaginary axis is a streamline of the complex potential you find.
2. Write down the complex potential for a single source or sink at point z_0 in $\Im(z) > 0$, with an impermeable wall along $\Im(z) = 0$. By taking the limit $\Im(z_0) \to 0^+$ find the complex potential for a source/sink located **on** the wall. How do you explain this result?

4.1.3 Conformal mapping and flow around obstacles

Conformal mappings can be used to transform very simple complex potentials, such as those derived above, to more complicated ones describing flows around obstacles. Suppose we have a complex potential $W(\zeta) = \Phi(\xi, \eta) + i\Psi(\xi, \eta)$ describing a flow within a region D in the (ξ, η)-plane, where $\zeta = \xi + i\eta$. Consider a univalent conformal mapping $z = f(\zeta)$, taking D onto some domain D' in the z-plane. Since f is univalent on D it is invertible, with $\zeta = f^{-1}(z)$ taking D' onto D. Then, corresponding to the complex potential W is a complex potential $w(z)$ such that

$$w(f(\zeta)) = W(\zeta), \quad \text{or equivalently,} \quad w(z) = W(f^{-1}(z));$$

and since W is analytic on D, and f, f^{-1} are analytic, w will be analytic on D'. A streamline γ of the flow in the ζ-plane will be mapped to some curve γ' in the z-plane, on which

$$\Im(w(z))|_{z\in\gamma'} = \Im\left(W(\zeta)\right)|_{\zeta\in\gamma} = \Psi|_{\zeta\in\gamma} = \text{constant},$$

because γ is a streamline. Thus, γ' is also a streamline for the flow represented by w in the z-plane; that is, streamlines are mapped to streamlines under f.

Coupled with the observation above, that a closed streamline is equivalent to a rigid obstacle in the flow, this conformal mapping of the complex potential may be used to generate flows around obstacles.

Case study: Flow around a flat plate We consider the Joukowski transformation

$$z = f(\zeta) = \zeta + \frac{a^2}{\zeta},$$

introduced in (92), firstly in the special case in which the circle of radius a in the ζ-plane is mapped to the slit $-2a \le \Re(z) \le 2a$ along the horizontal axis in the z-plane. Then, if we take the complex potential of example 4.7 above,

$$W(\zeta) = u_0\left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta}e^{i\alpha}\right), \tag{110}$$

which we know has $|\zeta| = a$ as a streamline, this will map to a flow around a flat plate along $-2a \le \Re(z) \le 2a$ in the z-plane. The complex potential $w(z)$ in the z-plane is given by

$$w(z) = W(f^{-1}(z)),$$

where W is as in (110) and f^{-1} is the mapping inverse to (92), given by

$$\zeta = f^{-1}(z) = \frac{z}{2} + \frac{1}{2}(z^2 - 4a^2)^{1/2}. \tag{111}$$

This mapping f^{-1} has branch-points at $z = \pm 2a$, and we can define it as a single-valued analytic function on the z-plane cut along $-2a \le \Re(z) \le 2a$ if we take the definition of the square-root as

$$(z^2 - 4a^2)^{1/2} = |z^2 - 4a^2|^{1/2}\exp[i(\operatorname{Arg}(z-2a) + \operatorname{Arg}(z+2a))/2],$$
$$\operatorname{Arg}(z-2a),\ \operatorname{Arg}(z+2a) \in [0, 2\pi).$$

Substituting for $\zeta(z)$ from (111) in (110) gives a (single-valued) complex potential $w(z)$, describing flow at an angle α around a flat plate $-2a \leq \Re(z) \leq 2a$.

The flow streamlines in the z-plane can be extracted by plotting the level curves of

$$\begin{aligned}\psi &= \Im\left[u_0\left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta}e^{i\alpha}\right)\right] \\ &= u_0 \Im\left[\frac{e^{-i\alpha}}{2}[z + (z^2 - 4a^2)^{1/2}] + \frac{2a^2 e^{i\alpha}}{[z + (z^2 - 4a^2)^{1/2}]}\right], \qquad (112)\end{aligned}$$

with a, $\alpha \in \mathbb{R}$ and $z = x + iy$. Recalling the result (101), the complex conjugate velocity field is given in terms of ζ by

$$u - iv = \frac{dw}{dz} = \frac{W'(\zeta)}{f'(\zeta)} = \frac{u_0\left(e^{-i\alpha} - \frac{a^2}{\zeta^2}e^{i\alpha}\right)}{1 - \frac{a^2}{\zeta^2}},$$

so the velocity is infinite at the points corresponding to $\zeta = \pm a$, that is, the plate's endpoints $z = \pm 2a$; while at the points $\zeta = \pm a e^{i\alpha}$ the velocity is zero, so there are stagnation points of the flow at $z = \pm 2a\cos\alpha$ (using (92)). The stagnation points move to the plate ends only when the oncoming flow is parallel to the plate ($\alpha = 0$), and are at the plate's center only when the flow is perpendicular to the plate ($\alpha = \pi/2$).

The flow represented by (112) is rotationally symmetric (rotate through angle π to leave the streamlines unchanged). We can introduce asymmetry into the flow by adding **circulation** about the plate, since we know how to do this for the circular cylinder that we are mapping from. The appropriate complex potential in the ζ-plane is

$$W(\zeta) = u_0\left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta}e^{i\alpha}\right) + \frac{i\Gamma}{2\pi}\log\zeta, \qquad (113)$$

and again we retrieve the corresponding potential in the z-plane by substituting for ζ from (111). Now the complex velocity is given in terms of ζ by

$$u - iv = \frac{dw}{dz} = \frac{W'(\zeta)}{f'(\zeta)} = \frac{u_0\left(e^{-i\alpha} - \frac{a^2}{\zeta^2}e^{i\alpha}\right) + \frac{i\Gamma}{2\pi\zeta}}{1 - \frac{a^2}{\zeta^2}}, \qquad (114)$$

so that again in general the flow velocity is infinite at the plate's endpoints $\zeta = \pm a$ ($z = \pm 2a$). A point on the plate's surface may be parametrized by setting $\zeta = e^{i\theta}$, $\theta \in [0, 2\pi)$, and thus on the plate's surface the velocity is given by

$$\frac{dw}{dz} = \frac{u_0\left(e^{-i\alpha} - e^{i\alpha - 2\theta}\right) + \frac{i\Gamma}{2\pi a} e^{-i\theta}}{1 - e^{-2i\theta}}.$$

Away from the endpoints $\theta = 0$, π we may rearrange this expression, multiplying both numerator and denominator by $(1 - e^{2i\theta})$ to obtain

$$\frac{dw}{dz} = u - iv = \frac{u_0(\cos\alpha - \cos(2\theta - \alpha)) + \frac{\Gamma}{2\pi a}\sin\theta}{1 - \cos 2\theta}, \qquad \theta \neq 0,\ \pi,$$

confirming that the velocity at the plate's surface is always parallel to the plate as we would expect. At the endpoints, as already noted the denominator in dw/dz vanishes, so that in general the velocity is unbounded there. However, exceptionally, the numerator in dw/dz may also vanish at one endpoint for the right value of circulation Γ. From (114) we see that this happens for $\zeta = a$ when

$$\Gamma = 4\pi a u_0 \sin\alpha,$$

and when this is the case, the velocity at the right-hand end of the plate is in fact **finite**. This observation turns out to be very important in practical applications, for the following (non-rigorous!) reason. The infinite velocity at the front of the plate can be eliminated by rounding it slightly, which only slightly changes the flow found above. If we retain the sharp right-hand plate edge however, then the only way to make the flow velocity finite everywhere is for the circulation Γ to take the value found above; and it turns out that this is **always the case in practice** when such flows are generated (e.g. by taking a flat plate-like object and translating it through air). The required circulation is generated during the initial transient start-up period when the plate is first moved from rest.

The same methods can be used to find an exact solution for flow around an airfoil shape, with arbitrary circulation, using the observations at the end of §3.6 to find a conformal mapping to an airfoil.

This generation of exactly the right amount of circulation (vorticity, in fluid dynamics terminology) was predicted by early researchers, and is known as the **Kutta condition**.[5] The real importance for the theory of flight is that the **lift** generated on a 2D object in inviscid flow is directly proportional to the circulation around it. This is easily shown from the above exact solutions for a circular cylinder and for a flat plate, using a result due to Blasius:

Theorem 4.10 *(Blasius' theorem) The force $\boldsymbol{F} = (F_1, F_2)$ exerted on a 2D body B in inviscid flow is given by the contour integral*

$$F_1 - iF_2 = \frac{i\rho}{2} \oint_{\partial B} \left(\frac{dw}{dz} \right)^2 dz, \tag{115}$$

where ρ is the density of the fluid, and ∂B is the closed curve representing the boundary of the body.

To set the stage for proving this theorem, we return to the Euler equations that were introduced in §B, and provide some further motivation. Consider inviscid incompressible fluid of constant density ρ, in irrotational flow with velocity $\boldsymbol{u}$ We **define** the pressure p by the condition that the force exerted across a geometrical surface element $\boldsymbol{n}dS$ (with normal $\boldsymbol{n}$) within the fluid is $p\boldsymbol{n}dS$ – specifically, this is the force exerted on the fluid into which $\boldsymbol{n}$ points, by the fluid on the other side of the element. (This definition involves an assumption about the form of internal forces in the fluid, but this assumption is valid for an inviscid fluid.)

Consider a fixed closed volume V in the fluid, with surface S. The net volume rate at which fluid leaves V is

$$\int_S \boldsymbol{u} \cdot \boldsymbol{n} \, dS.$$

This quantity must be zero for an incompressible flow. Applying the Divergence theorem we then have

$$\int_V \nabla \cdot \boldsymbol{u} \, dV = 0.$$

Since this holds for all regions V we deduce that $\nabla \cdot \boldsymbol{u} = 0$ throughout the fluid, which is the incompressibility equation stated earlier.

[5] After German mathematician Martin Wilhelm Kutta, 1867-1944.

For the momentum equation of this arbitrary volume V, we note that the net force acting on V due to the rest of the fluid (no other forces are supposed to act) is

$$-\int_S p\boldsymbol{n}\,dS = -\int_V \nabla p\,dV,$$

again by application of the Divergence theorem. Applying Newton's second law to an arbitrary small volume δV then gives

$$\text{Force} \approx -\nabla p\delta V \approx \text{mass} \times \text{acceleration} \approx \rho\delta V\frac{d\boldsymbol{u}}{dt}.$$

We conclude that

$$\rho\frac{d\boldsymbol{u}}{dt} = -\nabla p,$$

where by the chain rule,

$$\frac{d\boldsymbol{u}}{dt} = \frac{\partial\boldsymbol{u}}{\partial t} + \frac{\partial\boldsymbol{u}}{\partial x}\frac{dx}{dt} + \frac{\partial\boldsymbol{u}}{\partial y}\frac{dy}{dt} \equiv \frac{\partial\boldsymbol{u}}{\partial t} + u\frac{\partial\boldsymbol{u}}{\partial x} + v\frac{\partial\boldsymbol{u}}{\partial y}.$$

Hence the Euler equations given earlier.

With the above definition of pressure in terms of internal fluid dynamical forces established as being consistent with the Euler equations, we now use it as a basis to prove Blasius' theorem.

Proof (of Blasius' theorem) We work with the dimensional form of the steady Euler equations,

$$uu_x + vu_y = -p_x, \tag{116}$$

$$uv_x + vv_y = -p_y, \tag{117}$$

$$u_x + v_y = 0, \tag{118}$$

noting that the Bernoulli condition associated with (116) and (117) is

$$p + \frac{\rho}{2}|\boldsymbol{u}|^2 = \text{constant}. \tag{119}$$

We use the tangent angle θ to parametrize the boundary ∂B of B, so that the anticlockwise tangent $\boldsymbol{t}$ and outward normal $\boldsymbol{n}$ to ∂B are given by

$$\boldsymbol{t} = (\cos\theta, \sin\theta), \qquad \boldsymbol{n} = (\sin\theta, -\cos\theta).$$

The force exerted on an arbitrary element $\boldsymbol{n}\delta s$ of ∂B is

$$\delta \boldsymbol{F} = -p\boldsymbol{n}\delta s,$$

so that if $\delta \boldsymbol{F} = (\delta F_1, \delta F_2)$ then

$$\delta F_1 - i\delta F_2 = -p(\sin\theta + i\cos\theta)\delta s = -pie^{-i\theta}\delta s.$$

Next note that ∂B is a streamline for the flow, so $\boldsymbol{u} = |\boldsymbol{u}|(\cos\theta, \sin\theta)$ on ∂B, and thus

$$\frac{dw}{dz} = u - iv = |\boldsymbol{u}|(\cos\theta - i\sin\theta) = |\boldsymbol{u}|e^{-i\theta} \quad \text{on } \partial B.$$

We now recall Bernoulli's result, which enables us to write the pressure appearing in the force in terms of the fluid speed, and hence in terms of the complex potential:

$$\begin{aligned}
\delta F_1 - i\delta F_2 &= -ipe^{-i\theta}\delta s = i\left(\frac{\rho}{2}|\boldsymbol{u}|^2 - k\right)e^{-i\theta}\delta s \\
&= i\frac{\rho}{2}\left(\frac{dw}{dz}\right)^2 e^{i\theta} - ike^{-i\theta}\delta s.
\end{aligned}$$

Since $z_s = x_s + iy_s = \cos\theta + i\sin\theta = e^{i\theta}$,

$$\delta F_1 - i\delta F_2 = \frac{i\rho}{2}\left(\frac{dw}{dz}\right)^2 \frac{\partial z}{\partial s}\delta s - ik\frac{\partial \bar{z}}{\partial s}\delta s,$$

so that, integrating,

$$F_1 - iF_2 = \frac{i\rho}{2}\oint_{\partial B}\left(\frac{dw}{dz}\right)^2 dz - ik\oint d\bar{z}.$$

Since the last integral here is just the change in $\bar{z}$ around the boundary ∂B, a closed curve, it is zero. The result follows.

We refer interested readers to the books by Acheson [3] or Batchelor [4] for more details of forces on bodies in inviscid flow.

Example 4.11 *Use Blasius' theorem to calculate the force acting on a circular cylinder in a uniform oncoming flow at angle α, with imposed circulation Γ.*

For this example

$$\frac{dw}{dz} = u_0\left(e^{-i\alpha} - \frac{e^{i\alpha}}{z^2}\right) + \frac{i\Gamma}{2\pi\zeta}$$

is singular only at $z = 0$. Hence

$$\begin{aligned}
F_1 - iF_2 &= \frac{i\rho}{2}\oint_{|\zeta|=1}\left(\frac{w}{z}\right)^2 \\
&= -\pi\rho \times \left[\text{Residue of } \left(\tfrac{dw}{dz}\right)^2 \text{ at } \zeta = 0.\right] \\
&= -\pi\rho \times \frac{i\Gamma u_0 e^{-i\alpha}}{\pi},
\end{aligned}$$

and

$$F_1 = -\rho u_0 \Gamma \sin\alpha, \qquad F_2 = \rho u_0 \Gamma \cos\alpha.$$

The force perpendicular to the free stream direction is

$$F_\perp = -F_1 \sin\alpha + F_2 \cos\alpha = \rho u_0 \Gamma,$$

while the force parallel to this direction is

$$F_\| = F_1 \cos\alpha + F_2 \sin\alpha = 0$$

Remark Note that Blasius' result can easily be used in conjunction with conformal mapping solutions to the inviscid flow problem, noting that if $z = f(\zeta)$ is a mapping (e.g. from the unit disk), and $w(z) = w(f(\zeta)) = W(\zeta)$, then

$$\begin{aligned}
F_1 - iF_2 &= \frac{i\rho}{2}\oint_{\partial B}\left(\frac{dw}{dz}\right)^2 dz = \oint_{|\zeta|=1}\left(\frac{dW}{d\zeta}\right)^2 \frac{1}{f'(\zeta)^2} f'(\zeta) d\zeta \\
&= \oint_{|\zeta|=1}\left(\frac{dW}{d\zeta}\right)^2 \frac{1}{f'(\zeta)} d\zeta.
\end{aligned}$$

This result for the circular cylinder is a special case of a much more general result known as the Kutta-Joukowski lift theorem:

Theorem 4.12 *The force on a 2D body B in a uniform flow (speed U at infinity, aligned with the x-axis), with superimposed circulation Γ, is*

$$\boldsymbol{F} = -\rho U \Gamma \boldsymbol{j}.$$

Proof The complex potential $w(z)$ is analytic outside B. Since the flow is uniform at infinity, the large-z Taylor expansion is such that

$$\frac{dw}{dz} \sim U + \frac{k_1}{z} + \frac{k_2}{z^2} + \cdots.$$

Since w and dw/dz are complex analytic outside B we can deform the integration contour in Blasius' result to a large circle, $z = R$, so that

$$F_1 - iF_2 = \frac{i\rho}{2} \oint_{|z|=R} \left(\frac{dw}{dz}\right)^2 dz = -2\pi\rho k_1 U,$$

by direct calculation. On the other hand, direct calculation also gives

$$2\pi i k_1 = \oint_{|z|=R} \frac{dw}{dz}\, dz = \oint_{\partial B} \frac{dw}{dz}\, dz = [w]_{\partial B} = [\phi]_{\partial B},$$

because ∂B is a streamline and so $[\psi]_{\partial B} = 0$. But, we can evaluate $[\phi]_{\partial B}$, because:

$$\Gamma = \oint_{\partial B} \boldsymbol{u} \cdot d\boldsymbol{s} = \oint_{\partial B} (\phi_x x_s + \phi_y y_s) ds = \oint_{\partial B} \frac{d\phi}{ds} ds = [\phi]_{\partial B}.$$

So, $k_1 = \Gamma/(2\pi i)$, and thus $F_1 - iF_2 = \rho U \Gamma i$. The result follows.

Homework: Use Blasius' result (115) to show that the force exerted on a circular cylinder in oncoming flow, speed u_0, with circulation Γ (note that for a cylinder you can take the flow angle $\alpha = 0$ without loss of generality) is $\rho u_0 \Gamma$. In which direction does this force act?
Also: Ablowitz & Fokas, problems for section 5.4, question 9.

4.2 Hele-Shaw flow

4.2.1 Preamble

A Hele-Shaw cell is an experimental fluid-dynamical set-up in which a viscous fluid is confined in the thin gap between two horizontal, parallel plates of glass (or perspex). We assume a cartesian coordinate system (x, y, Z), such that the glass plates lie in planes $Z = 0$, $Z = b$, while (x, y) are coordinates in the plane of the cell. The experimental set-up was introduced in 1898 by H.S. Hele-Shaw in order to visualize flow streamlines for flow around various obstacles within the cell; see [6]. If the gap between the plates is only partially filled with viscous fluid and the remainder is filled with air, then there will be a **free boundary** between the fluid and the air. Starting from the **Navier-Stokes** equations of fluid dynamics it can be shown (see for example [7]) that the flow is approximately two-dimensional; and if $(u(x,y,Z), v(x,y,Z))$ are the (x,y) components (respectively) of the fluid velocity then (in appropriately rescaled coordinates) the Z-averaged velocity

$$(\bar{u},\bar{v}) = \int_0^1 (u,v)dZ$$

is the gradient of a potential function p, the **fluid pressure**,

$$(\bar{u},\bar{v}) = -\nabla p, \tag{120}$$

and moreover, $(\bar{u},\bar{v})$ is divergence-free, so that p satisfies **Laplace's equation** within the 2D domain $D(t)$ occupied by fluid,

$$\nabla^2 p = 0 \quad \text{in } D(t). \tag{121}$$

Here t denotes time; since the boundary of the domain moves in time we make explicit the time-dependence of the domain shape. This PDE must be solved subject to appropriate boundary conditions.

Ordinarily we would just need one boundary condition to solve Laplace's equation on a given 2D domain, *e.g.* the values of p specified on the boundary $\partial D(t)$. However, the problem here is further complicated by the fact that the boundary $\partial D(t)$ of $D(t)$ is unknown *a priori*, and moreover can move in time if we **drive** the fluid, *e.g.* by injecting or sucking fluid from a given point in the domain. Thus here we need two boundary conditions to complete the problem specification. The first is a **dynamic boundary condition**, which in its simplest form states that the pressure p is continuous across $\partial D(t)$. Assuming constant pressure in the air domain, this condition takes the form

$$p = 0 \quad \text{on } \partial D(t), \tag{122}$$

without loss of generality. The second condition is a **kinematic boundary condition**, encoding the fact that, since the interface curve $\partial D(t)$ is made up of fluid particles, its velocity in the direction normal to $\partial D(t)$ must equal the component of the fluid velocity normal to $\partial D(t)$, V_n:

$$V_n = \boldsymbol{n}\cdot(\bar{u},\bar{v}) = -\boldsymbol{n}\cdot\nabla p \equiv -\frac{\partial p}{\partial n} \quad \text{on } \partial D(t). \tag{123}$$

Finally, for a moving boundary problem we must specify an initial condition on the domain, $\partial D(0) = \partial D_0$; and any driving mechanism for the flow must also be specified. This driving mechanism will typically appear as a singularity in the pressure field p, either at a finite point or at infinity (if the domain $D(t)$ extends to infinity). We shall first consider flows driven by a single point source or sink within the fluid domain $D(t)$. For a point source/sink at the origin $(x, y) = (0, 0)$ the appropriate singularity is

$$p \sim -\frac{Q}{2\pi} \log r \quad \text{as } r = (x^2 + y^2)^{1/2} \to 0, \tag{124}$$

for cylindrical polar coordinates (r, θ). One can easily verify that, for the velocity field given by (120), the above singularity gives

$$(\bar{u}, \bar{v}) \sim \frac{Q}{2\pi r} \boldsymbol{e}_r \quad \text{as } r \to 0,$$

where $\boldsymbol{e}_r$ is the unit vector in the radial direction. Thus the local flow field is a source when $Q > 0$ and a sink when $Q < 0$. The net fluid flux through any small circle C (centered on the origin and of radius $0 < \epsilon \ll 1$) around the singularity is given by

$$\text{Flux} = \oint_C (\bar{u}, \bar{v}) \cdot \boldsymbol{n} ds = \int_0^{2\pi} \frac{Q}{2\pi r} \boldsymbol{e}_r \cdot \boldsymbol{e}_r r \, d\theta = Q,$$

so Q is a measure of the strength of the source/sink.

Equation (121), subject to boundary conditions (122) and (123), the driving singularity (124), and the initial condition $\partial D(0) = \partial D_0$, completes the problem specification.

4.2.2 Hele-Shaw: The conformal mapping approach

The free boundary problem to be solved is

$$\begin{aligned}
\nabla^2 p &= 0 \quad \text{in } D(t), & (125)\\
p &= 0 \quad \text{on } \partial D(t), & (126)\\
\boldsymbol{n} \cdot \nabla p &= -V_n \quad \text{on } \partial D(t), & (127)\\
p &\sim -\frac{Q}{2\pi} \log r \quad \text{as } r = (x^2 + y^2)^{1/2} \to 0, & (128)\\
\partial D(0) &= \partial D_0. & (129)
\end{aligned}$$

The key observation is Lemma 1.8, which tells us that

$$p = \Re(w(z,t)), \tag{130}$$

where $w(z,t)$ is some analytic function of the complex variable $z = x+iy$. In fact $w(z,t)$, the **complex potential** of the flow, is analytic on $D(t)$, except at the singular point $z = 0$, where it has the local behavior

$$w(z,t) \sim -\frac{Q}{2\pi}\log z, \quad \text{as } z \to 0. \tag{131}$$

We cannot solve the Laplace equation directly on $D(t)$, because we do not know the position of the boundary on which we must apply the boundary conditions. Therefore, we invoke theorem 3.13 (the Riemann mapping theorem), which tells us that there exists a conformal mapping, $z = f(\zeta,\tau)$, from the unit disc in ζ-space onto $D(t)$ (considered as a domain in the complex z-plane).[6] We have freedom to insist that $f(0,\tau) = 0$ (and also that $f'(0,\tau) > 0$); see theorem 3.20, so that locally the mapping has the form

$$z = \zeta f_\zeta(0,\tau) + \cdots \quad \text{as } \zeta \to 0. \tag{132}$$

Corresponding to $w(z,t)$ is a complex potential in the ζ-plane,

$$W(\zeta) = w(f(\zeta,\tau),t) \quad \text{on } |\zeta| \leq 1, \tag{133}$$

analytic on $|\zeta| \leq 1$ except at $\zeta = 0$ where, by (131) and (132), it has the local behavior

$$W(\zeta) \sim -\frac{Q}{2\pi}\log(\zeta f_\zeta(0,\tau)+\cdots) \sim -\frac{Q}{2\pi}\log\zeta, \quad \text{as } \zeta \to 0. \tag{134}$$

By (126) W also satisfies

$$\Re(W(\zeta)) = 0 \quad \text{on } |\zeta| = 1. \tag{135}$$

Conditions (133)–(135) above are sufficient to uniquely define the real part of W; its harmonic conjugate $\Im(W)$ is then determined (up to an arbitrary additive constant; see lemma 1.8), and the final solution for W is

$$W(\zeta) = -\frac{Q}{2\pi}\log\zeta. \tag{136}$$

[6] It will be helpful later to use the time variable $\tau = t$, defined such that while $\partial/\partial t$ denotes the time derivative at fixed z, $\partial/\partial\tau$ denotes the time derivative at fixed ζ.

The next step in the solution procedure is to use the kinematic condition (127) to determine the time-dependent conformal map $f(\zeta, \tau)$ between the known domain in ζ-space and the unknown domain $D(t)$ in z-space.

We note first that, since $p = 0$ provides a functional description of the free boundary curve $\partial D(t)$, the gradient of p is normal to $\partial D(t)$, and thus a unit normal to $\partial D(t)$ is given by

$$\boldsymbol{n} = \frac{\nabla p}{|\nabla p|}. \tag{137}$$

Moreover, if $\boldsymbol{x}(t) = (x(t), y(t))$ denotes a point on the moving interface $\partial D(t)$, then since $p(x(t), y(t), t) = 0$ for all t we have

$$p_t + p_x \dot{x} + p_y \dot{y} = 0.$$

These two results give us an expression for V_n, the normal velocity of $\partial D(t)$, since

$$V_n = \boldsymbol{n} \cdot \frac{d\boldsymbol{x}}{dt} = \frac{1}{|\nabla p|}(p_x \dot{x} + p_y \dot{y}) = -\frac{p_t}{|\nabla p|}. \tag{138}$$

Using (137) and (138) in (127) yields

$$p_t = |\nabla p|^2 \quad \text{on } \partial D(t). \tag{139}$$

We now translate this result to the ζ-plane, noting that (130) gives

$$|w_z(z,t)|^2 = |\nabla p|^2$$

($w_z(z,t) = p_x - i p_y$ by the Cauchy-Riemann equations). Thus (139) can be written

$$\Re(w_t(z,t)) = |w_z(z,t)|^2 \quad \text{on } \partial D(t). \tag{140}$$

Since $w(z,t) = W(\zeta)$ and $z = f(\zeta, \tau)$ we have, with $W(\zeta)$ given by (136),

$$\begin{aligned} w_t(z,t) = W_\zeta(\zeta)\zeta_t, \quad \text{and} \quad 0 = f_\zeta(\zeta,\tau)\zeta_t + f_\tau(\zeta,\tau),& \\ \Rightarrow \quad w_t(z,t) = \frac{Q}{2\pi\zeta}\frac{f_\tau(\zeta,\tau)}{f_\zeta(\zeta,\tau)}& \qquad (141) \\ \text{and} \quad w_z(z,t) = \frac{W_\zeta(\zeta)}{f_\zeta(\zeta,\tau)} = \frac{-Q}{2\pi\zeta f_\zeta(\zeta,\tau)}.& \qquad (142) \end{aligned}$$

Substituting these results (141) and (142) into (140) gives

$$\Re\left(\frac{f_\tau(\zeta,\tau)}{\zeta f_\zeta(\zeta,\tau)}\right) = \frac{Q}{2\pi}\frac{1}{|\zeta f_\zeta(\zeta,\tau)|^2} \quad \text{on } |\zeta| = 1$$

which, noting that $\bar{\zeta}\zeta = 1$ defines $|\zeta| = 1$, is easily rearranged into the final form

$$\Re\left(\zeta f_\zeta(\zeta,\tau)\overline{f_\tau(\zeta,\tau)}\right) = \frac{Q}{2\pi} \quad \text{on } |\zeta| = 1. \tag{143}$$

Equation (143) is known as the **Polubarinova-Galin** equation after its original discoverers, and must be solved for $f(\zeta,\tau)$ subject to an initial condition.

4.2.3 The Polubarinova-Galin equation

It is a remarkable fact that several exact solutions to (143) may be written down. Any polynomial in ζ, with time-dependent coefficients, provides an exact solution. Let

$$f(\zeta,\tau) = \sum_{n=1}^{N} a_n(\tau)\zeta^n, \tag{144}$$

where $a_1(\tau) \in \mathbb{R}^+$ without loss of generality. Clearly this map is analytic on the unit disc; and, with suitable restrictions on the coefficients $a_n(\tau)$ it will be conformal and univalent as is required for physically-realistic solutions. Then the Polubarinova-Galin equation becomes

$$\begin{aligned} \Re\left(\sum_{n=1}^{N} n a_n e^{in\theta} \sum_{m=1}^{N} \dot{\bar{a}}_m e^{-im\theta}\right) &= \frac{Q}{2\pi} \\ \Rightarrow \quad \Re\left(\sum_{r=-(N-1)}^{N-1} c_r e^{ir\theta}\right) &= \frac{Q}{2\pi} \end{aligned} \tag{145}$$

where

$$c_r = \sum_{j=1}^{N-r}(r+j)a_{r+j}\dot{\bar{a}}_j \quad \text{and} \quad c_{-r} = \sum_{j=1}^{N-r} j a_j \dot{\bar{a}}_{r+j}, \quad r = 0,\, 1, \ldots, N-1. \tag{146}$$

Equation (145) rearranges to

$$
\begin{aligned}
& \sum_{r=-(N-1)}^{N-1} (c_r e^{ir\theta} + \overline{c}_r e^{-ir\theta}) = \frac{Q}{2\pi} \\
\Rightarrow \quad & c_0 + \overline{c}_0 + \sum_{r=1}^{N-1} (c_r + \overline{c}_{-r}) e^{ir\theta} + \sum_{r=1}^{N-1} (\overline{c}_r + c_{-r}) e^{-ir\theta} = \frac{Q}{\pi} \\
\Rightarrow \quad & \Re\left(c_0 + \sum_{r=1}^{N-1} (c_r + \overline{c}_{-r}) e^{ir\theta} \right) = \frac{Q}{2\pi}, \quad \forall \theta \in [0, 2\pi].
\end{aligned}
$$

It follows that

$$
\Re(c_0) = \frac{Q}{2\pi} \quad \text{and} \quad c_r + \overline{c}_{-r} = 0, \quad 1 \leq r \leq N-1,
$$

so that, returning to the definition (146), we have

$$
\frac{d}{dt}\left(\sum_{r=1}^{N} r |a_r|^2 \right) = \frac{Q}{\pi} \tag{147}
$$

$$
\sum_{j=1}^{N-r} \left[(r+j) a_{r+j} \dot{\overline{a}}_j + j \overline{a}_j \dot{a}_{r+j} \right] = 0, \quad 1 \leq r \leq N-1, \tag{148}
$$

and we just have a system of N first order ODEs ($N-1$ complex and one real, reflecting the fact that a_1 is real while $a_2, \ldots a_N$ are complex) to solve for the coefficients $a_r(\tau)$.

Remark Equation (147) represents **conservation of mass** for the fluid domain. This result follows from the expression

$$
A(t) \equiv \int\int_{D(t)} dx dy = \frac{1}{2i} \oint_{\partial D(t)} \bar{z} \, dz.
$$

for the area $A(t)$ of $D(t)$, which follows from Greens's theorem in the plane (**homework:** prove the above result). Thus,

$$
\begin{aligned}
A(t) &= \frac{1}{2i} \oint_{\partial D(t)} \bar{z} \, dz \\
&= \frac{1}{2i} \oint_{|\zeta|=1} \overline{f(\zeta, \tau)} f_\zeta(\zeta, \tau) \, d\zeta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2i}\oint_{|\zeta|=1}\sum_{m=1}^{N}\overline{a}_m\overline{\zeta}^m\sum_{n=1}^{N}na_n\zeta^{n-1}\,d\zeta \\
&= \frac{1}{2i}\oint_{|\zeta|=1}\sum_{m=1}^{N}\overline{a}_m\zeta^{-m}\sum_{n=1}^{N}na_n\zeta^{n-1}\,d\zeta \quad \text{using the fact that } \overline{\zeta}=\zeta^{-1} \text{ on } |\zeta|=1 \\
&= \pi\mathrm{Res}\left(\sum_{m=1}^{N}\overline{a}_m\zeta^{-m}\sum_{n=1}^{N}na_n\zeta^{n-1};\zeta=0\right) \quad \text{by Cauchy's residue theorem} \\
&= \pi\sum_{n=1}^{N}|a_n|^2.
\end{aligned}
$$

Since Q represents the rate at which fluid is injected into or removed from the domain $D(t)$, it is related to the rate of change of area by $dA/dt = Q$. Thus finally,

$$
\frac{dA}{dt} = \pi\frac{d}{dt}\left(\sum_{n=1}^{N}|a_n|^2\right) = Q,
$$

which is exactly equation (147).

Specific solution: Suppose we seek a solution in which only a_1 and a_N are nonzero in (144), so that

$$
f(\zeta,t) = a_1\zeta + a_N\zeta^N. \tag{149}
$$

Then only c_0 and c_{N-1} are nonzero, and with $a_1 \in \mathbb{R}$ equations (147) and (148) reduce to

$$
\begin{aligned}
\frac{d}{dt}(|a_1|^2 + N|a_N|^2) &= \frac{Q}{\pi}, \\
\frac{d}{dt}(a_1^N a_N) &= 0,
\end{aligned}
$$

so that the system is exactly integrable and

$$
a_N = \frac{k}{a_1^N}, \qquad a_1^2 + \frac{N|k|^2}{a_1^{2N}} = \frac{Qt + A_0}{\pi} \tag{150}
$$

where $k = a_1(0)^N a_N(0)$ is a constant, and $A_0 = \pi(a_1(0)^2 + N|a_N|^2)$ is the initial area of the fluid domain $D(0)$.

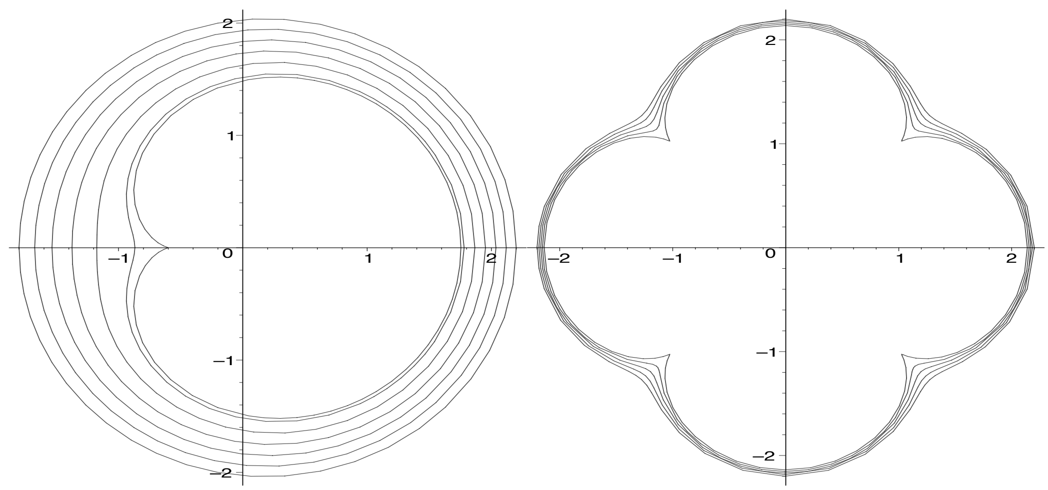

Figure 13: Solutions of the Hele-Shaw free boundary problem given by the conformal mapping (149), for $Q = -1$, in cases $N = 2$ (left-hand figure) and $N = 5$ (right-hand figure). Initial conditions are $a_1(0) = 2$, $a_N(0) = 0.2$ in both cases. The interface for $N = 2$ is shown at times $t = 0.\,1.2$, 2.4, 3.6, 4.8, 6.0, 6.37. The interface for $N = 5$ is shown at times $t = 0.\,0.3$, 0.6, 0.9, 1.22. In both cases the interface moves inwards, as fluid is removed through the sink.

This evolution is easily visualized using, for example, the software packages Maple or Mathematica. With a_1 and a_N given by the algebraic equations (150), the free boundary may be plotted parametrically at each timestep t by noting that $\zeta = e^{i\theta}$, $0 \le \theta \le 2\pi$ parametrizes the free boundary, and so the boundary is given by

$$\boldsymbol{x}(t) = \{(\Re(f(e^{i\theta}, t)), \Im(f(e^{i\theta}, t))) : 0 \le \theta \le 2\pi\}.$$

Examples of such evolution for cases $N = 2$ and $N = 5$ are given in figure 13. It is interesting to note that, even when the initial data corresponds to a univalent conformal mapping (so that the domain $D(0)$ has a smooth boundary) the case $Q < 0$ always leads to a non-univalent mapping within finite time t^*. This "suction" problem is unstable, and solutions undergo finite-time breakdown, normally via formation of a 3/2-power cusp in the free boundary, corresponding to a simple zero of $f_\zeta(\zeta, t)$ moving onto the boundary $|\zeta| = 1$ from the external region $|\zeta| > 1$, and loss of conformality. Subsequent evolution leads to a self-overlapping fluid domain, which is clearly unphysical. Both the examples shown in figure 13 exhibit this cusp formation. In general the map (149) with $Q < 0$ will lead to the formation of $N - 1$

cusps simultaneously in the free boundary at some finite time t^* before all fluid has been removed from the domain.

Homework: 1. Show that a simple zero of $f_\zeta(\zeta, t)$ on $|\zeta| = 1$ (at $\zeta = 1$, say) gives rise to a cusp singularity on the image curve $\partial D(t)$. Assume, for simplicity, that f and all derivatives evaluated at the point $\zeta = 1$ are real. **Hint:** Note that, near $\zeta = 1$ and on the boundary, $\zeta = e^{i\theta}$, with $|\theta| \ll 1$. Taylor expand this, and Taylor expand again in the conformal mapping, using the smallness of $|\theta|$ each time.
2. Investigate the shapes described by the quadratic mapping function $z = f(\zeta) = a\zeta + b\zeta^2$ from the unit disc $|\zeta| \leq 1$. Assuming a, $b \in \mathbb{R}$, can you determine for which values of these parameters the map is univalent on the unit disc, and for which values it is conformal?

4.2.4 The Schwarz function approach

Another approach to solving the free boundary problem, which is in many ways more elegant and powerful than that outlined above, is to use the concept of the **Schwarz function** of the free boundary. Given an analytic curve ∂D, its Schwarz function is defined to be that function $g(z)$ such that the equation

$$\bar{z} = g(z) \tag{151}$$

defines the curve ∂D. For a given curve ∂D, g is analytic, at least in some neighborhood of the curve, and unique (though it is often impossible to write down explicitly). If one has the equation of the curve in the form $F(x, y) = 0$ then in principle the Schwarz function may be obtained by substituting for $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/(2i)$, and solving for $\bar{z}$.

Example 4.13 *Find the Schwarz functions of (i) the straight line $y = mx + c$, and (ii) the circle $(x - a)^2 + (y - b)^2 = r^2$.*

(i) Substituting for x and y gives

$$\frac{z - \bar{z}}{2i} = \frac{m(z + \bar{z})}{2} + c \quad \Rightarrow \quad \bar{z} = z\left(\frac{1 - im}{1 + im}\right) - \frac{2ic}{1 + im} \equiv g(z).$$

(ii) Rewrite the equation of the circle in complex form as

$$|z - (a + ib)|^2 = r^2,$$

then

$$(z-(a+ib))(\bar{z}-(a-ib))=r^2 \quad \Rightarrow \quad \bar{z}=\frac{r^2}{z-(a+ib)}+(a-ib)\equiv g(z).$$

To implement the Schwarz function approach for the Hele-Shaw free boundary problem (125)–(129) we require the following

Proposition 4.14 *For the moving analytic curve $\partial D(t)$ with Schwarz function $g(z,t)$, the complex tangent and normal velocity to the boundary are given by*

$$z_s=\frac{\partial z}{\partial s}=\frac{1}{\sqrt{g_z(z,t)}}, \qquad V_n=-\frac{i}{2}\frac{g_t(z,t)}{\sqrt{g_z(z,t)}} \tag{152}$$

respectively.

Proof If $z=z(s,t)=x(s,t)+iy(s,t)$ is a complex point on the curve $\partial D(t)$, where s is arclength along $\partial D(t)$, then by the definition of the Schwarz function we have

$$\overline{z}_s=g_z(z,t)z_s \quad \Rightarrow \quad \frac{z_s}{\overline{z}_s}=\frac{1}{g_z} \quad \text{on } \partial D(t).$$

But, noting that arclength s is defined by $x_s^2+y_s^2=1$, we have $|z_s|^2=1$ on $\partial D(t)$, so that $\overline{z_s}=1/z_s$, and hence

$$z_s=\frac{1}{\sqrt{g_z(z,t)}}, \tag{153}$$

proving the first result claimed.

For the second result, we note that, since $\boldsymbol{t}=(x_s,y_s)$ is a unit tangent vector to the curve, $\boldsymbol{n}=(y_s,-x_s)$ is a unit outward normal vector. Thus, $V_n=y_s\dot{x}-x_s\dot{y}=\Im(z_s\dot{\bar{z}})$. Differentiating the Schwarz function equation for the curve (151) with respect to time gives

$$\dot{\bar{z}}=g_z(z,t)\dot{z}+g_t(z,t)=\overline{z}_s^2\dot{z}+g_t(z,t), \tag{154}$$

so that, using (154) and its complex conjugate, remembering that $z_s\overline{z_s}=1$, we have

$$\begin{aligned} V_n &= \Im(z_s\dot{\bar{z}})=\frac{1}{2i}(z_s\dot{\bar{z}}-\overline{z_s}\dot{z})=\frac{1}{2i}(\overline{z_s}\dot{z}+z_sg_t-z_s\dot{\bar{z}}-\overline{z_sg_t}) \\ &= -\frac{1}{2i}(z_s\dot{\bar{z}}-\overline{z_s}\dot{z})+\frac{1}{2i}(z_sg_t-\overline{z_sg_t}) \\ &= -V_n+\frac{1}{2i}(z_sg_t-\overline{z_sg_t}), \end{aligned}$$

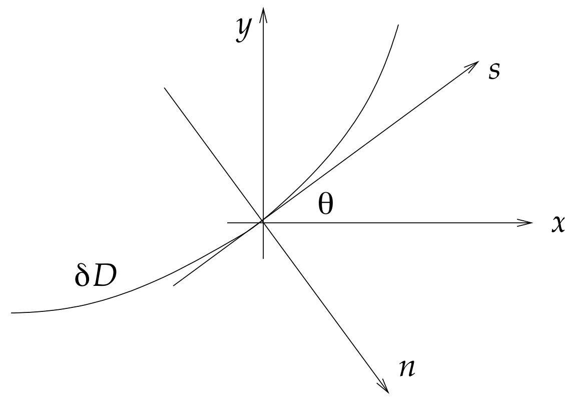

Figure 14: Local coordinates tangential and normal to the curve $\partial D(t)$.

where we have suppressed the (z,t) arguments in g for brevity. Thus,

$$2V_n = \frac{1}{2i}(z_s g_t - \overline{z_s g_t}). \tag{155}$$

Next we note that (154) and its complex conjugate also give

$$\begin{aligned} \dot{\overline{z}} = \overline{z}_s^2 \dot{z} + g_t &= \overline{z}_s^2 (z_s^2 \dot{\overline{z}} + \overline{g_t}) + g_t \\ \Rightarrow \quad 0 &= \overline{z}_s^2 \overline{g_t} + g_t \\ \Rightarrow \quad \overline{z}_s \overline{g_t} &= -z_s g_t. \end{aligned}$$

Using this last result in (155), together with (153), finally gives

$$V_n = -\frac{i}{2} \frac{g_t(z,t)}{\sqrt{g_z(z,t)}}, \tag{156}$$

proving the second result of the Proposition.

We are now ready to reformulate the Hele-Shaw problem in terms of the Schwarz function of its free boundary. In local coordinates (n,s) centered at an arbitrary point on $\partial D(t)$ (see figure 14), where θ is the angle between the x-axis and the tangent direction s, we have

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial s} + \sin\theta \frac{\partial}{\partial n}, \quad \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial s} - \cos\theta \frac{\partial}{\partial n},$$

so that the derivative of the complex potential $w(z,t)$ (see (130)) may be expressed as follows:

$$
\begin{aligned}
\frac{\partial w}{\partial z} &= p_x - ip_y = e^{-i\theta}p_s + ie^{-i\theta}p_n = \overline{z_s}(p_s + ip_n) \\
&= -\frac{iV_n}{z_s} \quad \text{on } \partial D \\
&= -\frac{1}{2}\frac{\partial g}{\partial t} \quad \text{on } \partial D,
\end{aligned}
$$

where we used the fact that the complex tangent $z_s = e^{i\theta}$, together with the boundary conditions (126) and (127) and the results (152) above.

This last equality has been demonstrated to hold just on the boundary, $\partial D(t)$. However, the complex potential is analytic on the whole of the domain $D(t)$, and we know that the Schwarz function is analytic at least in some neighborhood of the analytic curve $\partial D(t)$. Thus, the principle of **analytic continuation** holds (§1.7), and we deduce that in fact the equality

$$\frac{\partial w(z,t)}{\partial z} = -\frac{1}{2}\frac{\partial g(z,t)}{\partial t} \tag{157}$$

holds wherever either function is defined, and in particular, it holds throughout the fluid domain $D(t)$.

Equation (157) is key to finding exact solutions to the free boundary problem. The crucial observation is that we know $w(z,t)$ to be analytic on $D(t)$, except at the specified driving singularity (128). However, the Schwarz function of any boundary curve will in general have singularities within $D(t)$. The above equation tells us that, any such singularities of g, which do not coincide with a specified singularity of $w(z,t)$ **must not vary in time**, so that $\partial g/\partial t$ will not exhibit the singular behavior.

Before going ahead to solve the free boundary problem by this approach we introduce one final piece of machinery.

Definition 4.15 *(Complex conjugate function) Given a complex analytic function $F(\zeta)$, the complex conjugate function $\overline{F}(\zeta)$ is defined by*

$$\overline{F}(\zeta) = \overline{F(\overline{\zeta})}. \tag{158}$$

$\overline{F}(\zeta)$ is complex analytic if $\overline{\zeta}$ lies in the domain of analyticity of F.

Example 4.16 *If $F(\zeta) = \sum_{n=1}^{N} a_n\zeta^n$ then $\overline{F}(\zeta) = \sum_{n=1}^{N} \overline{a_n}\zeta^n$.*

Implementation of the Schwarz function approach

To implement the approach we return to the conformal mapping that we know exists between the unit disc $|\zeta| \leq 1$ and the domain $D(t)$, $z = f(\zeta, t)$. We again consider a flow driven by a single point source or sink, so that the only singularity of $w(z,t)$ is at $z = 0$, where w has the singular behavior (128). On the boundary the Schwarz function identity (151) leads to

$$g(f(\zeta,t),t) = \overline{f(\zeta,t)} = \overline{f}(\overline{\zeta},t) = \overline{f}(1/\zeta,t), \qquad \text{on } |\zeta| = 1,$$

using the definition of the complex conjugate function (158) above. Since we know that each of the Schwarz function and $\overline{f}$ are analytic in some neighborhood of $|\zeta| = 1$ we can use analytic continuation to deduce that the equality

$$g(f(\zeta,t),t) = \overline{f}(1/\zeta,t)$$

holds everywhere that these functions are defined, and therefore that $\overline{f}(1/\zeta,t)$ is the Schwarz function of the free boundary in the ζ-plane.

Given a conformal mapping then, with time-dependent coefficients, we can write down the Schwarz function in the ζ-plane explicitly. We can then find the location of its singularities $\zeta_k(t)$ within the unit disc, which will correspond to the singularities $z_k(t) = f(\zeta_k(t),t)$ of $g(z,t)$ within $D(t)$. Unless $z_k(t) = 0$ (the location of the only singularity of w), equation (157) tells us that $dz_k/dt = 0$, a condition that will yield a **trivially-integrable** ordinary differential equation for the coefficients of the mapping function. If the singularity corresponds to $z = 0$ then another differential equation may be obtained by noting that the local behavior of dw/dz in (157) is

$$\frac{dw}{dz} = -\frac{Q}{2\pi z} + O(1) \quad \text{as } z \to 0,$$

and matching the singular behavior appropriately.

The procedure is best illustrated by example, and we consider a case that would be very difficult to solve explicitly using the Polubarinova-Galin equation (143). We take

$$z = f(\zeta,t) = \sum_{n=1}^{N} \frac{a_n(t)\zeta}{\zeta - b_n(t)} + c(t)\zeta, \tag{159}$$

which maps the origin to the origin, and which will be analytic on $|\zeta| \leq 1$ provided $|b_n(t)| \geq 1$. The Schwarz function in the ζ-plane is given by

$$\overline{f}\left(\frac{1}{\zeta}\right) = \sum_{n=1}^{N} \frac{\overline{a_n(t)}}{1 - \overline{b_n(t)}\zeta} + \frac{\overline{c(t)}}{\zeta}, \tag{160}$$

which has singularities at points $\zeta = 0$, $\zeta_n(t) = 1/\overline{b_n(t)}$, within the unit disc, corresponding to singularities of $g(z,t)$ at $z = 0$, $z_n(t) = f(1/\overline{b_n(t)}, t)$ within $D(t)$.

Near $\zeta = 1/\overline{b_n(t)}$ the Schwarz function in the ζ-plane satisfies

$$\overline{f}\left(\frac{1}{\zeta}\right) = \frac{\overline{a_n(t)}}{1 - \overline{b_n(t)}\zeta} + O(1),$$

and a local Taylor expansion of the mapping function gives

$$\begin{aligned} z - z_n(t) &= (\zeta - 1/\overline{b_n(t)}) f_\zeta(1/\overline{b_n(t)}, t) + O((\zeta - 1/\overline{b_n(t)})^2) \\ \Rightarrow \quad \frac{\overline{a_n(t)}}{1 - \overline{b_n(t)}\zeta} &= -\frac{\frac{\overline{a_n(t)}}{\overline{b_n(t)}} f_\zeta\left(\frac{1}{\overline{b_n(t)}}, t\right)}{z - z_n(t)} + O(1). \end{aligned} \tag{161}$$

By (160), this last expression gives the local form of the Schwarz function in the z-plane, near the singularity at $z = z_n(t) = f(1/\overline{b_n(t)}, t)$. Since these singularities lie within $D(t)$, and the complex potential must be analytic at these points, any singular behavior on the right-hand side of (157) due to these poles must vanish. Clearly, near $z = z_n(t)$ equation (161) gives

$$\frac{\partial g}{\partial t} = -\frac{\dot{z}_n(t)}{(z - z_n(t))^2}\left[\frac{\overline{a_n(t)}}{\overline{b_n(t)}} f_\zeta\left(\frac{1}{\overline{b_n(t)}}, t\right)\right] - \frac{1}{z - z_n(t)}\frac{d}{dt}\left[\frac{\overline{a_n(t)}}{\overline{b_n(t)}} f_\zeta\left(\frac{1}{\overline{b_n(t)}}, t\right)\right] + O(1),$$

so the requirement that this singular behavior be eliminated in (157) leads to two equations for each singularity $z_n(t)$, namely

$$\frac{d}{dt}\left[f(1/\overline{b_n(t)}, t)\right] = 0, \quad n = 1, \dots, N, \tag{162}$$

$$\frac{d}{dt}\left[\frac{\overline{a_n(t)}}{\overline{b_n(t)}} f_\zeta\left(\frac{1}{\overline{b_n(t)}}, t\right)\right] = 0, \quad n = 1, \dots, N. \tag{163}$$

To complete the solution we must also match the singular behavior at $z = 0$ in (157), where we know $dw/dz = -Q/(2\pi z) + O(1)$. Near $\zeta = 0$ we have

$$\overline{f}\left(\frac{1}{\zeta}, t\right) = \frac{\overline{c(t)}}{\zeta} + O(1), \quad \text{and} \quad z = \zeta f'(0,t) + O(\zeta^2),$$

$$\Rightarrow \quad g(z,t) = \frac{\overline{c(t)} f'(0,t)}{z} + O(1),$$

so that matching the singular behavior in (157) requires

$$\frac{d}{dt}(\overline{c(t)} f'(0,t)) = \frac{Q}{2\pi}. \tag{164}$$

Equations (162), (163) and (164) are immediately integrable, and give $2N+1$ algebraic equations for the 2N+1 unknowns $a_n(t)$, $b_n(t)$, $c(t)$, completing the solution to the problem.

The evolution may again be depicted graphically using Maple or a similar package, as described for the polynomial mapping function example given earlier. Again, the generic behavior is that for $Q > 0$ (injection of fluid through a point source), singularities of the mapping function in $|\zeta| \geq 1$ (at $\zeta = b_n(t)$) move further away from the unit circle, corresponding to the domain boundary becoming smoother and more circular; while for $Q < 0$ (suction of fluid through a point sink within the flow domain) the $b_n(t)$ approach the unit circle $|\zeta| = 1$, always reaching it within finite time. When the first singularity reaches the unit circle the map ceases to be conformal on the unit disc; the boundary $\partial D(t)$ develops a singularity, and the solution breaks down.

An example of the evolution is shown in figure 15, for the case $N = 2$. To simplify the analysis we assumed real coefficients, and $a_1 = -a_2 = a$, $b_1 = -b_2 = b$. It is easy to check in this case that equations (162) are the same for $n = 1, 2$, as are equations (163), and so together with (164) we have just three (real) equations for the three (real) unknowns a, b, c. Once initial conditions are specified, the equations can be solved numerically. As the figure shows, the evolution breaks down in finite time via the formation of two cusps in the free boundary, at the images of $\zeta = 1$ and $\zeta = -1$.

Homework: 1. Derive the three equations satisfied by the coefficients $a(t)$, $b(t)$, $c(t)$ in the specific example illustrated by figure 15. Show that the cusps at points $\zeta = \pm 1$ form simultaneously (when they form). What is the

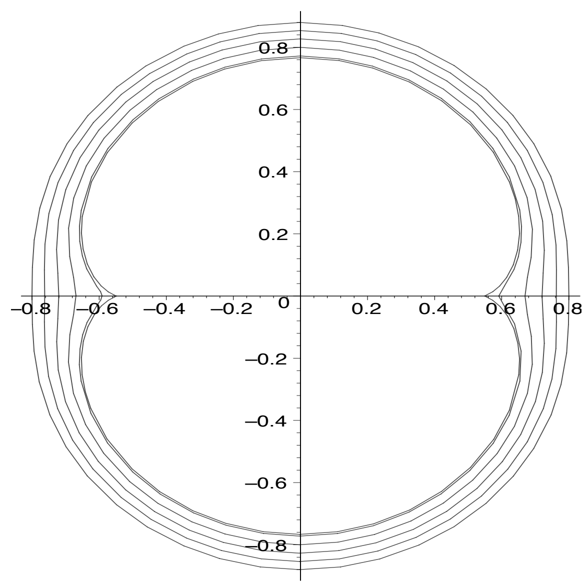

Figure 15: Evolution of a fluid domain described by the mapping (159), with $N = 2$, $a_1 = -a_2 = a$, $b_1 = -b_2 = b$ and real coefficients a, b, c. The flow is driven by a point sink of strength $Q = -1$, and initial conditions are $a(0) = 0.15$, $b(0) = 2$, $c(0) = 1$. The evolving interface is shown at times $t = 0$, 0.3, 0.6, 0.9, 1.2, 1.25, 1.26.

large-time behavior of the governing equations when $Q > 0$? What are the implications for the large-time behavior of the fluid domain?
2. Implement the Schwarz function approach for the simple mapping function $z = f(\zeta, t) = a(t)\zeta$.
3. (Harder) Try extending this to the flow described by the quadratic mapping function $z = f(\zeta, t) = a(t)\zeta + b(t)\zeta^2$.

4.3 Other applications

We conclude by mentioning some other real-world problems in which complex analysis approaches can be very useful.

4.3.1 2D Stokes flow (and elasticity)

Stokes flow, or slow viscous flow, describes the situation of a Newtonian fluid in which the **Reynolds number**[7] is very small, $\mathrm{Re} \ll 1$. This means that the inertial terms in the Navier-Stokes equations may be neglected, and the flow is dominated by viscous effects. The governing equations for the fluid velocity $\boldsymbol{u}$ and pressure p are, in suitably scaled variables,

$$\nabla p = \nabla^2 \boldsymbol{u}, \qquad \nabla \cdot \boldsymbol{u} = 0$$

and writing the velocity field in terms of a streamfunction ψ, $\boldsymbol{u} = (\psi_y, -\psi_x)$ (so that the incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ is automatically satisfied), we find

$$\nabla p = (\psi_{xxy} + \psi_{yyy}, -\psi_{xxx} - \psi_{xyy}).$$

Since the curl of a gradient field is always zero, $\nabla \wedge (\nabla p) = \mathbf{0}$, we obtain finally

$$\nabla^2(\nabla^2 \psi) = 0, \quad \text{or} \quad \nabla^4 \psi = 0.$$

This equation is known as the bilaplacian or biharmonic equation and, similarly to Laplace's equation, its solutions are expressible in complex variable form. To see this note that the Laplace operator may be written as

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}},$$

[7] The Reynolds number of a flow is a dimensionless number characterizing the flow, defined by $\mathrm{Re} = \rho U L/\mu$. where ρ is the fluid density, U is a typical flow velocity, L is a typical lengthscale associated with the flow, and μ is the fluid's viscosity.

so that if $\nabla^2\psi = q$,

$$\nabla^2 q = 0 \quad \Rightarrow \quad \frac{\partial^2 q}{\partial z \partial \bar{z}} = 0 \quad \Rightarrow \quad q = 4(f(z) + g(\bar{z})).$$

Then

$$\begin{aligned}
\frac{\partial^2\psi}{\partial z\partial\bar{z}} &= f(z) + g(\bar{z}) \\
\Rightarrow \quad \frac{\partial\psi}{\partial z} &= \bar{z}f(z) + \int^{\bar{z}} g(\bar{z})\,d\bar{z} + h(z) = \bar{z}f(z) + G(\bar{z}) + h(z) \\
\Rightarrow \quad \psi &= \bar{z}\int^{z} f(z)\,dz + zG(\bar{z}) + \int^{z} h(z)\,dz + j(\bar{z}) \\
&= \bar{z}F(z) + zG(\bar{z}) + H(z) + J(\bar{z}).
\end{aligned}$$

We note that, since ψ is a real function we must have $\psi = \overline{\psi}$, which requires $F(z) = \overline{G(\bar{z})} = \overline{G}(z)$ and $H(z) = \overline{J(\bar{z})} = \overline{J}(\bar{z})$, so that finally

$$\psi = 2\Re(\bar{z}F(z) + H(z)).$$

This is known as the **Goursat representation** of biharmonic functions. Once determined, the functions F and H, which are complex analytic on the flow domain away from any specified driving singularities of the flow, fully specify the fluid flow. In general, determination of F and H requires that we specify appropriate boundary conditions. For free boundary problems, like those discussed in §4.2 above, it turns out that (as was the case for the Hele-Shaw problem) the boundary conditions for a simply-connected flow domain have a convenient form when expressed in terms of the complex variable representation, and exact time-dependent solutions to moving boundary problems may be obtained.

The full details of this procedure are beyond the scope of this course. Here we just give a few examples of flows in unbounded domains, obtained by choosing suitable forms for F and H.

Example 4.17 *(Rigid-body rotation) Find the Goursat functions F and H for a flow in rigid-body rotation about the origin.*

Rigid-body rotation with constant angular velocity $\boldsymbol{\omega} = \omega\boldsymbol{k}$ corresponds to a velocity field

$$\boldsymbol{u} = \omega(-y, x)$$

so that the streamfunction $\psi = -\omega(x^2 + y^2) = -\omega z\bar{z}$, up to an arbitrary additive constant. Thus, $F(z) = -\omega z/2$ and $H(z) = 0$.

Example 4.18 *(Rigid-body translation) Find the Goursat functions F and H for a flow in rigid-body translation, $\boldsymbol{u} = (u_0, v_0)$.*

Here $\psi = u_0 y - v_0 x = \Re(-iu_0 z - v_0 z)$, so that $F(z) = 0$ and $H(z) = -i(u_0 - iv_0)z/2$.

A linear combination of rigid body rotation and translation may be obtained by adding the two streamfunctions obtained above, taking $F(z) = -\omega z/2$ and $H(z) = -i(u_0 - iv_0)z/2$.

Homework: Find the Goursat functions associated with:
1. The streamfunction for a vortex, $\psi = (\Gamma/2\pi)\log r$.
2. The streamfunction for a collection of vortices of strengths Γ_j at points (x_j, y_j) in the flow.

4.4 Hodograph Transformation and the potential plane

A frequently-used technique in fluid-dynamical problems for which a complex potential exists is to work in the complex plane of the complex potential, or its derivative with respect to z. One can then attempt to find the mapping $z(w)$ between the planes, and hence invert to obtain $w(z)$.

The procedure is best illustrated by example, and we use a classical problem from the Hele-Shaw free boundary literature, due to Saffman and Taylor [9]. These authors carried out an experiment in which air was forced into a long, rectangular Hele-Shaw cell, initially filled with viscous fluid (glycerine). The air is observed to form a steadily-propagating "finger", traveling down the rectangle, as shown in figure 16 (taken from the original paper; sub-figure 8).

The authors modeled the experiment using the theory of the classical Hele-Shaw free boundary problem presented in §4.2. The idealized geometry is shown in figure 17. The cell is taken to be infinitely long in the x-direction, with confining walls at $y = \pm 1$. We consider a semi-infinite, steadily-translating finger of air, moving with speed U and with its nose at the origin, and flat sides $y \to \pm\lambda$ as $x \to -\infty$.

In §4.2 we worked with the complex potential whose real part was the fluid pressure, $p = \Re(w)$, where $\boldsymbol{u} = -\nabla p$; here, to conform to standard usage, *Saffman & Taylor* *Proc. Roy. Soc. A*, *volume* 245, *plate* 2

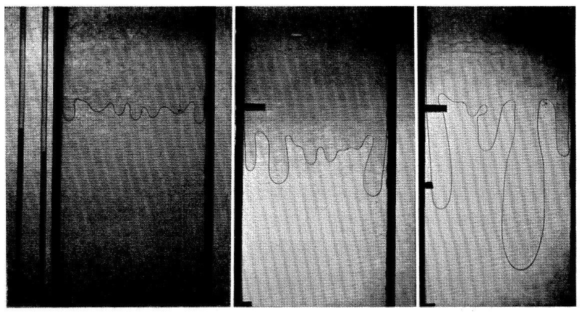

FIGURE 2 FIGURE 3 FIGURE 4

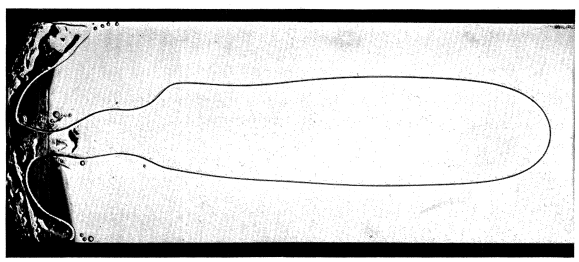

FIGURE 8

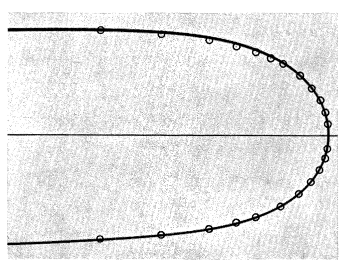

FIGURE 9

(*Facing p.* 318)

Figure 16:

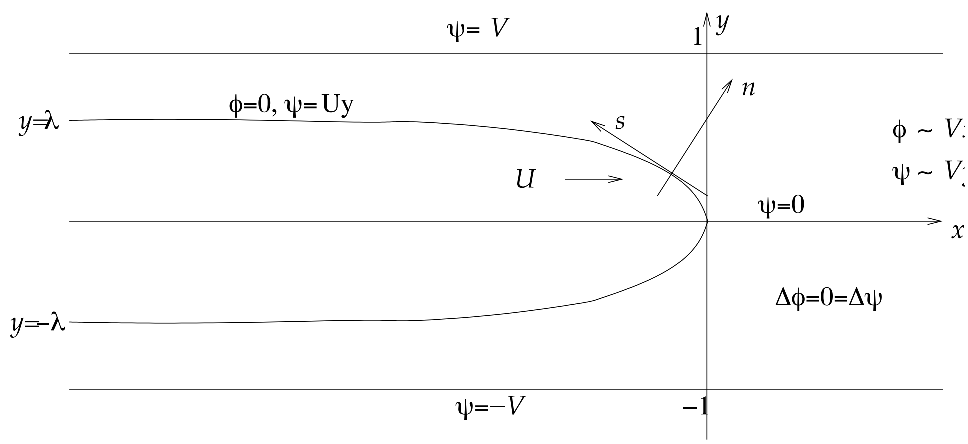

Figure 17: Schematic showing the idealized Saffman-Taylor problem geometry, together with the boundary conditions and asymptotic behavior for ϕ, ψ.

we shall use $\varpi = -w = \phi + i\psi$, where $\boldsymbol{u} = \nabla\phi$. Both ϕ and its harmonic conjugate ψ (a streamfunction for the flow) are harmonic functions on the flow domain $D(t)$. Far ahead of the moving finger the flow is uniform, so that

$$\phi \sim Vx \quad \text{as } x \to \infty, \tag{165}$$

for some V. The walls $y = \pm 1$ are streamlines for the flow, so ψ is constant on these lines (this condition is also immediate upon noting that, since no fluid can flow through the walls, $\partial\phi/\partial y = 0$ on $y = \pm 1$, so that $\partial\psi/\partial x = 0$ there also, by the Cauchy-Riemann equations). Then, observing that the above asymptotic behavior (165) for ϕ implies

$$\psi \sim Vy \quad \text{as } x \to \infty,$$

we must have

$$\psi = \pm V \quad \text{on } y = \pm 1.$$

The channel centerline $y = 0$ is also a streamline, on which $\psi = 0$ by symmetry.

On the free boundary ∂D we must satisfy the dynamic and kinematic boundary conditions, (122) and (123), which, noting that the interface is moving with velocity $\boldsymbol{v} = (U, 0)$, and that the normal to the interface is given by $\boldsymbol{n} = (\partial y/\partial s, -\partial x/\partial s)$ (where s is arclength along the interface, measured from the finger tip), translate to

$$\phi = 0 \quad \text{on } \partial D, \tag{166}$$

$$\frac{\partial \phi}{\partial n} = U \frac{\partial y}{\partial s}. \tag{167}$$

Since ψ is the harmonic conjugate to ϕ, the Cauchy-Riemann equations give $\phi_n = \psi_s$ on ∂D, and thus condition (167) may in fact be integrated to give

$$\psi = Uy \quad \text{on } \partial D, \tag{168}$$

using the fact that $\psi = 0$ on $y = 0$ to fix the constant of integration. Finally, we note that far behind the finger tip, the fluid is confined to the parallel-sided gap $\lambda < y < 1$ between the finger and the channel wall (similarly for the gap in $y < 0$). ψ changes from the value $U\lambda$ on the finger $y = \lambda$ (from (168) above) to the value V on $y = 1$. The flow here at $x = -\infty$ is in the x-direction only, $\boldsymbol{u} = (u, 0)$ so that the net flux of fluid in this gap is

$$\text{flux} = \int_\lambda^1 u \, dy = \int_\lambda^1 \frac{\partial \psi}{\partial y} \, dy = V - \lambda U.$$

But the net flux of fluid far behind the finger must be zero, so that in fact

$$V = \lambda U. \tag{169}$$

This completes the problem specification, summarized in figure 17 and below:

$$\nabla^2 \psi = 0 = \nabla^2 \phi \quad \text{in } D, \tag{170}$$

$$\psi = 0 \quad \text{on } y = 0, \tag{171}$$

$$\psi = \pm V \quad \text{on } y = \pm 1, \tag{172}$$

$$\psi = Uy, \quad \phi = 0, \quad \text{on } \partial D, \tag{173}$$

$$\psi \sim Vy, \quad \phi \sim Vx, \quad \text{as } x \to \infty. \tag{174}$$

We solve this problem by considering the mapping $\varpi = f(z)$ onto the potential plane. We shall use the symmetry in the problem, and solve only on the upper-half flow domain $0 \le y \le 1$, extracting the solution in $y \le 0$ by symmetry. The domain $\hat{D}^+$ corresponding to the upper-half flow domain D^+ has

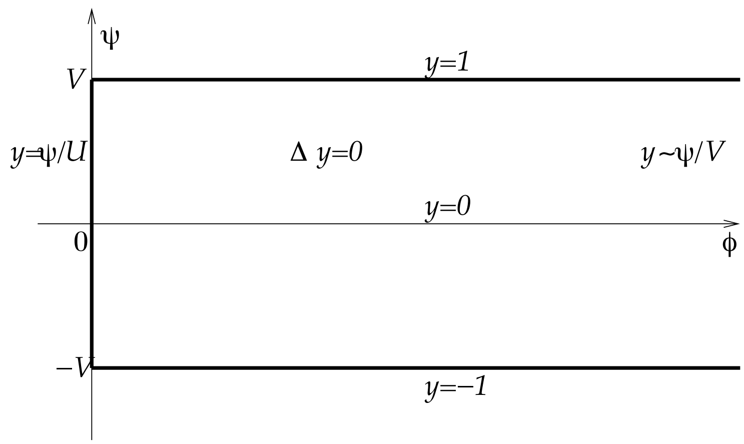

Figure 18:

a very simple geometry: it is a semi-infinite strip, bounded on two sides by $\psi = 0$, $\psi = V$, and on the remaining side by $\phi = 0$ (figure 18). Assuming an invertible transformation between z and ϖ planes, then $z = f^{-1}(\varpi)$ will be an analytic function of ϖ, it follows that x and y are harmonic functions of the real variables ϕ and ψ. In particular, the full specification of the problem for y is immediate, and is given by

$$\begin{aligned} \nabla^2 y &= 0 \quad \text{in } \phi \geq 0,\, 0 \leq \psi \leq V, &(175)\\ y &= 0 \quad \text{on } \psi = 0,\, \phi \geq 0 &(176)\\ y &= 1 \quad \text{on } \psi = V,\, \phi \geq 0 &(177)\\ y &= \frac{\psi}{U} \quad \text{on } \phi = 0,\, 0 \leq \psi \leq V, &(178)\\ y &\sim \frac{\psi}{V} \quad \text{as } x \to \infty. &(179) \end{aligned}$$

(see figure 18) Once we have solved for $y(\phi, \psi)$ we can construct its harmonic conjugate $x(\phi, \psi)$ and hence $z = f^{-1}(\varpi) = f^{-1}(\phi + i\psi)$. Since we know that $\phi = 0$ on the interface, this then enables the interface to be plotted via $z = f^{-1}(\psi)$, using $\psi \in (-V, V)$ as parameter.

The general solution to (175)–(177) and (179) is easily written down as

$$\psi = \frac{\psi}{V} + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi\psi}{V}\right) \exp\left(-\frac{n\pi\phi}{V}\right)$$

where the A_n are determined from the remaining boundary condition (178), which requires

$$\psi\left(\frac{1}{U}-\frac{1}{V}\right)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi\psi}{V}\right).$$

The coefficients are extracted by the usual Fourier series method of multiplying both sides of this equation by $\sin(m\pi\psi/V)$ and integrating between $\psi=0$ and $\psi=V$, which leads to

$$A_m=\frac{2(-1)^m}{m\pi}(1-\lambda),$$

recalling that $V=\lambda U$. Thus,

$$y=\frac{\psi}{V}+\frac{2(1-\lambda)}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\sin\left(\frac{m\pi\psi}{V}\right)\exp\left(-\frac{m\pi\phi}{V}\right),$$

which, noting that

$$\sin\left(\frac{m\pi\psi}{V}\right)\exp\left(-\frac{m\pi\phi}{V}\right)=-\Im\left[\exp\left(-\frac{m\pi}{V}(\phi+i\psi)\right)\right]$$

we recognise as

$$\begin{aligned} y &= \Im\left[\frac{\varpi}{V}+\frac{2(1-\lambda)}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\exp\left(-\frac{m\pi\varpi}{V}\right)\right] \\ &= \Im\left[\frac{\varpi}{V}+\frac{2(1-\lambda)}{\pi}\ln\left(1+\exp\left(-\frac{\pi\varpi}{V}\right)\right)\right]. \end{aligned}$$

We still have a real constant to choose when extracting $z=x+iy$, but recalling that we want $z=0$ when $\varpi=0$ fixes this, and we finally obtain

$$z=f^{-1}(\varpi)=\frac{\varpi}{V}+\frac{2(1-\lambda)}{\pi}\ln\left[\frac{1}{2}\left(1+\exp\left(-\frac{\pi\varpi}{V}\right)\right)\right]. \tag{180}$$

The complex potential ϖ is thus implicitly determined as a function of z and, as stated earlier, we can plot the interface by setting $\phi=0$ in (180), and plotting the parametric curve

$$(x,y)=\{(\Re(f^{-1}(\varpi)),\Im(f^{-1}(\varpi))):\varpi=i\psi,\quad\psi\in(-V,V)\}.$$

A The Schwarz-Christoffel transformation

As mentioned earlier, the Riemann mapping theorem, though it guarantees the existence of a conformal mapping between any two simply-connected domains in the complex plane, gives us no clue as to how to construct such mappings. There is, however, one notable and quite general case in which the conformal mapping can be constructed explicitly: when we are mapping from a nice canonical domain (the upper half-plane, in the usual theorem statement) onto the interior of an arbitrary polygon. The **Schwarz-Christoffel transformation** gives a recipe for doing exactly this. Before stating and (non-rigorously) proving the theorem, we outline some of the important ideas behind the construction of the transformation.

The theorem makes use of a variant of the Schwarz reflection principle, which enables us to analytically continue functions across boundaries or lines in the complex plane.

A.1 Schwarz reflection principle

Although this principle is really quite general, we shall consider it only in a couple of special cases. First, consider a function $f(\zeta)$, analytic on the upper half-plane D_+. Then, the function $\tilde{f}$, defined for any point ζ in the lower half-plane D_- by

$$\tilde{f}(\zeta) = \overline{f(\overline{\zeta})}, \tag{181}$$

is in fact **analytic** on D_-. That $\tilde{f}$ is defined on D_- is confirmed by checking that $\zeta \in D_- \iff \overline{\zeta} \in D^+$; and analyticity can be demonstrated by appealing to the Cauchy-Riemann equations. Writing $f(\zeta) = u(\xi,\eta) + iv(\xi,\eta)$ and $\tilde{f}(\zeta) = \tilde{u}(\xi,\eta) + i\tilde{v}(\xi,\eta)$, the definition (181) gives

$$\tilde{u}(\xi,\eta) = u(\xi,-\eta), \quad \tilde{v}(\xi,\eta) = -v(\xi,-\eta),$$

and hence for $\xi + i\eta \in D_-$, $\xi - i\eta \in D^+$, we have

$$\begin{aligned} \tilde{u}_\xi(\xi,\eta) &= u_\xi(\xi,-\eta) = v_\eta(\xi,-\eta) = \tilde{v}_\eta(\xi,\eta), \\ \tilde{u}_\eta(\xi,\eta) &= -u_\eta(\xi,-\eta) = v_\xi(\xi,-\eta) = -\tilde{v}_\xi(\xi,\eta), \end{aligned}$$

confirming that $\tilde{f}$ is analytic on the domain D_- as claimed. (We note for later reference that the function $\tilde{f}$ defined here is often written as $\overline{f}$, and is known as the **complex conjugate function** to f.)

A similar procedure may be carried out for the unit disc, $B(0;1)$. If $f(\zeta)$ is analytic on $|\zeta| \leq 1$ then we can define a function $\tilde{f}$, analytic outside the unit disc

$$\tilde{f}(\zeta) = \overline{f(1/\overline{\zeta})} \quad |\zeta| \geq 1.$$

In each of these examples, the new functions $\tilde{f}$ rely on "reflecting" the points in the boundary curve – in the first case, $\overline{\zeta}$ is the reflection of ζ in the real axis $\Im(\zeta) = 0$, while in the second case, $1/\overline{\zeta}$ is the reflection of ζ in the unit circle $|\zeta| = 1$. These are sensible definitions of reflections because, in either case, equality of the point with its reflection defines the boundary curve.

Returning to the first case, suppose we also know that the function f takes real values on the real axis, that is, $f(\xi) \in \mathbb{R}$ for $\xi \in \mathbb{R}$. Then in this case we have equality of f and $\tilde{f}$ on the boundary,

$$\tilde{f}(\xi) = \overline{f(\overline{\xi})} = \overline{f(\xi)} = f(\xi) \quad \text{for } \xi \in \mathbb{R},$$

and since both f and $\tilde{f}$ are analytic in some neighborhood of the boundary and coincide there, it follows that $\tilde{f}$ represents the analytic continuation of f into D_-.

We may generalize this notion somewhat. Suppose we only know that f is real along some line segment l of the real axis. Then the process of defining the function $\tilde{f}$ will again lead to an analytic continuation of f, but the extended function (which takes values $f(\zeta)$ for ζ in D^+ and $\tilde{f}(\zeta)$ for ζ in D_-) will be analytic on the real axis only along segment l, and not elsewhere on this axis. An obvious example of such a function is given by $f(\zeta) = \sqrt{\zeta}$, defined for points $\zeta \in D_+$ by

$$f(\zeta) = f(re^{i\theta}) = r^{1/2}e^{i\theta/2}, \quad 0 \leq \theta \leq \pi.$$

This is real along $l = [0, \infty)$ but not along $(-\infty, 0)$, and taking values at points $\zeta = re^{i\theta} \in D_-$ $(-\pi < \theta < 0)$ given by

$$\tilde{f}(re^{i\theta}) = \overline{f(re^{-i\theta})} = \overline{r^{1/2}e^{-i\theta/2}} = r^{1/2}e^{i\theta/2},$$

we see that the composite function defined by f and $\tilde{f}$ is analytic on the ζ-plane, except along the real negative axis (which of course represents a branch-cut in the plane), where it is discontinuous.

In the above, we used the fact that the function f was real along the real axis, or segments of it, in order to define the analytic continuation.

Considering $z = f(\zeta)$ as a conformal mapping, this means that (part of) the real axis in the ζ-plane maps to (part of) the real axis in the z-plane. However, since it is trivial to translate and rotate a line segment, we can see that we could easily carry out the same procedure of analytic continuation for the case in which $z = f(\zeta)$ maps the real ζ-axis, or part of it, to any straight line in the z-plane. The general equation of such a straight line is

$$\Re(\beta z) = \gamma, \quad \beta \in \mathbb{C} \quad (\beta \neq 0), \quad \gamma \in \mathbb{R},$$

and we suppose this is the image of $\zeta = \xi \in \mathbb{R}$ under f, so that we have

$$\beta f(\xi) + \overline{\beta}\overline{f(\xi)} = 2\gamma, \quad \xi \in \mathbb{R}. \tag{182}$$

Consider now the function $\tilde{f}$ defined for $\zeta \in D_-$ by

$$\tilde{f}(\zeta) = \frac{2\gamma}{\beta} - \frac{\overline{\beta}}{\beta}\overline{f(\overline{\zeta})}.$$

For $\zeta \in D_-$, $\overline{\zeta} \in D_+$, and as before, $\tilde{f}$ may be seen to define a function analytic on D_-. Moreover, letting ζ approach the boundary between D_+ and D_-, $\zeta = \xi \in \mathbb{R}$, we have

$$\tilde{f}(\xi) = \frac{2\gamma}{\beta} - \frac{\overline{\beta}}{\beta}\overline{f(\xi)} = \frac{2\gamma}{\beta} - \frac{1}{\beta}(2\gamma - \beta f(\xi)) = f(\xi).$$

Thus here, as before, $\tilde{f}$ represents an analytic continuation of the function f onto the lower half-plane D_-; and image points $z = \tilde{f}(\zeta)$, $\zeta \in D_-$, lie in a reflected region to the other side of the line $\Re(\beta z) = \gamma$ in the z-plane.

Before going on to state and prove the Schwarz-Christoffel theorem, we note that the above discussion extends to analytic continuation across line segments, as well as infinite lines. As with the simple example above, when the continuation is across line segments, the resulting composite function in general has branch-points.

A.2 The Schwarz-Christoffel theorem

Theorem A.1 *(Schwarz-Christoffel) Let Γ be the piecewise linear boundary of a polygon in the z-plane, with vertices A_j $(j = 1, \ldots, n)$ and associated interior angles $\pi\alpha_j$. Then, the transformation $z = f(\zeta)$ defined by*

$$\frac{df}{d\zeta} = C\Pi_{j=1}^{n}(\zeta - a_j)^{\alpha_j - 1}, \tag{183}$$

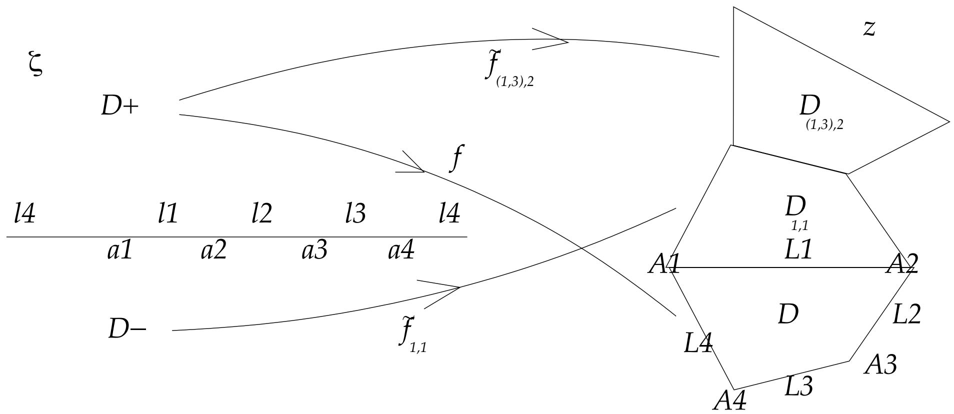

Figure 19: Schematic showing the action of the conformal map f and its various analytic continuations

where $C \in \mathbb{C}$ and $a_j \in \mathbb{R}$, gives a mapping from D_+, the upper half ζ-plane, to D, the interior of the polygon, such that the real axis maps onto Γ, with $f(a_j) = A_j$. $f(\zeta)$ is a one-to-one transformation on D_+, conformal and analytic except at the points $\zeta = a_j$ on the real axis.

Proof (Non-rigorous.) We know that such a mapping f exists, by the Riemann mapping theorem. We prove the result by construction of the desired function f. Consider the line segment $l_j = [a_j, a_{j+1}]$ of the real ζ-axis, that maps to the line segment $L_j = [A_j, A_{j+1}]$ under f. By the Schwarz reflection method discussed above, we can analytically continue f across l_j, to a function $\tilde{f}_{j,1}$, analytic on the lower half ζ-plane D_-, and such that f and $\tilde{f}_{j,1}$ together form a function analytic on the two half-planes, and analytic across l_j. The image of D_- under $\tilde{f}_{j,1}$ will be a reflected polygon $D_{j,1}$, reflected across segment $[A_j, A_{j+1}]$. We refer to the sketch in figure 19 for a specific example. In the same manner, we may analytically continue $\tilde{f}_{j,1}$ across another line segment l_k, to obtain a new function $\tilde{f}_{(j,k),2}$, analytic on D_+. Again, $\tilde{f}_{j,1}$ and $\tilde{f}_{(j,k),2}$ together give a function analytic on D_-, D_+ and across l_k; and the image of D_+ under $\tilde{f}_{(j,k),2}$ will be the domain $D_{(j,k),2}$, a reflection of $D_{j,1}$ across another of its sides. Thus, $D_{(j,k),2}$ is nothing more than the original polygon D, translated and rotated. (In reality, $\tilde{f}_{(j,k),2}$ is nothing more than another branch of the original function f.) We may repeat this process as often as we like, analytically continuing functions across arbitrary line segments; and provided we do this an even number of times, the resulting function $\tilde{f}_{\boldsymbol{j},2m}(\zeta)$ (where $\boldsymbol{j}$ is a vector of length $2m$ representing the sequence of Schwarzian reflections used) takes the upper half ζ-plane D_+ onto a rotation and translation of D. Therefore, $\tilde{f}_{\boldsymbol{j},2m}(\zeta)$ is linearly related to f,

$$\tilde{f}_{\boldsymbol{j},2m}(\zeta) = B_{\boldsymbol{j},2m} f(\zeta) + C_{\boldsymbol{j},2m}, \qquad B_{\boldsymbol{j},2m} = \exp(i\beta_{\boldsymbol{j},2m}),$$

and it follows that the function

$$g(\zeta) = \frac{\tilde{f}''_{\boldsymbol{j},2m}(\zeta)}{\tilde{f}'_{\boldsymbol{j},2m}(\zeta)} \equiv \frac{f''(\zeta)}{f'(\zeta)}$$

is in fact single-valued, regardless of whether we use f or any one of the $\tilde{f}_{\boldsymbol{j},2m}$ to define it.

We now note that $g(\zeta)$ is in fact real-valued on the real ζ-axis. This follows because the piecewise linear nature of the boundary mapping means that $f(\xi) = F(\xi)e^{i\gamma_j}$, for $F(\xi)$ a real function, and γ_j a real constant on each line segment l_j. Therefore, on the real axis $f'(\zeta) = f'(\xi) = F'(\xi)e^{i\gamma_j}$ and $f''(\zeta) = f''(\xi) = F''(\xi)e^{i\gamma_j}$, so that

$$g''(\zeta) = \frac{f''(\xi)}{f'(\xi)} = \frac{F''(\xi)}{F'(\xi)} \in \mathbb{R}.$$

This fact, together with the uniqueness of g, tells us that, although there are many possible branches for f when analytically continued across line segments, there is one unique analytic function $g(\zeta) = f''(\zeta)/f'(\zeta)$; and this function may be analytically continued across the entire real axis, since it takes real values there, by the usual prescription

$$\tilde{g}(\zeta) = \overline{g(\overline{\zeta})},$$

to give a function analytic on the entire ζ-plane except at the points $\zeta = a_j$.

Near a_j we know that the map f must describe the polygonal vertex A_j, and so locally f has the form

$$f(\zeta) = A_j + (\zeta - a_j)^{\alpha_j}(c_j^{(0)} + c_j^{(1)}(\zeta - a_j) + \cdots),$$

where $c_j^{(0)} \in \mathbb{C} \neq 0$ (we know that f maps a_j to A_j, and that at this point angles are multiplied by the factor α_j, since the straight line, with "angle"

π, must map to a corner of angle $\pi\alpha_j$). It follows that $g(\zeta) = f''(\zeta)/f'(\zeta)$ has a simple pole at the point a_j, of residue $(\alpha_j - 1)$,

$$g(\zeta) = \frac{\alpha_j - 1}{\zeta - a_j}\left(1 + O(\zeta - a_j)\right). \tag{184}$$

Apart from these poles, g has no other singularities, and

$$G(\zeta) := g(\zeta) - \sum_{j=1}^{n} \frac{\alpha_j - 1}{(\zeta - a_j)}$$

defines an entire function. Since we assume for the moment that no vertex of the polygon is at infinity, then $f(\zeta)$ is analytic at infinity and so G is in fact a bounded entire function, and so is equal to some constant c, by Liouville's theorem. But since f itself is bounded and analytic at infinity, it has a regular Taylor series in powers of $1/\zeta$ there; and it follows that $g(\zeta) = f''(\zeta)/f'(\zeta) \to 0$ as $|\zeta| \to \infty$, and hence also that $G(\zeta) \to 0$ as $|\zeta| \to \infty$. Thus $G(\zeta) \equiv 0$, and

$$\frac{f''(\zeta)}{f'(\zeta)} = \sum_{j=1}^{n} \frac{\alpha_j - 1}{(\zeta - a_j)}.$$

Integration of this result leads to (183).

A.2.1 Notes on the theorem

1. Since the sum of the exterior angles of a polygon is 2π, and the exterior angle at the jth vertex is $\pi - \pi\alpha_j$, it follows that

$$\sum_{j=1}^{n} \alpha_j = n - 2. \tag{185}$$

2. The correspondence of three, and only three, boundary points, can be prescribed arbitrarily (see theorem 3.20). So, for example, given a polygon with vertices A_j, we can fix a_1, a_2 and a_3 at will, but the remaining a_j (sometimes called *accessory parameters*) cannot be chosen and must be determined somehow (this can be difficult).

3. Although the proof as given does not carry through to the case in which one of the vertices A_j lies at infinity (consider the step where we argue for the local form of f near one of the a_j), the result of the theorem is good for this case also. The case $\alpha_j = 0$ corresponds to a (parallel-sided) strip, while an open sector-type domain corresponds to $\alpha_j < 0$. We just need to argue that, local to a_j, $g(\zeta) = f''(\zeta)/f'(\zeta)$ has the appropriate form as given by (184). This is straightforward enough to do for the "open sector" case, where the local form of f may again be written down, but requires a little more thought for the strip-like case.

4. The next possible flaw in the theorem's proof is: What if one of the preimage points a_k lies at infinity? We saw in point 2 above that we cannot always specify the a_j at will, so we cannot be sure that they are always at finite points. To address this point, we consider the transformation

$$\zeta = a_k - \frac{1}{w},$$

which takes $\zeta = a_k$ to $w = \infty$, while the other a_j are mapped to finite points $b_j = 1/(a_k - a_j)$ on the real w-axis. We can now investigate the form of the mapping $z = F(w) = f(\zeta(w))$ from the upper half w-plane to the polygon. We have

$$f'(\zeta)\zeta'(w) = F'(w), \quad \text{and} \quad \zeta'(w) = \frac{1}{w^2}.$$

Also, for $j \neq k$,

$$\zeta - a_j = \frac{a_k - a_j}{w}(w - b_j), \quad \text{while} \quad \zeta - a_k = -\frac{1}{w}.$$

Substituting these results in (183) gives

$$w^2 F'(w) = \tilde{C}\left(\frac{1}{w}\right)^{\alpha_k - 1} \Pi_{j\neq k}\left(\frac{1}{w}\right)^{\alpha_j - 1}(w - b_j)^{\alpha_j - 1}$$
$$\Rightarrow \quad F'(w) = \tilde{C}\Pi_{j\neq k}(w - b_j)^{\alpha_j - 1},$$

using remark (1) above. Near infinity we see that

$$F'(w) = \tilde{C}w^{-(\alpha_k+1)}\left(1 + \frac{p_1}{w} + \frac{p_2}{w^2} + \cdots\right),$$

so for $\alpha_k > 0$ $w = \infty$ maps to a finite point, as usual.

A.2.2 Examples

Example A.2 *Find a mapping from the upper half ζ-plane to the semi-infinite strip $-k < \Re(z) < k$, $\Im(z) > 0$.*

Here we have freedom to choose the correspondence between points, and writing $A_1 = -k$, $A_2 = k$ and $A_3 = \infty$, we take $a_1 = -1$, $a_2 = 1$, $a_3 = \infty$. With interior angles $\pi/2$ at A_1 and A_2 we have $\alpha_1 = \alpha_2 = 1/2$, and formula (183) then gives

$$f'(\zeta) = C(\zeta+1)^{-1/2}(\zeta-1)^{-1/2} = \frac{\tilde{C}}{\sqrt{1-\zeta^2}}$$

which, on integrating, gives

$$f(\zeta) = \tilde{C}\sin^{-1}\zeta + C_1.$$

Knowing that $f(\pm 1) = \pm k$ fixes $C_1 = 0$, $\tilde{C} = 2k/\pi$, so that

$$z = f(\zeta) = \frac{2k}{\pi}\sin^{-1}\zeta, \quad \text{or} \quad \zeta = \sin\frac{\pi z}{2k}.$$

B Fluid dynamics

The state of a flowing liquid (or gas) may be characterized by specifying the velocity of each *fluid element* $\boldsymbol{x}(t)$, as a function of time:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(\boldsymbol{x}, t)$$

(think of a fluid element as an infinitesimal dyed blob of fluid). The equations governing a two-dimensional viscous incompressible fluid with velocity field $\boldsymbol{u}(\boldsymbol{x}, t) = (u(x, y, t), v(x, y, t))$ are, in the absence of external body forces such as gravity,

$$\rho \frac{du}{dt} \equiv \rho(u_t + uu_x + vu_y) = -p_x + \mu(u_{xx} + u_{yy}), \tag{186}$$

$$\rho \frac{dv}{dt} \equiv \rho(v_t + uv_x + vv_y) = -p_x + \mu(v_{xx} + v_{yy}), \tag{187}$$

$$\nabla \cdot \boldsymbol{u} = u_x + v_y = 0, \tag{188}$$

where the scalar function $p(x, y, t)$ is the *pressure* in the fluid at any given point. Equations (186) and (187) represent the momentum balance for the fluid (essentially Newton's 2nd law, "mass times acceleration equals force" for the fluid, in both x and y directions), while (188) represents the incompressibility condition. These equations, which collectively are known as the **Navier-Stokes** equations, can be derived by considering the rates of change of momentum and mass for an arbitrary material fluid volume; see for example Ockendon & Ockendon [7].

We may transform the problem to dimensionless form by introducing representative scalings for each of the physical quantities. Suppose that L is a representative lengthscale for the flow, and that U is a representative velocity scale (for example, if we are considering the flow generated by moving a sphere through a large volume of stationary fluid then the representative lengthscale would be the sphere radius, while the velocity scale would be the speed of translation of the sphere). We anticipate that the representative timescale for the disturbances generated will be $T = L/U$. If we choose an arbitrary pressure scaling Π for the moment, then introducing our scalings

$$(x, y) = L(x', y'), \quad (u, v) = U(u', v'), \quad t = (L/U)t', \quad p = \Pi p'$$

into (186) gives (on dropping the primes)

$$\frac{\rho U^2}{L}(u_t + uu_x + vu_y) = -\frac{\Pi}{L}p_x + \frac{\mu U}{L^2}(u_{xx} + u_{yy})$$

$$\Rightarrow \quad \frac{\rho U L}{\mu}(u_t + uu_x + vu_y) = -\frac{\Pi L}{\mu U} p_x + (u_{xx} + u_{yy}). \tag{189}$$

The quantity $\rho UL/\mu$ appearing on the left-hand side here is a very important parameter in fluid mechanics: the *Reynolds number* of the flow, which we write as Re. The left-hand side of this equation represents the *inertia* of the fluid. With our assumptions of no external forces acting, the flow is always being driven by the pressure gradient, so the term p_x will always be important in the equations. The remaining 2nd derivative terms on the right-hand side are the viscous forces (essentially internal friction) within the fluid. One possibility is that each of these three effects (inertia, pressure, viscous) could be equally important – in this case there is no real simplification of the problem, and it's very difficult to find exact solutions. Another possibility is that the pressure terms balance only one of the other two: inertial or viscous terms. We consider the two cases below.

B.1 Case 1: Inviscid fluid dynamics

If pressure balances inertia, with viscous effects much smaller, then this means that the Reynolds number must be very large, $\mathrm{Re} \gg 1$. In this case the appropriate scaling for the pressure is $\Pi = \mu U \mathrm{Re}/L = \rho U^2$, and (189) becomes

$$(u_t + uu_x + vu_y) = -p_x + \frac{1}{\mathrm{Re}}(u_{xx} + u_{yy}).$$

For $\mathrm{Re} \ll 1$ this can be approximated by the "inviscid" form of the equation,

$$(u_t + uu_x + vu_y) = -p_x, \tag{190}$$

with (187) similarly becoming

$$(v_t + uv_x + vv_y) = -p_y,$$

and (188) remaining unchanged,

$$u_x + v_y = 0.$$

Suppose now that we have the situation considered in §4.1, where in addition to being 2D and incompressible, the flow is also irrotational, $\boldsymbol{\omega} = \nabla \cdot \boldsymbol{u} = \boldsymbol{0}$. Then with this condition and (188) we have

$$u_x + v_y = 0, \quad v_x - u_y = 0,$$

so that $u - iv$ is an analytic function, by the Cauchy-Riemann theorem. Since analytic functions are infinitely differentiable (and integrable) we may write

$$\frac{dw}{dz} = u - iv, \tag{191}$$

(as usual $z = x + iy$) for some complex analytic function $w(z) = \phi(x, y, t) + i\psi(x, y, t)$. Since, differentiating w with respect to z we have

$$\frac{dw}{dz} = \phi_x + i\psi_x = \phi_x - i\phi_y \tag{192}$$

(on use of the Cauchy-Riemann equations). Comparing (191) and (192), we see that

$$(u, v) = (\phi_x, \phi_y) = \nabla\phi$$

and the function ϕ is a velocity potential for the flow. The function $w = \phi + i\psi$ is the complex potential. As the real part of a complex analytic function, ϕ is harmonic, and thus solutions of the 2D Laplace equation represent possible solutions to inviscid fluid dynamics. Returning to (190) and (191), the pressure field satisfies

$$\frac{\partial}{\partial x}\left(p + \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2)\right) = 0 = \frac{\partial}{\partial y}\left(p + \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2)\right).$$

Integration of these leads to the well-known Bernoulli condition for inviscid flow which, in the special case of steady flow ($\partial/\partial t = 0$) takes the form

$$p = \text{constant} - \frac{1}{2}(\phi_x^2 + \phi_y^2) = \text{constant} - \frac{1}{2}|\boldsymbol{u}|^2.$$

B.2 Case 2: Very viscous flow (Stokes flow)

Returning to (189), the other possibility we mentioned is that the driving pressure gradient balances viscous forces, with inertial terms being much smaller. In this case the Reynolds number is very small, $\mathrm{Re} \ll 1$, and we choose the pressure scale Π to be $\Pi = \mu U/L$, so that (189) becomes

$$\mathrm{Re}(u_t + uu_x + vu_y) = -p_x + (u_{xx} + u_{yy}),$$

with a similar expression arising from the equation (187):

$$\mathrm{Re}(v_t + uv_x + vv_y) = -p_y + (v_{xx} + v_{yy}),$$

Neglecting the left-hand side in these equations gives the equations of (two-dimensional) slow viscous flow, known as the Stokes equations:

$$p_x = u_{xx} + u_{yy}, \quad p_y = v_{xx} + v_{yy}, \quad u_x + v_y = 0.$$

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